11.6 Absolute Convergence, Ratio and Root Tests

Given the series $\sum a_n$, we define series with positive values $\sum |a_n| = |a_1| + |a_2| + ...|a_n|$. Now we define the absolute convergence, Ratio and Root tests

1. **ABSOLUTELY CONVERGENT**: The series $\sum a_n$ is absolutely convergent, if the series $\sum |a_n|$ is convergent.

Example: $\sum a_n = \sum \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent because $\sum |a_n| = \sum \frac{1}{n^2}$ is convergent p-series (p=2).

2. **CONDITIONALLY CONVERGENT**: The series $\sum a_n$ is conditionally convergent, if the series $\sum a_n$ is convergent but $\sum |a_n|$ is not convergent.

Example: $\sum a_n = \sum \frac{(-1)^{n-1}}{n}$ is convergent (Using Alternating Series Test)**, but it is NOT absolutely convergent** because $\sum |a_n| = \sum \frac{1}{n}$ is divergent (p=1).

NOTE: if $\sum a_n$ **is absolutely convergent, then it must be convergent**.

Example: Determine if the series $\sum \frac{Cosn}{n^2}$ is convergent or divergent or absolutely convergent.

The give series is alternating (has positive and negative terms). We can see that $\sum \left|\frac{Cosn}{n^2}\right| = \sum \frac{|\cos n|}{n^2}$. Since $|\cos n| \le 1$ for all n. We see $\frac{|\cos n|}{n^2} \le \frac{1}{n^2}$ (convergent). By Comparison Test $\sum \left|\frac{Cosn}{n^2}\right|$ is Convergent. Therefore $\sum \frac{Cosn}{n^2}$ is absolutely convergent.

RATIO TEST: Given the series $\sum a_n$ **,**

1. if $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ is absolutely convergent.
2. if $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum a_n$ is divergent.

3. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then test fails.

Example: Test the convergence of $\sum a_n = \sum \frac{(-1)^n n^3}{3^n}$

The series is alternating. We can apply ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3 3^n}{n^3\ 3^{n+1}} = \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1 \Rightarrow \text{The series is absolutely convergent}$$

ROOT TEST: Given the series $\sum a_n$

1. if $\lim_{n\to\infty}\sqrt{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.
2. if $\lim_{n\to\infty}\sqrt{|a_n|} = L > 1 \text{ or } \infty$, then $\sum a_n$ is divergent.
3. if $\lim_{n\to\infty}\sqrt{|a_n|} = 1$, then test fails.

Exercises: Determine if the series is absolutely convergent, conditionally convergent or divergent.

Ex1(book-4). $\sum \frac{(-1)^{n-1} 2^n}{n^4}$

This is an alternating series ($b_n = \frac{2^n}{n^4}$) and it is decreasing.

$$\lim_{n\to\infty}\frac{2^n}{n^4} \equiv \lim_{x\to\infty}\frac{2^x}{x^4} = \lim_{x\to\infty}\frac{2^x}{4x^3} = \lim_{x\to\infty}\frac{2^x}{4.3.x^2} = \lim_{x\to\infty}\frac{2^x}{4.3.2.1} = \infty$$ (Using Hospital Rule, differentiate 4 times) ➔ The series is divergent (using divergent test)

Ex2(book-9). $\sum \frac{1}{2n!}$. This is a series with positive terms. We can apply the ratio test to cancel factorial.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{2n+2!}}{\frac{1}{2n!}} = \lim_{n\to\infty}\frac{2n!}{2n+2!} = \lim_{n\to\infty}\frac{\cancel{2n!}}{(2n+2)(2n+1)\ \cancel{2n!}} = 0(<1)$$

By Ratio Test, it is Absolutely Convergent and therefore convergent.

Ex3(book-27). $\sum \frac{2.4.6..........(2n)}{n!}$

It can be written as $\sum \frac{(2.1)(2.2)(2.3)..........(2n)}{n!} = \sum \frac{2^n.n!}{n!} = \sum 2^n$

The series DIVERGES, because $\lim_{n\to\infty} 2^n = \infty$

Ex4(book-33). (a) Show that $\sum \frac{x^n}{n!}$ converges for all

(b) Deduce form (a) that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

(a) Since the factorial is in the series, we may apply Ratio test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$. By Ratio Test it is convergent

(b) Since $\sum \frac{x^n}{n!}$ is convergent, then limit of nth term must be zero… $\lim_{n\to\infty}\frac{x^n}{n!} = 0$

Ex5(book-24). $\sum \frac{(-1)^n}{\left(\tan^{-1} n\right)^n}$

Apply Root test: $\lim_{n\to\infty}\sqrt{|a_n|} = \lim_{n\to\infty}\frac{1}{\tan^{-1} n} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} < 1$. Therefore the series is absolutely convergent.

Ex6(book-26). $\frac{2}{5} + \frac{2.6}{5.8} + \frac{2.6.10}{5.9.11} + \frac{2.6.10.14}{5.9.11.14} + ...$

In the summation form it can be written as: $\sum \frac{1.6.10.14............(4n-2)}{5.8.11.14........................(3n+2)}$

We can try Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{2.6.10.....(4n-2)}(4n+2)}{\cancel{5.8.11........(3n+2)}(3n+5)}\frac{\cancel{5.8.11..(3n+2)}}{\cancel{2.6.10....(4n-2)}}\right| = \lim_{n\to\infty}\frac{4n+2}{3n+5} = \frac{4}{3} > 1$$

The series is DIVERGENT.