Sequence and series

① Show that sequence $\left\{(2n+1)\sin\frac{7}{n}\right\}_{n=1}^{\infty}$ converges to 14

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (2n+1)\sin\frac{7}{n}$$

$$= \lim_{n\to\infty}\left(2n\sin\frac{7}{n} + \sin\frac{7}{n}\right)$$

$$= \lim_{n\to\infty}\left[14\cdot\underbrace{\frac{\sin\left(\frac{7}{n}\right)}{\left(\frac{7}{n}\right)}}_{\downarrow\, 1} + \underbrace{\sin\frac{7}{n}}_{\downarrow\, 0}\right] \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$= 14$$

② Show that $\int_{-\infty}^{\infty} e^{-|x|}\,dx$ converges and its value is 2

$$I = \underbrace{\int_{-\infty}^{0} e^{x}dx}_{I_1} + \underbrace{\int_{0}^{\infty} e^{-x}dx}_{I_2}$$

$$I_1 = \int_{-\infty}^{0} e^{x}\,dx = \lim_{t\to-\infty}\left[e^{x}\right]_t^0 = \lim_{t\to-\infty}(1-e^{t}) = 1$$

$$I_2 = \int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}(1-e^{-t}) = 1$$

$$\Rightarrow\ I = I_1 + I_2 = 2$$

③ $\int_0^1 \frac{dx}{e^x-1}$ diverges $\Rightarrow$ Improper, type II

$$= \lim_{t\to 0^+}\int_t^1 \frac{dx}{e^x-1} = \lim_{t\to 0^+}\left[\ln|e^x-1|\right]_t^1$$

$$= \lim_{t\to 0^+}\Big[\underbrace{\ln|e^1-1|}_{+ve} - \underbrace{\ln|e^t-1|}_{-\infty}\Big)$$

$$= +\infty \quad \text{Diverges}$$

④ $\int_1^{\infty}\frac{\ln x}{x^2} \underset{=}{\text{by parts}} \lim_{t\to\infty}\left(\ln x\cdot\left(-\frac{1}{x}\right) + \int\frac{1}{x^2}dx\right)$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{x}\right]_1^t + \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$

$$= \lim \left[-\frac{\ln t}{\ \ }\right] + \lim \left[-\frac{1}{t}+1\right]$$

(5) Show that $\int_0^1 \frac{dx}{3x-2}$ diverges

$$= \int_0^{2/3} \frac{dx}{3x-2} + \int_{2/3}^1 \frac{dx}{3x-2}$$

$$= \lim_{t \to \frac{2}{3}^-} \left(\int_0^t \frac{dx}{3x-2} \right) + \lim_{t \to \frac{2}{3}^+} \left(\int_t^1 \frac{dx}{3x-2} \right)$$

$$= \lim_{t \to \frac{2}{3}^-} \left[\frac{1}{3} \ln |3x-2| \right]_0^t + \lim_{t \to \frac{2}{3}^+} \left[\frac{1}{3} \ln |3x-2| \right]_t^1$$

$$= \lim_{t \to \frac{2}{3}^-} \left(\frac{1}{3} \underbrace{\ln |3t-2|}_{-\infty} - \frac{1}{3} \ln 2 \right) + \lim_{t \to \frac{2}{3}^+} \left(\frac{1}{3} \ln \underset{2}{[\text{scribbled}]} - \underset{-\infty}{\ln |3t-2|} \right)$$

$\Rightarrow -\infty$ Diverges

(6) Find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^n}{3^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} + \sum \left(\frac{2}{3}\right)^n$$

Geometrical series with $a = \frac{1}{3}$, $r = -\frac{1}{3}$

Geometrical Series $a = \frac{2}{3}$, $r = \frac{2}{3}$

$$= \frac{\frac{1}{3}}{1 + \frac{1}{3}} + \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{1}{4} + \frac{2}{1} = \boxed{\frac{9}{4}}$$

(7) Find the sum of series $\sum \frac{2}{(n+1)(n+3)}$ if converges

$$S_n = \sum_{i=1}^{n} \frac{2}{(i+1)(i+3)} = \sum_{n=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3} \right)$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6} \right) + \cdots \left(\frac{1}{n+1} - \frac{1}{(n+3)} \right)$$

$$\lim_{n \to \infty} S_n = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

(8) Test the convergence / Divergence of $\sum_{n=1}^{\infty} \frac{5-\sqrt{n}}{n^3}$

$\Rightarrow \sum \frac{5}{n^3} - \sum \frac{\sqrt{n}}{n^3}$

$\Rightarrow 5 \sum \frac{1}{n^3} - \sum \frac{1}{n^{5/2}}$

Convergent $p=3$ Convergent $p = 5/2$

$\Rightarrow$ Convergent

(9) Consider $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$,

(a) Show it is convergent using integral Test.

