# MATH102 -073 FINAL

1. The area of the region enclosed by the curves $y = 2x^2\text{-}1$ and $y = -2x\text{-}1$ is 1/3

Hint: Find the point of intersection of curves, x=0 and x=-1

Area = $\int_{-1}^{0} \left[(-2x\text{-}1)\text{-}(2x^2 - 1)\right] dx\text{....}complete$ yourself

==============================

2. If the region enclosed by the curves y = sin x and y = 0 between x = 0 and $x = \pi$ is resolved about y-axis, then the volume of the solid generated is equal to: $2\pi^2$

Hint: Use the method of cylindrical cells, Volume about y-axis from x = 0 to $x = \pi$ , V= $\int_0^{\pi} 2\pi.radius\ of\ cell.height.thickness = \int_0^{\pi} 2\pi.x.\sin x dx...Complete$ yourself

==============================

3. The series $\sum_{n=1}^{+\infty}\left(\frac{n^3+1}{2n^3+n}\right)^n$ Converges by the Root test.

Hint: Apply root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n^3+1}{2n^3+n}\right)\right|^n} = \lim_{n\to\infty}\frac{n^3+1}{2n^3+n} = 1/2 < 1, Convergent)$$

==============================

4. Show that $\sum_{n=0}^{+\infty}\frac{(-1)^n + 2^{n+1}}{3^n}$ is equal to 27/4.

Hint:

=

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{3^n} + \sum_{n=0}^{+\infty}\frac{2^{n+1}}{3^n} = GS, a=1, r=-1/3 + GS, a=1, r=2/3 = \frac{1}{1+1/3} + 2.\frac{1}{1-3/3} = 27/4$$

==============================

5. Show that $\int \frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}dx$ is equal to $x + \sqrt{1-x^2} + C$

Hint: $= \int\left(1 - \frac{x}{\sqrt{1-x^2}}\right)dx$ , Let 1-x$^2 = u$ and complete.

==============================

6. Show that $\int_0^{\pi} |\sin 2x| dx$ is equal to 2

Hint:

$$\int_0^{\pi/2} \sin 2x dx - \int_{\pi/2}^{\pi} \sin 2x dx..Complete\ yourself\ (|\sin 2x| = \sin 2x, 0 \le 2x \le \pi \ or\ 0 \le x \le \pi/2$$

and $(|\sin 2x| = -\sin 2x, \pi \le 2x < 2\pi \ or\ \pi/2 \le x \le \pi$

==============================

7. The set of all values of p for which the series $\sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{(n-1)^{5p-1}}$ is $\left(\frac{1}{5}, \infty\right)$ ,

converges is given by the interval $\left(\frac{1}{5}, \infty\right)$

Hint:

=

$\sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{(n-1)^{5p-1}}$ *is Alternating series,* dereasing and $b_k = \frac{1}{(k-1)^{5p-1}}$. If $b_k = 0$ *then* 5p-1>0

p will be in $\left(\frac{1}{5}, \infty\right)$.

==============================

8. The area of the surface obtained by rotating the curve

$y = \sqrt{3 - x^2}$, $0 \le x \le 1$, about the x-axis is equal to $2\pi\sqrt{3}$

Hint: Surface area

$$= \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2\pi x \sqrt{1 + \frac{x^2}{3 - x^2}} dx = -\pi \int_3^2 \frac{du}{u^{1/2}} \quad (3\text{-}x^2 = u, -2x dx = du)$$

9. Show that $\int \tan^{-1}\left(\frac{1}{x}\right) dx$ is equal to $\tan^{-1}\left(\frac{1}{x}\right) + \ln\sqrt{1 + x^2} + C$

Hint : Integral By parts

$$\int \tan^{-1}\left(\frac{1}{x}\right).1\, dx = \tan^{-1}\left(\frac{1}{x}\right).x - \int \frac{1}{1 + \frac{1}{x^2}} \left(\frac{-1}{x^2}\right) x\, dx = \text{x} \tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\int \frac{2x}{1 + x^2} dx, Let\ 1 + x^2 = u$$

and complete.

