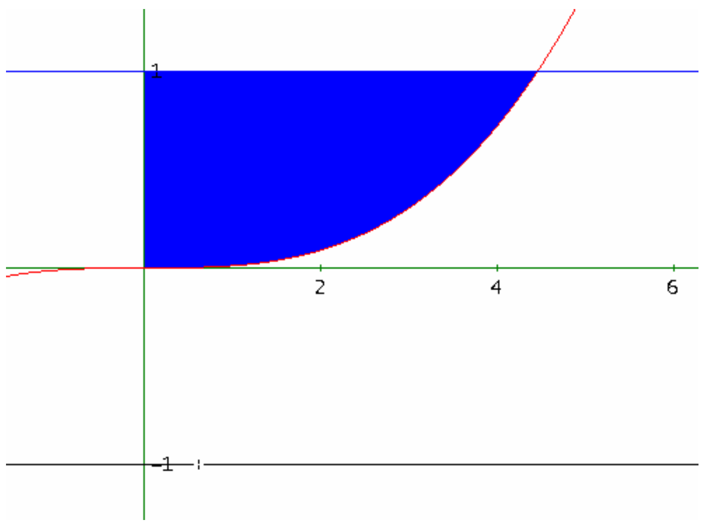
King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences
Semester (052) Math 102-1&5 Second Major Exam
Date: May 26, 2006 **Time: 6:20-7:50 pm**

Name: ______________________________ ID#:____________

Q1.(12points) Use cylindrical shells to find the volume of the solid generated when the region enclosed by

$$y=\left(\frac{\pi x}{14}\right)^3,\ y=1,\ and\ x=0, \text{ is revolving about } y=-1.$$



$$v=\int_0^1 2\pi(y+1)\left(\frac{14}{\pi}y^{\frac{1}{3}}\right)dy$$

$$=28\int_0^1\left(y^{\frac{4}{3}}+y^{\frac{1}{3}}\right)dy$$

$$=28\left[\frac{3}{7}y^{\frac{7}{3}}+\frac{3}{4}y^{\frac{4}{3}}\right]_0^1=28\left[\left(\frac{3}{7}+\frac{3}{4}\right)-0\right]=33$$

Q2. (12 points) Find the exact arc length of the parametric curve

$$x = 2e^{t}, \quad y = \frac{1}{2}e^{2t} - t, \, 0 \le t \le 1.$$

$$x'(t) = 2e^{t}, \quad y'(t) = e^{2t} - 1$$

$$\left(x'(t)\right)^2 + \left(y'(t)\right)^2 = 4e^{2t} + e^{4t} - 2e^{2t} + 1 = e^{4t} + 2e^{2t} + 1 = \left(e^{2t} + 1\right)^2$$

$$L = \int_0^1 \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)}\, dt = \int_0^1 \sqrt{\left(e^{2t} + 1\right)^2}\, dt = \int_0^1 \left(e^{2t} + 1\right) dt$$

$$= \frac{e^{2t}}{2} + t \Big]_0^1 = \left(\frac{e^2}{2} + 1\right) - \left(\frac{1}{2} + 0\right) = \frac{e^2}{2} + \frac{1}{2}$$

Q3. (12 points) Find the area of the surface that generated by revolving the curve

$$x = \sqrt{9 - y^2}, \, -2 \le y \le 2, \text{ about } \mathbf{y\text{-axis}}$$

$$\frac{dx}{dy} = \frac{-y}{\sqrt{9 - y^2}}, \quad 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 = 1 + \frac{y^2}{9 - y^2} = \frac{9 - y^2 + y^2}{9 - y^2} = \frac{9}{9 - y^2}$$

$$S = \int_c^d 2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\, dy = \int_{-2}^{2} 2\pi \sqrt{9 - y^2} \sqrt{\frac{9}{9 - y^2}}\, dy$$

$$= \int_{-2}^{2} 2\pi \sqrt{9}\, dy = 6\pi \int_{-2}^{2} dy = 24\pi$$

Q4. (50 points) Evaluate the following

$$a)\ \int (\ln x)^2 dx$$

$$u = (\ln x)^2 \qquad dv = 1dx$$

$$du = \frac{2\ln x}{x}dx \qquad v = x$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int x\,\frac{2\ln x}{x}dx = x(\ln x)^2 - \int 2\ln x\, dx$$

$$u = 2\ln x \qquad dv = 1dx$$

$$du = \frac{2}{x} \qquad v = x$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \left[2x\ln x - \int \frac{2}{x}x\,dx\right]$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

$$b)\ \int_0^{\ln 2} \coth^2 x\, dx$$

$$\int_0^{\ln 2} \coth^2 x\,dx = \int_0^{\ln 2}\left(\operatorname{csch}^2 x + 1\right)dx = \lim_{k\to o+}\int_k^{\ln 2}\left(\operatorname{csch}^2 x + 1\right)dx$$

$$= \lim_{k\to o+}\left(-\coth x + x\right)\Big]_k^{\ln 2} = \lim_{k\to o+}\left[\left(-\coth\ln 2 + \ln 2\right) - \left(-\coth k + k\right)\right]$$

