Integrals: Extra Problems

① $\displaystyle\int_1^2 \frac{dx}{\sqrt{x}\sqrt{4-x}}$, Let $\sqrt{x} = u \Rightarrow x = u^2,\ dx = 2u\,du$

$\displaystyle\Rightarrow \int_1^{\sqrt{2}} \frac{2\cancel{u}\,du}{\cancel{u}\sqrt{4-u^2}} \Rightarrow \left[2\sin^{-1}\frac{u}{2}\right]_1^{\sqrt{2}} = 2\left[\frac{\pi}{4} - \frac{\pi}{6}\right] = \boxed{\frac{\pi}{6}}$

② $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$, Let $x^2 - 9 = u^2 \Rightarrow 2x\,dx = 2u\,du$

$\displaystyle\Rightarrow \int \frac{u^2}{u^2+9}\,du \Rightarrow \int\left(1 - \frac{9}{u^2+9}\right)du \Rightarrow$ Now you can complete....

③ $\displaystyle\int \frac{dx}{\sqrt{x} + x^{3/2}}$, let $\sqrt{x} = u$, $x = u^2 \Rightarrow dx = 2u\,du$

$\displaystyle= \int \frac{2u\,du}{u + u^3} = 2\int \frac{du}{1+u^2} \Rightarrow$ complete yourself

④ $\displaystyle\int \frac{dx}{1+\sqrt{x}}$. Let $\sqrt{x} = w \Rightarrow x = w^2 \Rightarrow dx = 2w\,dw$

$\displaystyle= 2\int \frac{w\,dw}{1+w} = 2\int\left(1 - \frac{1}{1+w}\right)dw \Rightarrow$ complete yourself.

⑤ $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$, Let $\sqrt{x} = u$, $x = u^2 \Rightarrow dx = 2u\,du$

$\displaystyle= \int 2\sqrt{1+u}\;u\,du$, let $\sqrt{1+u} = w \Rightarrow 1+u = w^2 \Rightarrow du = 2w\,dw$

$\displaystyle= 2\int w\cdot(w^2-1)\;2w\,dw = 4\int w^2(w^2-1)\,dw$

..... complete yourself ...

⑥ $\displaystyle\int \frac{2\,dx}{x + \sqrt[3]{x}}$, let $\sqrt[3]{x} = u \Rightarrow x = u^3 \Rightarrow dx = 3u^2\,du$

$\displaystyle= \int 6\frac{u^2}{u^3+u}\,du = 6\int \frac{u\,du}{u^2+1}$, Let $u^2+1 = w$ and complete.....

⑦ $\displaystyle\int_1^{16} \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$, Let $\sqrt[4]{x} = u \Rightarrow x = u^4,\ dx = 4u^3\,du$

$\displaystyle\Rightarrow \int_1^4 \frac{4u^3\,du}{u^2+u} = 4\int_1^4\left(\frac{u^2}{u+1}\right)du = 4\int_1^4\left(u - 1 + \frac{1}{u+1}\right)du$

Long Division

(8) $\displaystyle\int \frac{x^3+2x+3}{x^2+x-2}\,dx$, ~~use~~ Long Division and Partial Fraction . . .

$$= \int \left(x-1 + \frac{5x-1}{(x+2)(x-1)} \right) dx \longrightarrow \frac{5x-1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Find A, B and complete. . . .

(9) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+5}}\,dx$, Let $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta\, d\theta$

$$= \frac{1}{5}\int \frac{\sec\theta\, d\theta}{\tan^2\theta} = \frac{1}{5}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

, let $\sin\theta = w$

$\Rightarrow$ complete yourself. . .

(10) $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}} = \int \frac{dx}{\sqrt{(x+4)^2+3^2}}$, Let $x+4 = 3\tan u$

$dx = 3\sec^2 u\, du$

$$= 3\int \frac{\sec^2 u\, du}{\sqrt{3^2\tan^2 u + 3^2}}$$

$$= \int \sec u\, du \Rightarrow \text{complete. . . .}$$

(11) $\displaystyle\int \sqrt{2x-x^2}\,dx = \int \sqrt{1-(x-1)^2}\,dx$ (Complete square)

$$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

, $x-1 = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \cos^2\theta\, d\theta$$

, use $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

and complete. . .