==============================

10. Show that $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt[3]{n^4 + 5}}$ converges by the comparision test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Hint: We may take

$$b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}, \text{ since } 0 \le \cos^2 n \le 1 => \frac{\cos^2 n}{\sqrt[3]{n^4+5}} \le \frac{1}{\sqrt[3]{n^4}} (a \text{ convergent p-series})$$

=> the given series is convergent by comparison test.

11. $\displaystyle\int_0^{1/\sqrt{2}} \frac{1}{\left(1-x^2\right)^{5/2}} dx = \frac{4}{3}.$

Hint: Let

$$x = \sin\theta, dx = \cos\theta d\theta ==> \int_0^{\pi/4} \frac{\cos\theta d\theta}{\left(1-\sin^2\right)^{5/2}} = \int_0^{\pi/4} \frac{d\theta}{\cos^4\theta} = \int_0^{\pi/4} \left(\sec^2\theta + \sec^2\theta \tan^2\theta\right) d\theta$$

$$\left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\pi/4} \text{ ........} complete \text{ yourself}$$

=============================

12. The radius of convergence R and the interval of convergence I of the power series $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n (2x-3)^n}{n \;.\; 4^n}$ is given by R=2, I=$(-1/2, 7/2]$

Hint: Use Ratio Test

$$\lim_{n->\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n->\infty} \frac{(2x-3)^{n+1}}{(n+1) \;.\; 4^{n+1}} \frac{(n)\; 4^n}{(2x-3)^n} = \lim_{n->\infty} \frac{(2x-3)}{4}\left(\frac{n}{n+1}\right) = \frac{(x-3/2)}{2}$$

$$\left|\frac{(x-3/2)}{2}\right| < 1 => |(x-3/2)| < 2 ==> R = 2 \text{ and } -2 \le x - 3/2 \le 2 => (-1/2, 7/2)$$

The series diverges when x = -1/2 ( put in the given series ( harmonic)

The series converges when x = 7/2 ( put in the given series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n (4)^n}{n \;.\; 4^n} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n \;.} \text{ ( Convergent, by Alternating series test)}$$

*Interval is* $(-1/2, 7/2]$

=============================

13. If the region enclosed by the curves $y = e^x, x = 0, x = \ln 2$ and $y = 0$ is revolved about the line y = - 1, then the volume of the solid generated is equal to $\dfrac{7\pi}{2}$.

Hint: Using the method of cylindrical shells -

$$V = \pi \int_0^{\ln 2} \left[ \left(1+e^x\right)^2 - 1^2 \right] dx ...\text{Complete...}$$

==============================

14. If $ax + bx^2 + cx^3$ is the sum of the first three terms of the Maclarium Series of $e^{2x}\sin x$, then a + b + c = 29/6.

The MS for $e^{2x}\sin x = 1 + \frac{f^{'}(0)}{1!} + \frac{f^{''}(0)}{2!}x + \frac{f^{'''}(0)}{3!}x^2 + \frac{f^{''''}(0)}{4!}x^4 + ...$

$$a + b + c = \frac{f^{''}(0)}{2!} + \frac{f^{'''}(0)}{3!} + \frac{f^{''''}(0)}{4!}$$

Find $f^{''}(0), f^{'''}(0), f^{''''}(0)$ from $\text{f(x)=e}^{2x}\sin x$ to get a + b + c

==============================

15. The sequence $\left\{ (2n+1)\sin\frac{7}{n} \right\}$ converges to 14 .

Hint: $\lim_{n-\infty} a_n = \lim_{n-\infty}\left( (2n+1)\sin\frac{7}{n} \right) = \lim_{n-\infty}\left( 14\frac{\sin\frac{7}{n}}{\frac{7}{n}} \right) + \lim_{n-\infty}\sin\frac{7}{n} = 14$

==============================

16. The integral $x > e,$ then $\frac{d}{dx}\int_1^{\sqrt{\ln x}} e^{t^2}\,dt = \frac{1}{2\sqrt{\ln x}}$

Hint: Use FC, $\frac{d}{dx}\int_1^{\sqrt{\ln x}} e^{t^2}\,dt = \frac{d}{du}\int_1^{u} e^{t^2}\,dt\,\frac{du}{dx}$ ( Let $u=\sqrt{\ln x}$) .. Complete it..

