11.10 Taylor and Maclaurin Series

Question: Which functions have power series representation ?

We know that the power series of $\frac{1}{1-x}$ is $1+x+x^2+x^3+.... +x^n+...|x|<1$

The functions which satisfy certain conditions can be represented as power series.

$$Let\ f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+....=\sum_{n=0}^{\infty}c_n(x-a)^n \text{ where } |x-a|<R$$

If f(x) has derivative of <u>all order</u> at x=a

$$f(a)=c_0,\ \frac{f'(a)}{1!}=c_1,\ \ldots\ldots \frac{f^n(a)}{n!}=c_n$$

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+...=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$$

Taylor Series (Power Series): If f has derivative of all orders at a , then Taylor series of f at a is defined by

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$$

Maclaurin Series (Power Series): In the Taylor series , if a = 0 then, the Maclaurin Series of f is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n \text{ with convention } f^0(0)=f(0), f^0(a)=f(a)$$

Finding Maclaurin Series (MS) of a function

Example: Find MS for $f(x)=e^x$

$e^x=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$, We have to find the derivatives and the values of

$f^0(0), f^1(0),.....,f^n(0)$

$f(x)=e^x$ ==> $f^n(x)=e^x$ ➔ $f^0(0)=1, f^1(0)=1,.....,f^n(0)=1$ ➔

$$f(x)=e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}$$

Example: Find the Taylor Series of $f(x)=\frac{1}{x}$ at x=2

The Taylor series of f(x) at a is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n} = f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(2)}{n!}(x-2)^{n}$$

Also, $f(x)=x^{-1},\ f'(x)=(-1)x^{-2},\ f''(x)=(-1)(-2)x^{-3}.....f^{n}(x)=(-1)(-2)....(-n)x^{-(n+1)}$

$$f^{n}(2)=\frac{(-1)^{n}n!}{2^{n+1}}$$

$$f(x)=1/x=\sum_{n=0}^{\infty}\frac{(-1)^{n}n!}{2^{n+1}}\frac{1}{n!}(x-2)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^{n}}{2^{n+1}}(x-2)^{n}$$

Important Maclaurin Series (Memorize)

$$\frac{1}{1-x} = 1+x+x^{2}+x^{3}+....+x^{n}+...|x|<1$$

$$e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}\quad \text{x in}(-\infty,\infty)\text{ and R}=\infty$$

$$\sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-........=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n+1}}{2n+1!}\quad \text{where x in}(-\infty,\infty)$$

$$\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-........=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{2n!}\quad \text{where x in}(-\infty,\infty)$$

$$\tan^{-1}x=x-\frac{x^{3}}{3}+\frac{x^{5}}{5!}-........=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n+1}}{2n+1}\quad \text{where } |x|\le 1$$

The above MS can be used to find the power series of other functions.

Example 1: Find the MS Series of $f(x)=x\cos 2x$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{2n!}\quad \text{where x in}(-\infty,\infty)$$

$$\cos 2x=\sum_{n=0}^{\infty}\frac{(-1)^{n}(2x)^{2n}}{2n!}\quad \text{where 2x in}(-\infty,\infty)$$

$$x\cos 2x=\sum_{n=0}^{\infty}\frac{(-1)^{n}2^{2n}x^{2n+1}}{2n!}\quad \text{where x in}(-\infty,\infty)$$

Example 2: Evaluate $\int e^{x^{2}}dx$

Use the power series to evaluate the Integral

$$e^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}\quad |x|<\infty \rightarrow e^{x^{2}}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}\quad |x^{2}|<\infty$$

$$\int e^{x^2}dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\ \text{dx} \ = \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{2n}\ \text{dx} \ = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^{2n+1}}{2n+1} \quad |x| \ < \infty$$

Example 3: Find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$

Compare it with series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ==> $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

Example 4: Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4.16^n\,(2n+1)!}$

Compare it with series $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1!}$

$$\frac{\pi^{2n+1}}{4.16^n} = \frac{\pi^{2n+1}}{4^{2n+1}} = \left(\frac{\pi}{4}\right)^{2n+1} \rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4.16^n\,(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Example 5: If the series $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{\sqrt{n}x^n}{e^n+2}$ is the MS for some function f(x).

Find $f^n(0)$

The MS $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$. Compare it with given series, we find

$$\frac{f^n(0)}{n!} = \frac{(-1)^{n+1}\sqrt{n}}{e^{n+2}} \implies f^n(0) = \frac{(-1)^{n+1}\sqrt{n}\ n!}{e^{n+2}}$$

Example 6: Use the series to evaluate $\lim_{x\to 0} \frac{x-\tan^{-1}x}{x^3}$

$$\lim_{x\to 0} \frac{x-\tan^{-1}x}{x^3} = \lim_{x\to 0} \frac{\left\{x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\right)\right\}}{x^3} = \frac{1}{3}$$

EX1 (book:18) Find the Taylor Series for $f(x) = x^{-2}$ at a =1, assume that **f** has a power series expansion.

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$, We find derivatives of $f(x) = x^{-2}$

$f^{'}(x) = -2x^{-3}, f^{''}(x) = 6x^{-4}, f^{'''}(x) = -24x^{-5}..$

At a =1 values are 1,-2,6,-24…

$$f(x) = x^{-2} = \sum_{n=0}^{\infty} \frac{f^{n}(1)}{n!}(x-1)^{n} = 1-2(x-1)+\frac{6(x-1)^{2}}{2!}-\frac{24(x-1)^{3}}{3!}+..=\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(x-1)^{n}$$

EX1 (book:30) Use definition to obtain Maclaurin Series for $f(x) = \cos^2 x$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^{n}(2x)^{2n}}{2n!}\right] = \frac{1}{2}\left[1+1+\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}(x)^{2n}}{2n!}\right]$$

$$=1+\left[\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n-1}(x)^{2n}}{2n!}\right] \quad \text{where } R=\infty$$

EX1 (book:48) Find the limit $f(x) = \lim_{x->0}\frac{1-\cos x}{1+x-e^{x}}$

$$=\lim_{x->0}\frac{1-\left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-...\right)}{1+x-\left(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+...\right)} = \lim_{x->0}\frac{\frac{x^{2}}{2!}-\frac{x^{4}}{4!}+\frac{x^{6}}{6!}-..}{-\frac{x^{2}}{2!}-\frac{x^{3}}{3!}-...} = \lim_{x->0}\frac{1/2-0}{-1/2-0} = -1$$

EX1 (book:60) Find the sum of series

$$1-\ln 2+\frac{(\ln 2)^{2}}{2!}-\frac{(\ln 2)^{3}}{3!}+\frac{(\ln 2)^{4}}{4!}-$$

$$\Rightarrow = \sum_{n=0}^{\infty}\frac{(-\ln 2)^{n}}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \frac{1}{2}$$

EX1 (book:59) Find the sum of series

$$3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+.......$$

$$\Rightarrow = \frac{3}{1!}+\frac{3^{2}}{2!}+\frac{3^{3}}{3!}+\frac{3^{4}}{4!}+.......=\sum_{n=1}^{\infty}\frac{3^{n}}{n!}=\sum_{n=0}^{\infty}\frac{3^{n}}{n!}-1=e^{3}-1$$

EX1 (book-42) Evaluate the Integral $\int\frac{e^{x}-1}{x}dx$ as infinite series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} ==> e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} ==> \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

$$\int \frac{e^x - 1}{x} = C + \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = C + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!} dx + \sum_{n=1}^{\infty} \frac{x^n}{n+1!}$$