(12) $\displaystyle\int \frac{\sqrt{x^2-4x}}{x-2}\,dx = \int \frac{\sqrt{(x-2)^2-4}}{x-2}\,dx$, Let $x-2 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$$\Rightarrow \int \frac{2\cdot \tan\theta \cdot 2\sec\theta\tan\theta}{2\sec\theta}\,d\theta$$

$$= 2\int \tan^2\theta = 2\int (\sec^2\theta - 1) \ldots \text{Complete} \ldots$$

(13) $\int \frac{dx}{x^3 - x} = \int \frac{dx}{x(x-1)(x+1)}$, $\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

find A, B, C and complete

(14) $\int \frac{x^2 + 2x - 1}{x^3 - x}\, dx = \int \frac{(x-1)^2}{x(x-1)(x+1)} = \int \frac{(x-1)}{x(x+1)}$

... complete using partial fraction ..

(15) $\int \frac{(\sqrt{1-x^2} - x)}{\sqrt{1-x^2}}\, dx = \int \left(\frac{1}{\cancel{1}} - \frac{x}{\sqrt{1-x^2}} \right) dx$

Let $1 - x^2 = u$ and complete ..

(16) $\int \tan^{-1}\left(\frac{1}{x}\right) dx$

$\Rightarrow \int \underbrace{\tan^{-1}\left(\frac{1}{x}\right)}_{\text{First}} \cdot \underbrace{1}_{\text{Second}}\, dx = \tan^{-1}\left(\frac{1}{x}\right) \cdot x - \int \frac{1}{1+\frac{1}{x^2}} \left(-\frac{1}{x^2}\right) \cdot x\, dx$

$= x \tan^{-1}\left(\frac{1}{x}\right) + \int \frac{x\, dx}{1+x^2}$

Let $1 + x^2 = u$ .. Complete

(17) $\int_0^{\pi/4} \sqrt{\frac{1+\sin x}{1-\sin x}}\, dx \Rightarrow \int_0^{\pi/4} \sqrt{\frac{1+\sin x}{1-\sin x}} \sqrt{\frac{1+\sin x}{1+\sin x}}\, dx$

$\Rightarrow \int_0^{\pi/4} \frac{1+\sin x}{\sqrt{1-\sin^2 x}}\, dx = \int_0^{\pi/4} \left(\frac{1}{\sin x} \cancel{dx} + \frac{\sin x}{\cos x} \right) dx$

$= \int (\sec x + \tan x)\, dx$

... Complete

(18) $\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x-3)^2}}$, Let $x - 3 = 3 \sin\theta$

$dx = 3\cos\theta\, d\theta$

$= \int d\theta$ ... complete yourself ..

(19) $\int_0^2 \ln(x^2+1)\,dx$

$$\Rightarrow \int_0^2 \underbrace{\ln(x^2+1)}_{\text{First}} \cdot \underbrace{1}_{\text{second}}\,dx = \left[x\ln(x^2+1)\right]_0^2 - 2\int \frac{x^2}{x^2+1}\,dx$$

$$= 2\ln 5 - 2\int_0^2 \left(1 - \frac{1}{x^2+1}\right)dx$$

$\Rightarrow$ you can complete now . . . . .

(20) $$\int \frac{dx}{x^2(1+x^2)} = \int \frac{dx}{x^2} - \int \frac{dx}{1+x^2} \cdots$$ complete yourself . .

