Mr. Paint and Mrs. Correct

Uwe Schauz

Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
schauz@kfupm.edu.sa

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Abstract

We introduce the game of Mr. Paint and Mrs. Correct, in which each vertex v of a graph G owns a stack of $\ell_v - 1$ erasers. In each round of this game Mr. Paint takes a thus far unused color, and colors some of the thus far uncolored vertices. Typically, he incorrectly colors adjacent ones with it. However, Mrs. Correct is positioned next to him, and corrects his incorrect coloring, i.e., she uses up some of the erasers – while stocks (stacks) last – to partially undo his assignment of the new color. If she has a winning strategy, i.e., she is able to enforce a correct and complete final graph coloring, then we say that G is ℓ-paintable.

The game provides an adequate game theoretic approach to list coloring problems. It turns out that ℓ-paintability is stronger then ℓ-list colorability, and in fact strictly stronger. However, many deep theorems about list colorability still remain true in the context of paintability, e.g., those of Thomassen, of Galvin, of Shannon and of Brooks. Our sharpening of Brook's Theorem is even sharper than the version of Borodin and of Erdős, Rubin and Taylor. It is based on a paintability version of Alon and Tarsi's Theorem, for which we present a purely combinatorial proof.

Furthermore, the concept of paintability and our sharpening of Alon and Tarsi's Theorem can be generalized to hypergraphs and even to polynomials. This leads to a paintability version of the Combinatorial Nullstellensatz.

Introduction

There are many papers about graph coloring games. Originally, these games were introduced with the aim to provide a game theoretic approach to coloring problems. The hope was to obtain good bounds for the chromatic number of graphs, in particular with regards to the four color problem (see, e.g., [BGKZ] and the literature cited there). However, there is a fundamental problem with these games which means that they cannot fulfill their original purpose. Typically, these games require many more colors than those actually needed for a correct graph coloring, so there is a large gap between the corresponding *game chromatic numbers* and the *chromatic* or the *list chromatic number* (i.e., the minimal size of given color lists L_v, assigned to the vertices v of a graph G, which ensures the existence of a correct vertex coloring $\lambda\colon v \longmapsto \lambda_v \in L_v$ of G). (See [Al], [Tu] and [KTV] in order to get an overview of list colorings.)

The game of Mr. Paint and Mr. Correct, introduced in Section 1 (in Game 1.1 and its reformulation Game 1.6), is different. It provides an adequate game theoretic approach to list coloring problems. The existence of a winning strategy for Mrs. Correct, which we call *ℓ-paintability* (see Definition 1.2 or the reformulated recursive Definition 1.8), comes very close to *ℓ-list colorability* (Definition 1.3). ℓ-paintability is stronger than ℓ-list colorability (Preposition 1.4), but not by much. Although Example 1.5 shows that there is a gap between these two notions, most theorems about list colorability hold for paintability as well. Therefore, good bounds for the *painting number* – which may be found using game theoretic approaches – are usually good bounds for the list chromatic number as well. The reason for all this is that (as described after Definition 1.3) paintability can be seen as a dynamic version of list colorability, where the lists of colors are not completely fixed before the coloring process starts.

All list coloring theorems – whose proofs are exclusively based on coloring extension techniques, on the existence of kernels, and on Alon and Tarsi's Theorem – can be transferred into a paintability version. These three techniques are the main techniques in the theory of list colorings. In addition, for colorings in the classical sense, there is the important recoloring technique (Kempe-chain technique). It is used for example in the proofs of Vizing's Theorem, and works with neither list colorings nor with paintability.

In Section 2 we prove several lemmas that can be used as a replacement for coloring extension techniques. They are based on a technique, called the *pre-use* of additional erasers, which is described in Preposition 2.1. We demonstrate the application of these replacements in the proof of Theorem 2.6, a sharpening of Thomassen's Theorem about the 5-list colorability of planar graphs.

In Section 3 (Lemma 3.1), we sharpen Bondy, Boppana and Siegel's Kernel Lemma. Afterwards, we apply it in the proof of Galvin's celebrated theorem about the list chromatic index of bipartite graphs (Theorem 3.2), and in Borodin, Kostochka and Woodall's sharpening of Galvins's result (Theorem 3.3). This leads also to a sharpening of their refinement of Shannon's bound for the list chromatic index of multigraphs (Theorem 3.5).

