MATH102 -071 FINAL

1. The value of the integral $\int_0^{\pi/4} \frac{\sin(2x)}{[1+\cos(2x)]^3}dx$ is $3/16$

Hint: $-\frac{1}{2}\int_2^1 \frac{du}{u^3}$ (Let 1+cos 2x = u , -2sin 2x dx= du . Now you can complete)

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2. If $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2}du$, then find $\frac{dy}{dx}$. (Ans: $\frac{3(1-3x)^3}{1+(1-3x)^2}$)

Hint: By Fundamental Theorem of Calculus

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3. The area of the region bounded by the graphs of $y = x^2 - 2$ and $y = x$ is 9/2.

Hint: Point of intersection of $y = x^2 - 2$ and $y = x$ is x = -1 and x = 2

Area= $\int_{-1}^{2}\left[(x)-(x^2-1)\right]dx = ...Now\ you\ can\ complete$

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4. The sum of the series $1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} - ..$ is 1/3.

Hint: $e^{-\ln 3} = 1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} - ..$ =1/3

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5. The volume of the solid generated by rotating the region enclosed by the curves $y = x$ and $y = \sqrt{x}$ about the y-axis is $\pi\int_0^1 (y^2 - y^4)dy$

Hint: Use Washer Method. Point of intersection is (0,0) and (1,1). The outer radius **x** ($x = y^2$)is given by $y = \sqrt{x}$ and inner radius **x** by $y = x$

Volume = $\pi\int_0^1 (outer\ radius^2 - inner\ radius^2)dy = \pi\int_0^1 (y^4 - y^2)dy$

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6. The sequence $\{(2-e)^n\}_{n=1}^{+\infty}$ converges to 0.

Hint: The sequence is of the form

r^n, where $r = (2-e), and$ -1<r <1==> $\lim_{n\to\infty} r^n = 0$

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7. If the n-th partial sum of the series

$\sum_{n=1}^{+\infty} a_n$ is $s_n = 2 - \frac{(-1)^n}{n^2}$, then the series $\sum_{n=1}^{+\infty} a_n$ converges and its sum is 2.

Hint: Alternating series Test

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2-\frac{(-1)^n}{n^2}\right) = 2 \text{ (since } \lim\left|\frac{(-1)^n}{n^2}\right| = 0 \text{ and therefore } \lim\frac{(-1)^n}{n^2} = 0)$$

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8. The series $\sum_{n=1}^{+\infty}\frac{(-3)^{n+1}}{2^{3n}}$ converges and its sum is 9/11.

Hint:

$$s_n = \frac{(-3)^2}{2^3}+\frac{(-3)^3}{2^6}+\frac{(-3)^4}{2^9}+..\text{It is geometrical series with a=}\frac{(-3)^2}{2^3} = 9/8, r = -3/8$$

$$s = \frac{a}{1-r} = \frac{9/8}{1+3/8} = 9/11$$

9. The series $1+\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}+\frac{1}{4^2\sqrt{4}}+\dots.$ is convergent p-series with p=5/2.

Hint: The series is $\sum_{n=1}^{+\infty}\frac{1}{2^{\frac{5}{2}}}$ is convergent p-series with p= 5/2

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10. Suppose that $f(1)=1, f(4)=7, f'(1)=-1, f''(1)=3, \textit{ and } f''$ is continuous.

Then the value of $\int_1^4 x\, f''(x)dx$ is equal to 7. (Integration by parts)

11. The average value of the function $f(x)=\frac{x}{(x+3)^3}$ over the interval [-1, 1] is -1/64.

Hint: $f_{avg} = \frac{1}{2}\int_{-1}^{1}\frac{x}{(x+3)^3} = \frac{1}{2}\int_{2}^{4}\left(\frac{u}{u^3}-\frac{3}{u^3}\right)du = \textit{...Now you may complete.}$

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12. The series $\sum_{n=2}^{+\infty}\frac{1}{n\ln n}$ diverges by the integral test.

Hint: $f(x)=\dfrac{1}{x\ln x}$ is positive, continious and decreasing for n>/2

$$\lim_{t->\infty}\int_2^t \frac{1}{x\ln x}dx=\lim_{t->\infty}\Big[\ln\left|\ln x\right|\Big]_2^t=\lim_{t->\infty}\Big[\ln\left|\ln t\right|-\ln 2\Big]=\infty.$$ Therefore, divergent.