(21) $$\int_0^{\pi^2/4} \cos\sqrt{x}\,dx$$, Let $\sqrt{x} = w \Rightarrow x = w^2,\ dx = 2w\,dw$

$\downarrow \frac{\pi^2}{4} \quad \downarrow \frac{\pi}{2}$

$$= \int_0^{\pi/2} \underset{\text{FIRST}}{\underset{\downarrow}{w}} \cos \underset{\text{SECOND}}{\underset{\downarrow}{w}}\,dw = \left[w \sin w\right]_0^{\pi/2} - \int_0^{\pi/2} \sin w\,dw$$

$$= \cancel{w \sin w} \frac{\pi}{2} + \left[\cos w\right]_0^{\pi/2} = \boxed{\frac{\pi}{2} - 1}$$

(22) $$\int \frac{dx}{5-4x+4x^2} = \frac{1}{4}\int \frac{dx}{x^2 - x + \frac{5}{4}} = \frac{1}{4}\int \frac{dx}{(x-\frac{1}{2})^2+1}$$

~~Let $x - \frac{1}{2} = \tan\theta$~~

$$= \frac{1}{4} \cdot \tan^{-1}\left(x - \frac{1}{2}\right) + C$$

(23) $$\int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int_1^3 \frac{e^{2x}-1}{e^{2x}+1}\,dx$$, Let $e^x = w$

$e^x dx = dw$

$dx = \frac{dw}{w}$

$$= \int_1^3 \frac{w^2-1}{w^2+1} \cdot \frac{dw}{w}$$

$$\frac{w^2-1}{w(w^2+1)} = \frac{A}{w} + \frac{B\cancel{w}+C}{w^2+1}$$

$\Rightarrow$ Find A, B, C

$A = -1,\ C = 0,\ B = 2$

$$= \int_1^3 \left[-\frac{1}{w} + \frac{2w}{w^2+1}\right]dw$$

$$= \left[-\ln w + \ln(w^2+1)\right]_1^3$$

$$= -\ln 3 + \ln 10 - \ln 2 = -\ln 6 + \ln 10$$

$$= \ln\frac{10}{6} = \boxed{\ln\frac{5}{3}}$$

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$$\int_{\ln 2}^{\ln \frac{2}{\sqrt{3}}} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = \cancel{\int \frac{dx}{\sqrt{e^{2x}-1}}} \quad \cancel{e^{x}} =$$

$$e^{-x} = u \Rightarrow \frac{1}{u} = e^{x}$$

$$-e^{-x}\,dx = du$$

$$= -\int \frac{du}{\sqrt{1-u^2}} = -\left[\sin^{-1} u\right]_{1/2}^{\sqrt{3}/2} \,.$$

$$= -\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \boxed{-\pi/6}$$

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$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \sqrt{\frac{1-x}{1+x}} \cdot \sqrt{\frac{1-x}{1-x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\,dx \quad (\text{Let } 1-x^2 = w$$

...... complete yourself..

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$$\int_0^1 \frac{1}{1+e^{-x}}\,dx = \int_0^1 \frac{e^{x}}{e^{x}+1}\,dx, \quad \text{Let } e^{x}+1 = u$$

complete yourself..

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$$\int \frac{e^{-x}\,dx}{e^{-2x}+3e^{-x}+2} = \int \frac{e^{x}\,dx}{1+3e^{x}+2e^{2x}} \quad \left[\text{Multiply by } \frac{e^{x}}{e^{x}}\right]$$

$$= \int \frac{du}{2u^2+3u+1}, \quad e^{x} = u, \; e^{x}\,dx = du$$

$$= \int \frac{du}{(2u+1)(u+1)}$$

.... use method of partial fractions and complete.

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$$\int \frac{dx}{1-\cos x} = \int \frac{1+\cos x}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1+\cos x}{\sin^2 x}\,dx$$

$$= \int \csc^2 x\,dx + \int \frac{\cos x}{\sin^2 x}\,dx$$

$$\sin x = w$$

$$\cos x\,dx = dw$$

$\boxed{28}$ $\int \sin 3x \cos 2x \, dx$

$\sin 3x = 3\sin x - 4\sin^3 x$
$\cos 3x = 4\cos^3 x - 3\cos x$
$\cos 2x = 2\cos^2 x - 1$

$= \int (3\sin x - 4\sin^3 x)(2\cos^2 x - 1)\, dx$

$= \int (3 - 4\sin^2 x)(2\cos^2 x - 1)\sin x \, dx$, Let $\cos x = w$

$-\sin x \, dx = dw$

$= -\int (3 - 4(1-w^2))(2w^2 - 1)\, dw$

$\Rightarrow$ Multiply and find the integral of each term.

