10.2 Monotone Sequences

10.3.1 DEFINITION. A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

increasing if $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$

strictly decreasing if $a_1 > a_2 > a_3 > \cdots > a_n > \cdots$

decreasing if $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$

SEQUENCE	DESCRIPTION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	Strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	Strictly decreasing
$1, 1, 2, 2, 3, 3, \ldots$	Increasing; not strictly increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	Decreasing; not strictly decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing nor decreasing

Testing for Monotonicity

DIFFERENCE BETWEEN SUCCESSIVE TERMS	CLASSIFICATION
$a_{n+1} - a_n > 0$	Strictly increasing
$a_{n+1} - a_n < 0$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	Increasing
$a_{n+1} - a_n \leq 0$	Decreasing

RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1}/a_n \leq 1$	Decreasing

Example 1 Show that the sequence is a strictly increasing

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$$

Monotonic By Derivative *f(x)*

10.3.2 DEFINITION. If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, then the original sequence is said to have that property ***eventually***.

For example $3, -4, 17, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Example 2

Show the sequence $\left\{\frac{4^n}{n!}\right\}_{n=1}^{\infty}$ is eventually decreasing

Example 3

Show the sequence $\{2n^2 - 7n\}_{n=1}^{\infty}$ is eventually increasing

Converge of Monotone Sequences

10.3.3 THEOREM. *If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:*

(a) *There is a constant M, called an* **upper bound** *for the sequence, such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.*

(b) *No upper bound exists, in which case $\lim_{n \to +\infty} a_n = +\infty$.*

10.3.4 THEOREM. *If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:*

(a) *There is a constant M, called a* **lower bound** *for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.*

(b) *No lower bound exists, in which case $\lim_{n \to +\infty} a_n = -\infty$.*

Example 4

Show that the sequence $\left\{ \frac{4^n}{n!} \right\}_{n=1}^{\infty}$ converge and find its limit.