$\Rightarrow \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^2} = \cancel{\lim_{t\to\infty} \left[\frac{\ln x}{(\ln x)^2} - \int \frac{(-2)}{(\ln x)^3} \cdot \frac{1}{x} \cdot \ln x \right.}$

(Let $\ln x = u$, $\frac{1}{x} dx = du$, $\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u}$

$\Rightarrow \lim_{t\to\infty} \left[-\frac{1}{\ln x} \right]_2^t = \lim_{t\to\infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$

$\Rightarrow$ Convergent

(b) How many terms of the series will be added so that error in the sum is within 0.01?

$\Rightarrow \quad R_n \le \int_n^{\infty} \frac{dx}{x(\ln x)^2} < 0.01$

$\Rightarrow \frac{1}{\ln n} < 0.01$ { See part (a) replace 2 with n }

$\Rightarrow 1/\ln n < \frac{1}{100}$

$\Rightarrow \boxed{n > e^{100}}$

(10) Test the convergence of sequence $\{(2-e)^n\}_{n=1}^{\infty}$

$\Rightarrow \{r^n\}$, $|r| < 1$, $r = 2-e$

$\lim_{n\to\infty} |r^n| = 0 \Rightarrow$ Converges to 0.

(11) Show that $\sum_{n=1}^{\infty} a_n$ is convergent with sum $= 2$ whose sum of n terms is given by $S_n = 2 - \frac{(-1)^n}{n^2}$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{(-1)^n}{n^2}$$

$$= 2 \quad , \text{ Since } \lim_{n\to\infty} \left|\frac{(-1)^n}{n^2}\right| = 0$$

(12) Find sum of the series $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

$$S_n = \frac{(-3)^2}{2^3} + \frac{(-3)^3}{2^6} + \cdots\cdots$$

It is Geometrical Series $\quad a = \frac{(-3)^2}{2^3} = \frac{9}{8}$, $r = \frac{-3}{2^3} = \frac{-3}{8}$

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{-9/8}{1+3/8} = \boxed{\frac{9}{11}}$$

(13) Show that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is divergent, by integral Test

$f(x) = \frac{1}{x \ln x}$ is +ve, continuous, decreasing

$$\lim_{t\to\infty} \int_2^t \frac{1}{x \ln x} = \lim_{t\to\infty} \Big[\ln|\ln x|\Big]_2^t$$

(Integrate
let $\ln x = w$
$\frac{1}{x}\,dx = dw$
$\int \frac{1}{x \ln x} = \ln|\ln x|$

$$= \lim_{t\to\infty} \Big[\ln \ln t - \ln 2\Big]$$

$$= \infty$$

(14) Find $\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\,dx$ if exists?

$$= \lim_{2^-} \int_0^t \frac{x^3}{\sqrt{4-x^2}} = \cancel{\lim_{2^-} \Big[-2\ln|4-x^2| + \tfrac{1}{2}(4-x^2)\Big]_0^t}$$

($4 - x^2 = u$
$-2x\,dx = du$

$$\int \frac{x^3\,dx}{\sqrt{4-x^2}} = \int \frac{x^2 \cdot x\,dx}{\sqrt{4-x^2}}$$

$$= \int -\frac{1}{2}\left(\frac{4-u}{\sqrt{u}}\right) du$$

$$= \cancel{\lim_{2^-} \Big[-2\ln|4-t^2| + \tfrac{1}{2}(4-t^2)} + 2\ln 4 - 2\Big]$$

$$\lim_{2^-} \Big[-4\sqrt{4-t^2} + \tfrac{1}{3}(4-t^2)^{3/2} + 8 - 8/3\Big]$$