CHAPTER THREE

THE DERIVATIVE

3.1 SLOPES AND RATES OF CHANGE

3.1.3 Definition. If $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$ *average velocity over* $[x_0, x_1]$

3.1.4 Definition. . If $y = f(x)$, then the instantaneous rate of change of y with respect to x *when* $x = x_0$ is

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

* *Instantaneous velocity at* x_0

* *Slope of the curve at the point* $P(x_0, f(x_0))$

3.2 THE DERIVATIVE

Objectives:

- *Slope of Curve and Tangent lines.*
- *The Derivative.*
- *Differentiability.*
- *Differentiability and continuity.*
- *Derivative Notation.*
- *Derivatives at the Endpoints of an Interval.*

3.2.1 Definition. Suppose that x_0 is a number in the domain of a function *f*. If

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \quad \text{exists,}$$

then the value of the limit is called the ***derivative of f at*** $\boldsymbol{x=x_0}$ and is denoted by $f'(x_0)$. That is

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

and $f'(x_0)$ is the ***slope of the graph of f at the point*** $\boldsymbol{P(x_0, f(x_0))}$ (or at $x=x_0$). If the limit does not exist, then the slope is ***undefined*** at P (or at $x=x_0$).

3.2.2 Definition. Suppose that x_0 is a number in the domain of a function *f*. If

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists,}$$

then we define the equation of the ***tangent line to the graph of f at the point*** $\boldsymbol{P}(\boldsymbol{x}_0, \boldsymbol{f}(\boldsymbol{x}_0))$ by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

3.2.3 Definition. The function f' denoted by the formula

$$f'(x) = \lim_{\omega \to x} \frac{f(\omega) - f(x)}{\omega - x}$$

is called the ***derivative of f with respect to x.***

DIFFERENTIABILITY

- For a number x_0 in the domain of a function f, we say that f is ***differentiable at*** x_0 or that ***the derivative of f at*** x_0, if

$$\lim_{\omega \to x_0} \frac{f(\omega) - f(x_0)}{\omega - x_0}$$

exists. If x_0 is not in the domain of f or the limit does not exist, the we say that f is ***not differentiable at*** x_0, or that ***the derivative of f does not exist at*** x_0.

- If f is differentiable at every value of x an open interval (a , b), then we say that f is ***differentiable on*** (a , b).
- In the case where f is differentiable on $(-\infty , \infty)$ we will say that f is ***differentiable everywhere.***
- f is not differentiable at x_0 where the graph of f fas
    - *a corner,*
    - *a vertical tangent line, or*
    - *a discontinuity.*

3.2.3 Theorem. *If f if differentiable at* $x = x_0$*, then f must also continuous at* $x = x_0$.

Derivative Notation

- $\dfrac{d}{dx}[f(x)] = f'(x)$
- $\dfrac{d}{dx}[f(x)]\Big|_{x=x_0} = f'(x_0)$

  if $y = f(x)$
- $\dfrac{dy}{dx} = f'(x)$ *and* $\dfrac{dy}{dx}\Big|_{x_0} = f'(x_0)$
- $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Derivative at the Endpoints of an Interval

We define derivative from the left

$$f'_{-}(x) = \lim_{\omega\to x^{-}} \frac{f(\omega) - f(x)}{\omega - x}$$

Derivative from the right

$$f'_{+}(x) = \lim_{\omega\to x^{+}} \frac{f(\omega) - f(x)}{\omega - x}$$

$f(x)$ is differentiable on $[a,b]$ if $f(x)$ is differentiable on (a,b) and differentiable from the left and right.