$$= \lim_{k\to o+}\left[-\frac{e^{\ln 2} + e^{-\ln 2}}{e^{\ln 2} - e^{-\ln 2}} + \ln 2 - \left(-\frac{e^{k} + e^{-k}}{e^{k} - e^{-k}} + k\right)\right]$$

$$= -\frac{2+\frac{1}{2}}{2-\frac{1}{2}} + \ln 2 + \frac{2}{0} = \infty$$

$$c)\int_{o}^{\sqrt{3}}\frac{x^5}{\sqrt{x^2+1}}dx=I$$

$$x=\tan\theta\rightarrow dx=\sec^2\theta d\theta$$

$$\int\frac{x^5}{\sqrt{x^2+1}}dx=\int\frac{\tan^5\theta}{\sqrt{\tan^2\theta+1}}\sec^2\theta d\theta$$

$$=\int\frac{\tan^5\theta}{\sqrt{\sec^2\theta}}\sec^2\theta d\theta=\int\tan^5\theta\sec\theta d\theta$$

$$=\int\tan^4\theta\tan\theta\sec\theta d\theta=\int\left(\tan^2\theta\right)^2\tan\theta\sec\theta d\theta$$

$$=\int\left(\sec^2\theta-1\right)^2\tan\theta\sec\theta d\theta,\ u=\sec\theta\rightarrow du=\sec\theta\tan\theta d\theta$$

$$=\int\left(u^2-1\right)^2du=\int\left(u^4-2u^2+1\right)du$$

$$=\frac{u^5}{5}-\frac{2u^3}{3}+u+c=\frac{\sec^5\theta}{5}-\frac{2\sec^3\theta}{3}+\sec\theta+c$$

$$I=\left.\frac{\left(\sqrt{1+x^2}\right)^5}{5}-\frac{2\left(\sqrt{1+x^2}\right)^3}{3}+\left(\sqrt{1+x^2}\right)\right]_0^{\sqrt{3}}=\frac{38}{15}$$

$$d)\ \int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} dx$$

$$= \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{k\to 1+} \int_k^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{\substack{l\to\infty \\ l\to 1+}} \int_2^l \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{k\to 1+} \left(\sec^{-1} x \Big]_k^2 \right) + \lim_{l\to\infty} \left(\sec^{-1} x \Big]_2^l \right)$$

$$= \lim_{k\to 1+} \left(\sec^{-1} 2 - \sec^{-1} k \right) + \lim_{l\to\infty} \left(\sec^{-1} l - \sec^{-1} 2 \right)$$

$$= \sec^{-1} 2 - \sec^{-1} 1 + \sec^{-1} \infty - \sec^{-1} 2$$

$$= -\sec^{-1} 1 + \sec^{-1} \infty = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$e) \int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$$

$$u = x^{\frac{1}{6}} \to du = \frac{1}{6} x^{-\frac{5}{6}} dx \to dx = 6x^{\frac{5}{6}} du = 6u^5 du$$

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5 du}{u^3 - u^2} = 6\int \frac{u^3}{u-1} du, \text{ by long division}$$

$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1} \right) du$$

$$= 6\left(\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1| \right) + c$$

$$= 2x^{1/2} + 3x^{1/3} + x^{1/6} + \ln\left|x^{1/6} - 1\right| + c$$

Q5. <u>(14 points)</u> Evaluate $\displaystyle\int \frac{x^3+3x^2+x+9}{\left(x^2+9\right)\left(x^2+3\right)}dx$

$$\frac{x^3+3x^2+x+9}{\left(x^2+9\right)\left(x^2+3\right)}=\frac{Ax+B}{x^2+9}+\frac{Cx+D}{x^2+3}$$

$$=\frac{(Ax+B)\left(x^2+3\right)+(Cx+D)\left(x^2+9\right)}{\left(x^2+9\right)\left(x^2+3\right)}$$

$coeff.\,of\;\; x^3 \rightarrow 1=A+C \qquad (1)$

$coeff.\,of\;\; x^2 \rightarrow 3=B+D \qquad (2)$

$coeff.\,of\;\; x \rightarrow \;\; 1=3A+9C \qquad (3)$

$cons\tan t \rightarrow \;\; 9=3B+9D \qquad (4)$

$from\ (1)\ \&\ (3) \rightarrow \underline{c=-1/3}\ \&\ \underline{A=4/3},$

$from\,(2)\ \&\ (4) \rightarrow \underline{D=0}\ \&\ \underline{B=3}$

$$Then\ I=\int \frac{\frac{4}{3}x+3}{x^2+9}dx+\int \frac{\frac{-1}{3}x}{x^2+3}dx$$

$$=\frac{4}{3}\int \frac{x}{x^2+9}dx+3\int \frac{1}{x^2+9}dx-\frac{1}{3}\int \frac{x}{x^2+3}dx$$

$$=\frac{2}{3}\ln(x^2+9)+\tan^{-1}\left(\frac{x}{3}\right)-\frac{1}{6}\ln(x^2+3)+c$$