17. $\int \frac{x^2 + x + 3}{(x-1)\left(x^2+2x+2\right)}dx = \ln|x-1| - \tan^{-1}(x+1) + C$

Hint: Use Partial Fractions

$$\int \frac{x^2 + x + 3}{(x-1)\left(x^2+2x+2\right)}dx = \int\left[\frac{1}{x-1} - \frac{1}{x^2+2x+2}\right]dx = \int\left[\frac{1}{x-1} - \frac{1}{(x+1)^2+1}\right]dx$$

.. Complete yourself

18. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^5 6^n}{(n+1)!}$

Hint: Use Ratio Test… The series is absolutely convergent

==============================

19. The value of the limit $\lim_{n->\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\left(\cos\frac{i\pi}{2n}\right)^2$ on the interval $\left[0,\frac{\pi}{2}\right]$ is $\frac{\pi}{8}$

Hint: $a=0, b=\pi/2, \Delta x=\frac{\pi}{2n}, x_i=a+i\Delta x, f(x_i)=\cos(x_i)=\cos(\frac{i\pi}{2n})$

A= $\frac{1}{2}\int_0^{\pi/2}\cos^2 x dx$ .. complete yourself..

==============================

20. The series $\sum_{n=2}^{+\infty}\frac{(-1)^n}{n\ln n}$ converges conditionaly.

Hint: Use Alternating series Test, it converges.

If we use Integral Test on $|a_n|=\sum_{n=2}^{+\infty}\frac{1}{n\ln n}$ , $\lim_{t->\infty}\int_2^t\frac{1}{x\ln x}dx=\infty$ Integral does not exists. It means it is conditionally convergent.

==============================

21. The $\int_0^{\pi/4}\sqrt{\frac{1+\sin x}{1-\sin x}}dx$ is $\ln\left(2+\sqrt{2}\right)$

Hint:

=

$$\int_0^{\pi/4}\sqrt{\frac{1+\sin x}{1-\sin x}}dx=\int_0^{\pi/4}\sqrt{\frac{1+\sin x}{1-\sin x}}\sqrt{\frac{1+\sin x}{1+\sin x}}dx=\int_0^{\pi/4}\frac{1+\sin x}{\sqrt{1-\sin^2 x}}dx=\int_0^{\pi/4}(\sec x+\tan x)dx$$

==============================

22. The improper integral $\int_{-\infty}^{+\infty}e^{-|x|}\,dx$ converges to 2

Hint: $\int_{-\infty}^{+\infty}e^{-|x|}\,dx=\int_{-\infty}^{0}e^{x}\,dx+\int_{0}^{+\infty}e^{-x}\,dx,\quad \int_{-\infty}^{0}e^{x}\,dx=\lim_{t->\infty}\int_t^0 e^x\,dx=1$

Other part is also 1. Answer= 1+1=2

==============================

23. Using the power series of $\int\frac{dx}{1+x^2}$, then the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+3}$ is equal to $\frac{\pi}{4}-\frac{2}{3}$

Hint: $\int \frac{dx}{1+x^2} = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + .. \Rightarrow \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ..$

$$\frac{\pi}{4} - 1 + \frac{1}{3} = \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - .. = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$

==============================

24. The improper integral $\int_0^1 \frac{1}{e^x - 1} dx$ diverges.

Hint: $\lim_{t \to 0+} \int_t^1 \frac{1}{e^x - 1} dx = \lim_{t \to 0+} \int_t^1 \frac{1}{e^x - 1} dx$ ( let $e^x - 1 = u$) and integrate...

==============================

25. If $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$, then $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

Hint: $Let\ f(\sin x) = \frac{x \sin x}{1+\cos^2 x}, \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$

$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$ ( Let cos x = u )

==============================

26. For the convergent alternating series S = $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(n+1)^3}$, what is the smallest number of terms needed to guarantee that $S_n$ approximates S within $\frac{1}{125} X\ 10^{-6}$. Answer 499

$$b_{n+1} \le \frac{10^{-6}}{125} \Longrightarrow \frac{1}{(n+1)^3} \le \frac{1}{125 X 10^6} \Rightarrow (n+1) \ge 500$$

$n \ge 499$