In Section 4, we give a purely combinatorial proof of a sharpening of Alon and Tarsi's Theorem (Theorem 4.1) about colorings and orientations of graphs. The result is then used in Section 5 to provide paintability versions of Alon and Tarsi's bound of the list chromatic number of bipartite and planar bipartite graphs (Theorem 5.3 and the corollaries 5.4 and 5.5). Also presented in the sharpened paintability version are: Fleischner and Stiebitz' Theorem 5.6 about certain 4-regular Hamiltonian graphs, Häggkvist and Janssen's bound (Theorem 5.7) for the list chromatic index of the complete graph K_n, and Ellingham and Goddyn's confirmation of the list coloring conjecture for planar r-regular edge r-colorable multigraphs (Theorem 5.8).

In Section 6, we work out a sharpening of Brooks' Theorem (Corollary 6.6), which can be proved using the Alon-Tarsi-Theorem. Our underlying Theorem 6.5 is even sharper than the, independently proven, version of Borodin and of Erdős, Rubin and Taylor. Its proof is based on our sharpening 4.1 of Alon and Tarsi's Theorem, and uses the existence of an induced even circuit with at most one chord (Lemma 6.4 and Lemma 6.3). The proof of Corollary 6.6 (Brooks' Theorem) is based on this Theorem 6.5, and on the slightly weaker but more general Theorem 6.1. The restrictive class of Gallai Trees (exceptions in some of these results) and some other terms are provided in Definition 6.2.

In Section 7, we generalize the graph theoretic paintability and list colorability to polynomials (Definitions 7.1, 7.2, 7.3 and Proposition7.5). Our Example 7.8 concerning the gap between these two notions, is easier to understand than the purely graph theoretical Example 1.5; and Proposition 7.6 – which shows that paintability is the stronger notion – is more general than Proposition 1.4. We prove in Theorem 7.9 a paintability version of the Combinatorial Nullstellensatz, which works for polynomials of arbitrarily high degree, without any degree restriction. It may be seen as a generalization of the Alon-Tarsi-Theorem 4.1, and can be used for example to examine hypergraph colorings (as in the following section) or matrix colorings (as defined in [Scha2, Section 5]).

Finally, in Section 8, we examine hypergraphs as specialization of polynomials and generalization of graphs. Indeed, Theorem 8.1 may be seen as a step between the Alon-Tarsi version 4.1 and the Combinatorial Nullstellensatz version 7.9. This observation is based on a connection between the *permanent* of the incidence matrix in Theorem 8.1, the orientations in Alon and Tarsi's Theorem and the coefficient of the graph polynomial in the Combinatorial Nullstellensatz.

1 Mr. Paint and Mrs. Correct

The game of Mr. Paint and Mrs. Correct is a game with complete information, played on a fixed given graph $G = (V, E)$. It is defined as follows: $G = (V, E)$

Game 1.1 (Mr. Paint and Mrs. Correct)**.** *Mr. Paint has many different colors, at least one for each round of the game. In each round he uses a new color that cannot be used again. Mrs. Correct has a finite stack* S_v *of erasers for each vertex* $v \in V$ *of the* S_v *underlying graph* G*. They are lying at the corresponding vertices, ready for use.*

The game of Mr. Paint and Mrs. Correct works as follows:

1P: *Mr. Paint starts, and in the first round he uses his first color to color some vertices of* G *(at least one).*

1C: *Mrs. Correct may use – and hereby use up – for each newly colored vertex* v *one eraser from* S_v *(if* $S_v \neq \varnothing$ *) to clear* v*. It is the job of Mrs. Correct to avoid monochromatic edges, i.e., edges with ends of the same color.*

2P: *In the second round Mr. Paint uses his second color to color some (at least one) of the by now uncolored vertices of* G*.*

2C: *Mrs. Correct, again, uses up erasers from some stacks* S_v *belonging to the newly colored vertices* v*, to avoid monochromatic edges.*

⋮ ⋮

End: The game ends when one player cannot move anymore, and hence loses.

Mrs. Correct cannot move if not anymore enough erasers are available with which she could avoid monochromatic edges, so that the remaining partial coloring would be incorrect.

Mr. Paint loses if all vertices have already been colored when it is his turn.