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13. The error in approximating the sum of the series $\sum_{n=1}^{+\infty}\dfrac{(-1)^n}{5^n}$ by the sum of the first four terms is less than or equal to $\dfrac{1}{5^4}$

Hint: $|R_n|\le b_{n+1}$ ==> $|R_4|\le b_{4+1}=\dfrac{5}{5^5}=\dfrac{5}{5^4}$ (5 th term in the series)

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14. The length of the curve $y=\ln(\sec x),\ 0\le x\le\dfrac{\pi}{4}$, is $\ln(1+\sqrt{2})$

Hint:Arc Length

$$=\int_0^{\pi/4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\pi/4}\sqrt{1+\tan^2 x}dx=\int_0^{\pi/4}\sec dx...You\ can\ complete...$$

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15. The improper integral $\displaystyle\int_0^2\frac{x^3}{\sqrt{4-x^2}}dx$ has the value 16/3 .

Hint: $=\displaystyle\int_2^0\frac{-\left(4-u^2\right)u}{u}du=\left[-4u+\frac{u^3}{2}\right]_2^0=16/3$ (Let $\sqrt{4-x^2}$=u)

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16. The integral $\displaystyle\int\frac{e^{-x}}{e^{-2x}+3e^{-x}+2}dx$ is equl to $\ln\left(\dfrac{2+e^{-x}}{1+e^{-x}}\right)+C$

Hint: $=\displaystyle\int\frac{e^x}{2e^x+3e^x+1}dx=\int\frac{u}{2u^2+3u+1}du=\int\frac{1}{(2u+1)(u+1)}du$ (Let $e^x=u$)

Use Method of Partial Fractions to complete

17. The value of the integral $\displaystyle\int_1^{16}\frac{1}{\sqrt{x}+\sqrt[4]{x}}dx$ is equal to $2+4\ln(1.5)$.

Hint: $=\displaystyle\int_1^4\frac{4u^3}{u^2+u}du=4\int_1^4\left(u-1+\frac{1}{u+1}\right)du$ (Let $\sqrt[4]{x}=u, x=u^4, dx=4u^3du$)

(Method of long division.. you can complete)

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18. The series $\sum_{n=1}^{+\infty} n\sin\left(\frac{1}{n}\right)$ diverges.

Hint:

$\lim_{n->\infty} a_n = \lim_{n->\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$ (It is not zero, so by divergence test, it is divergent)

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19. The series $\sum_{n=1}^{+\infty} \frac{n^2+1}{n^5+n^4+1}$ is convergent.

Hint: Use comparison test with

$\sum b_n = \sum \frac{n^2+1}{n^5} = \sum \frac{1}{n^3} + \sum \frac{1}{n^5}$ (both are convergent p-series)

Since $a_n \le b_n$ and b_n is convergent =>a_n is convergent

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20. The series $\sum_{n=1}^{+\infty} \frac{(-1)^n 3n}{4n-1}$ is divergent.

Hint: See page 737, the given series is alternating series, decreasing but $\lim_{n->\infty} b_n = 3/4(Not\ zero)$, the convergent test fails. However it is divergent by divergent test)

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21. The integral for the area of the surface obtained by rotating the curve y = tan x from (0, 0) to $\left(\frac{\pi}{4}, 1\right)$ about the y-axis is $2\pi \int_0^{\pi/4} x\sqrt{1+\sec^4 x}\ dx$

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22. The area between the x-axis and the curve $y = \frac{x}{e^x}$ for $x \ge 0$ is 1

A= $\int_0^{\infty} \frac{x}{e^x}\,dx = \lim_{t->\infty} \int_0^{t} \frac{x}{e^x}\,dx = \lim_{t->\infty} \left[\frac{-x}{e^x} - \frac{1}{e^x}\right]_0^t$, you can complete...