3.3 Techniques of Differentiation

Theorem.

a) *The derivative of a constant function is 0*

$$\frac{d}{dx}[c]=0.$$

b) **(The Power Rule)**. *If* n *is any integer, then*

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

c) *If f is differentiable at x*

$$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$$

d) *If f and g are differentiable at x, then so are f+g and f-g*

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)];$$

$$(f+g)'=f'+g'$$

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)];$$

$$(f-g)'=f'-g'$$

e) **(The Product Rule)** *If f and g are differentiable at x then so is fg*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)];$$

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

f) **(The Quotient Rule)** *If f and g are differentiable at x, and* $g(x) \neq 0$, *so is* f / g

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2};$$

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

Higher Derivative

$$f',\ f'' = (f')',\ \ f''' = (f'')',\ldots$$

In general

$f^{(n)}$ *The nth derivative of* f

$f^{(n)} = \dfrac{d^n}{x^n}$; *The nth derivative of* f *w.r.t.* x

$y = f(x)$

$\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots$

3.4 Derivative of Trigonometric Functions

- $\dfrac{d}{dx}[\sin x] = \cos x$ $\qquad \dfrac{d}{dx}[\cos x] = -\sin x$
- $\dfrac{d}{dx}[\tan x] = \sec^2 x$ $\qquad \dfrac{d}{dx}[\sec x] = \sec x \tan x$
- $\dfrac{d}{dx}[\cot x] = -\csc^2 x$ $\qquad \dfrac{d}{dx}[\csc x] = -\csc x \cot x$

3.5 The Chain Rule

Objectives:

Derivative of Compositions

3.5.1 Problem. *If we know the derivative of f and g, how can we use this information to find the derivative of fog?*

Let $y = (fog)(x) = f(g(x))$ *and* $u = g(x)$

So that $y = f(u)$

$$\frac{dy}{du} = f'(u) \quad and \quad \frac{du}{dx} = g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

3.5.2 Theorem(The Chain Rule). *If g is differentiable at x and f is differentiable at g(x), then fog is differentiable at x. Moreover,*

$$(fog)'(x) = f'(g(x))g'(x)$$

Alternatively. If

$$y = f(g(x)) \quad and \quad u = g(x)$$

then $y = f(u)$ *and*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note: An alternative Approach to Using the Chain Rule

The derivative of f(g(x)) is the derivative of the outside function evaluate at the inside function times the derivative of the inside function.

3.6 Implicit Differentiation

Note: An equation of the form $y = f(x)$ is said to define y ***explicit*** as a function of x because the variable y appears alone on one side of the equation.

For example $yx + y + 1 = x$ is not explicit, is called y implicit as a function of x.

3.7 Related Rates

A Strategy for Solving Related Rates Problems

1. Identify the rates of change that are given and those to be found. Interpret each rate of change as a derivative of a variable w.r.t. t and provide a description of each variable involved.
2. Find an equation relating the quantities whose rates are identified in Step 1. (Draw a graph, if applicable).
3. Differentiate the equation in Step 2 with respect to t.
4. Evaluate the equation found in Step 3 using the known values for the quantities and the rates of change at the moment in question.
5. Solve for the value of the remaining rate of change at this moment.

Example #1(17/224)
A 13-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at the rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

Example #2 (25/224)

A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at rate of 20 $ft^3/\min$, how fast the depth of the water increasing when the water is 16 ft deep.

Example #3 (24/224)

An aircraft is flying horizontally at the constant height of 4000ft above a fixed observation point. At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 mi/h.

(a) How fast is θ decreasing at this instant? Express the result in units of degree/s.

(b) How fast is the distance between the aircraft and the observation point change et this instant? Express the result in units of ft/s. Use 1mi=5280ft.

Example #4 (34/225)

An aircraft is flying at a constant altitude with constant speed 0f 600mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point *P*. At the instant the aircraft is 2 mi from the impact point *P* the missile is 4 mi from *P* and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing.

3.8 Local Linear Approximation; Differentials

Definition

Assume that a function *f* is differentiable at $x = x_0$. ***The local linear approximation of f at*** $x = x_0$. is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

$$Let\ \Delta x = x - x_0,\ then$$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Example #1

a) Find the local linear approximation of $f(x) = \sqrt{1+x}$ $at\ x_0 = 0$, and use it to approximate $\sqrt{1.1}$ and $\sqrt{0.9}$.

b) Graph f and its tangent line at x_0 together.

