10.10 Differentiating and Integrating Power Series

PROBLEM. Suppose that a function f is represented by a power series on an open interval. How can we use the power series to find the derivative of f on that interval?

THEOREM (***Differentiation of Power Series***). *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

Then:

(a) *The function f is differentiable on the interval $(x_0 - R, x_0 + R)$.*

(b) *If the power series representation for f is differentiated term by term, then the resulting series has radius of convergence R and converges to f' on the interval $(x_0 - R, x_0 + R)$; that is,*

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}[c_k(x - x_0)^k] \qquad (x_0 - R < x < x_0 + R)$$

Example 1

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad R = \infty$$

$$\frac{d}{dx}[\sin x] = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!} = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad R = \infty$$

$$= \cos x$$

Example 2

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{d}{dx}\left[e^x\right] = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad = \sum_{k=0}^{\infty} \frac{d}{dx}\left[\frac{x^k}{k!}\right] \qquad = \sum_{k=0}^{\infty} \frac{kx^{k-1}}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} \qquad = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k\left(k-1\right)!} \qquad = \sum_{k=1}^{\infty} \frac{x^{k-1}}{\left(k-1\right)!}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{\left(k\right)!} = e^x$$

THEOREM (*Integration of Power Series*). *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

(a) *If the power series representation of f is integrated term by term, then the resulting series has radius of convergence R and converges to an antiderivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$; that is,*

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \left[\frac{c_k}{k+1} (x - x_0)^{k+1} \right] + C \qquad (x_0 - R < x < x_0 + R)$$

(b) *If α and β are points in the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from α to β, then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and*

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{k=0}^{\infty} \left[\int_{\alpha}^{\beta} c_k (x - x_0)^k\,dx \right]$$

Example 3

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\int \cos x \, dx = \int \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] dx$$

$$= \left[x - \frac{x^3}{3(2!)} + \frac{x^5}{5(4!)} - \frac{x^7}{7(6!)} + \cdots \right] + c$$

$$= \left[x - \frac{x^3}{(3!)} + \frac{x^5}{(5!)} - \frac{x^7}{(7!)} + \cdots \right] + c$$

$$= \sin x + c$$

Example 4 Find Maclaurin series of $\tan^{-1} x$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\tan^{-1} x + c = \int \frac{1}{1+x^2}\,dx = \int \left[1 - x^2 + x^4 - x^6 + x^8 - \cdots\right] dx$$

$$\tan^{-1} x = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right] - c$$

$$\tan^{-1} 0 = 0 \rightarrow \qquad c = 0$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \qquad -1 < x < 1$$

Example 4 $If \ \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad -1 < x < 1.$

Find Miclaurin series of

$a) \frac{1}{1-x^2} \qquad b) \frac{x^2}{1+x}$