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23. The value of $\int_{1/2}^{3/2} \frac{1}{5-4x+4x^2}dx$ is $\pi/16$

Hint: Complete the square…

$$\int_{1/2}^{3/2} \frac{1}{5-4x+4x^2}dx = \frac{1}{4}\int_{1/2}^{3/2} \frac{1}{x^2-x+5/4}dx = \frac{1}{4}\int_{1/2}^{3/2} \frac{1}{(x-1/2)^2+1}dx = You \text{ can complete}$$

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24. The series $\sum_{n=1}^{+\infty}\left(\frac{1+\ln n}{n^2+3}\right)^n$ is convergent by the root test.

Hint: $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{1+\ln n}{n^2+3}\right) = 0$ (Use L'Hospital) <1, so it is convergent.

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25. The interval of convergence and the radius of convergence R of the power series $\sum_{n=0}^{+\infty}\frac{(-3)^{n+1}(2x+1)^n}{\sqrt{n+1}}$ is $\left(\frac{-2}{3}.\frac{-1}{3}\right]$; R=1/6

Hint: Use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+2}(2x+1)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{3^{n+1}(2x+1)^n} = 3|(2x+1)|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 6|(x+1/2)|$$

If it converges, then

$$\frac{-1}{6} < x+\frac{1}{2} < \frac{1}{6} \Rightarrow \frac{-2}{3} < x < \frac{-1}{3} \Rightarrow R = 1/6 \text{ and Interval } (-2/3,-1/3)$$

If x = - 2/3,

$$\sum_{n=0}^{+\infty}\frac{(-1)^n 3^n(-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{+\infty}\frac{1}{\sqrt{n+1}} (\textit{divergent} \text{ use limit comp test } a_n = \frac{1}{\sqrt{n+1}}, b_n = \frac{1}{\sqrt{n}})$$

Since $b_n = \frac{1}{\sqrt{n}}$ is divergent, therefore $a_n = \frac{1}{\sqrt{n+1}}$ is also divergent.

If x = - 1/3, $\sum_{n=0}^{+\infty}\frac{(-1)^n}{\sqrt{n+1}}$ is convergent by Alternating Series Test

Interval of convergence is (-2/3,-1/3]

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26. The value of the integral $\int_0^{1/3}\frac{x^2}{1+x^7}dx$ is *equal to* $\sum_{n=0}^{+\infty}\frac{(-1)^n}{(7n+3)3^{7n+3}}$

Hint: $\frac{1}{1-x} = \sum_{n=0}^{+\infty}(x)^n \Rightarrow \frac{1}{1+x^7} = \sum_{n=0}^{+\infty}\left(-x^7\right)^n \Rightarrow \frac{x^2}{1+x^7} = \sum_{n=0}^{+\infty}(-1)^n x^{7n}$

$$\int_0^{1/3}\frac{x^2}{1+x^7}dx = \int_0^{1/3}\sum(-1)^n x^{7n+2} = \left[C+\frac{\sum(-1)^n x^{7n+3}}{7n+3}\right]_0^{1/3} = \frac{\sum(-1)^n x^{7n+3}}{(7n+3)3^{7n+3}}$$

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27. If the region bounded by the curves $y=\sqrt{x-1}, y=0,$ and $x=5$ is rotated about the line y = 3, then the volume of the solid generated is 24π

Hint: Volume about y=3 from x=1 to x=5

Volume

$$=\int_1^5 \pi\left(3^2-\left(3-\sqrt{x-1}\right)^2\right)dx = 36\pi - \int_1^5 \pi\left(3-\sqrt{x-1}\right)^2 dx = 36\pi - 12\pi = 24\pi$$

Where $\int_1^5 \pi\left(9-6\sqrt{x-1}+x-1\right)dx = 24\pi$

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28. The Maclaurin series of $f(x) = x\cos\left(x^3\right)$ is $\sum_{n=0}^{+\infty}(-1)^n \frac{x^{6n+1}}{(2n)!}$

Hint:

$$\cos\left(x^3\right) = \sum_{n=0}^{+\infty}(-1)^n \frac{\left(x^3\right)^{2n}}{(2n)!} \Rightarrow x\cos\left(x^3\right) = \sum_{n=0}^{+\infty}(-1)^n \frac{x(x)^{6n}}{(2n)!} = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{6n+1}}{(2n)!}$$