$\boxed{29}$ $\int \frac{\sin 3x}{\cos x} dx = \int \frac{3\sin x - 4\sin^3 x}{\cos x} dx$

$= 3\int \tan x - 4\int \frac{(1-\cos^2 x)}{\cos x} \cdot \sin x \, dx$

Let $\cos x = u$

$-\sin x \, dx = du$

$\Rightarrow$ Complete yourself

$\boxed{30}$ $\int_0^{\pi/2} \cos^2 x \sin 2x \, dx = 2\int_0^{\pi/2} \cos^2 x \cdot \sin x \cos x \, dx$

Let $\cos x = u$

$\Rightarrow$ Complete yourself

$\boxed{31}$ $\int \sin^2 3x \, dx = \int \frac{1 - \cos 6x}{2} dx$

Complete yourself

$\boxed{31}$ $\displaystyle\int \frac{\sin 2x}{[1+\cos 2x]^3}\,dx$, Let $1+\cos 2x = u$

$-2\sin 2x = \frac{du}{dx}$

$= -\frac{1}{2}\displaystyle\int \frac{du}{u^3} \Rightarrow$ Complete yourself

$\boxed{32}$ $\displaystyle\int e^x \sin x \cos x\, dx$ | Type $\int e^{ax}\sin bx$ or $\int e^{ax}\cos bx$ use by parts twice ...

$I = \frac{1}{2}\underbrace{\int e^x \sin 2x\, dx}_{I_1}$

Let $I_1 = \displaystyle\int \underbrace{e^x}_{\text{First}}\, \underbrace{\sin 2x}_{\text{Second}}\, dx = -e^x\, 2\cdot\cos 2x + 2\int \frac{e^x}{F}\frac{\cos 2x}{S}dx$

$= -2e^x\cos 2x + 2(2e^x \sin 2x) - 4\underbrace{\int e^x \sin 2x\, dx}_{I_1})$

$I_1 = -2e^x\cos 2x + 4e^x\sin 2x - 4I_1$

$5I_1 = (-2\cos 2x + 4\sin 2x)e^x$

$I_1 = \frac{1}{5}(-2\cos 2x + 4\sin 2x)e^x$

$I = \frac{1}{10}(-2\cos 2x + 4\sin 2x)e^x + C.$

$\boxed{33}$ $\displaystyle\int \sin 3x \sin 6x\, dx$

$\int \sin mx \cos nx$ or $\int \sin mx \sin nx$ or $\int \cos mx \cos nx$ is changed using

$\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$

$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

$= \frac{1}{2}\displaystyle\int [\cos(3x-6x) - \cos(3x+6x)]dx$

$= \frac{1}{2}\displaystyle\int (\cos 3x - \cos 9x]dx$

$\{\cos(-\theta) = \cos\theta\}$

$= \frac{1}{2}\left[\frac{1}{3}\sin 3x - \frac{1}{9}\sin 9x\right] + C$

$\boxed{34}$ $\displaystyle\int \frac{dx}{4\sin x - 3\cos x}$

$\{$ Type $\displaystyle\int \frac{dx}{a\sin x + b\cos x}$

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

$t = \tan x/2 \quad dt = \frac{1}{2}\sec^2 \frac{x}{2}\, dx$

$= \displaystyle\int \left(\frac{2\,dt}{1+t^2}\right) \Big/ \left(\frac{4\cdot 2t}{1+t^2} - 3\frac{(1-t^2)}{1+t^2}\right)$

$\Rightarrow$ Use method of Partial fractions

$$\Rightarrow 2\left[\int\left(\frac{3}{10(3t-1)} - \frac{1}{10(t+3)}\right) dt\right.$$

$$\Rightarrow \frac{2}{10}\left[\int\frac{3\,dt}{(3t-1)} - \int\frac{dt}{(t+3)}\right]$$

$\Rightarrow$ complete yourself - - - -