Example #2

By using local linear approximation at $x_0 = 0$, show that

$$\frac{1}{x+1} \approx 1 - x.$$

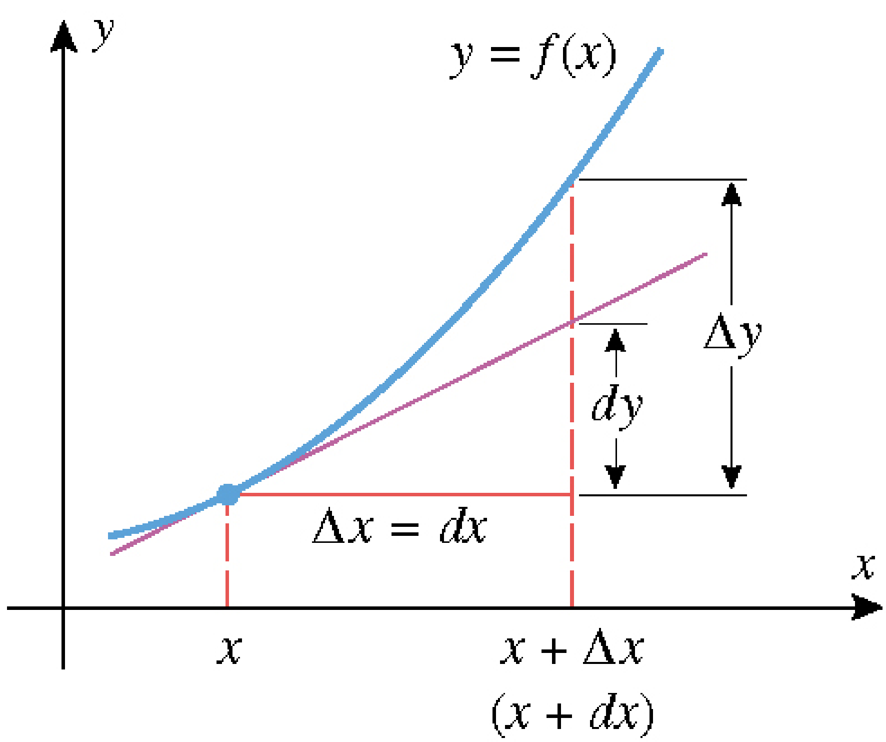
Differentials

If f is differentiable, then $\frac{dy}{dx} = f'(x)$, dx is called ***differential of x*** and ***the differential of f*** is

$dy = f'(x)dx$.

Example # 3 Given the function $y = x^3$, interpret the relationship between the differential dy *and* dx *when* $x = 2$.

Let a function $y = f(x)$, we defined $\Delta y = f(x + \Delta x) - f(x)$ to denoted the change in y from its value at some initial number x to its value at new number $x + \Delta x$.



Note: If Δx is closed to 0, then $\Delta y \approx dy$

Example # 4 Use dy to approximate Δy if $y = x\sqrt{8x+1}$; when x change from $x = 3$ *to* $x = 3.05$

Error Propagation in Application

If P is the true value and ΔP is the error in some measured or calculated of P then

a) $\frac{\Delta P}{P}$ is called ***relative error*** in calculation.

b) $\frac{\Delta P}{P}\%$ is called ***percentage error.***

Example # 5 (47/233)

The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be 30^o, a with possible error of $\pm 1^o$.

(a) Use differentials to estimate the error in the sides opposite and adjacent to measure angle.

(b) Estimate the percentage errors in the sides.

Example # 6 (54/234)
The volume of the sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within ±3%.($V = \frac{4}{3}\pi r^3$)

Example # 7Use Local Linear Approximation Method to estimate $\tan(62^\circ)$.

3.7(47/226)

Coffee is poured at a uniform rate of $20 cm^3 / s$ into a cup whose inside is shaped like truncated cone. If the upper and lower radii if the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up.

32/225
A man 6 ft tall is walking at the rate of $3\ ft/s$ toward a streetlight 18 ft high.

(a) At what rate is his shadow length changing?
(b) How fast is the trip of his shadow moving?

40/218 At what points is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

42/218 Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to x-axis.