This game ends after at most $\sum_{v \in V}(|S_v| + 1)$ rounds. If Mrs. Correct wins, then the game results in a proper coloring of G. In this case, Mrs. Correct has rejected the color of each vertex $v \in V$ up to $|S_v|$ times. Put another way, we could imagine that Mr. Paint uses real paint and varnishes the vertices with it, and that Mrs. Correct uses sandpaper pieces to roughen the paint surface. In this way we obtain up to $\ell_v := |S_v| + 1$ layers of paint on each $v \in V$, which leads us to the following terminology:

Definition 1.2 (Paintability)**.** Let $\ell = (\ell_v)_{v \in V}$ be defined by $\ell_v := |S_v| + 1$. If there is ℓ, ℓ_v a winning strategy for Mrs. Correct, then we say that G is *ℓ-paintable*. We also say that G^ℓ is *paintable*, where G^ℓ is the graph G together with $\ell_v - 1$ erasers at each vertex G^ℓ $v \in V$ (the *mounted graph*, as we call it).

For $n \in \mathbb{N}$ we always write n-"something" for $(n\mathbf{1})$-"something", where $\mathbf{1} = (1)_{v \in V}$. $\mathbf{1}$

There is a connection to list colorings, which are defined as follows:

Definition 1.3 (List Colorings)**.** A product $L = \prod_{v \in V} L_v$ of sets L_v (called *lists*) of ℓ_v elements (called *colors*) is an *ℓ-product* (where $\ell := (\ell_v)_{v \in V}$). L, L_v

If there is a (proper) coloring $\lambda \in L$ of G – i.e., if $\lambda_u \neq \lambda_v$ for all $uv \in E$ – then we say that G is *L-colorable.* If G is L-colorable for all ℓ-products L, then we say that G is *ℓ-list colorable* or just *ℓ-colorable.*

Imagine that Mr. Paint writes down the colors he suggests for the vertex v in a list L_v. At the end of the game the list L_v has at most $\ell_v := |S_v| + 1$ entries, since $|S_v|$ is the maximal number of rejections at v. Furthermore, if v "wears" a color at the end of the game, then its color lies in the list L_v. Hence, paintability may be seen as a dynamic version of list colorability, where the lists L_v are not completely fixed before the coloration process starts. Thus we have the following connection to the usual list colorability:

Proposition 1.4. *Let G be a graph and $\ell \in \mathbb{N}^V$.*

$$\boxed{G \text{ is } \ell\text{-paintable.} \implies G \text{ is } \ell\text{-list colorable.}}$$

The following example shows the strictness of this statement. Another one, in the more general frame of polynomial, is presented in Section 7:

Example 1.5. The graph G in Figure 1 below is ℓ-list colorable but not ℓ-paintable, where $\ell_v := 2$ for all vertices $v \in V$ except the center v_5, for which $\ell_{v_5} := 3$:

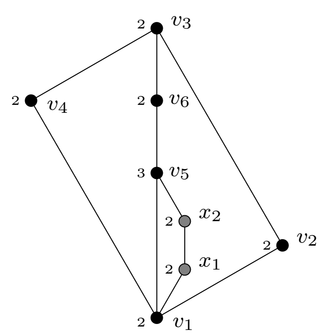


Figure 1: An ℓ-list colorable but not ℓ-paintable graph.

Proof. We start with the unpaintability of G: In order to prevail, Mr. Paint colors the vertices x_1 and x_2 in his first move. If Mrs. Correct then clears x_1, Mr. Paint can win as the induced subgraph $G[x_1, v_1, v_2, v_3, v_4]$ is not even L-colorable for $G[U]$

$$L = L_{x_1} \times L_{v_1} \times L_{v_2} \times L_{v_3} \times L_{v_4} := \{1\} \times \{1,2\} \times \{2,3\} \times \{3,4\} \times \{4,2\} . \qquad (1)$$

Indeed, this argument shows that the whole remaining uncolored part $G\backslash x_2$ of G is not list colorable for updated list sizes; and uncolorability implies unpaintability, as we have seen in Proposition 1.4 . Thus, Mrs. Correct cannot find a strategy for the remaining uncolored part $G\backslash x_2$ of G . (See also the recursive description of the game below).

If Mrs. Correct sands off x_2 , then Mr. Paint can win for the same reason. In this case there is an odd circuit in the remaining uncolored part $G\backslash x_1$ which cannot be colored with 2 colors, and the third color of v_5 can be "neutralized" through its neighbor x_2 . Summarizing, Mr. Paint wins in any case, and G is not ℓ-paintable.

We come now to the ℓ-list colorability, and have to examine all possible ℓ-products L : If

$$L_{x_1} = L_{x_2} \quad \text{or} \quad L_{x_1} \cap L_{x_1} = \varnothing \tag{2}$$

then each proper coloring of $G \setminus \{x_1, x_2\}$ extends to a proper coloring of G . It is thus sufficient to examine the more difficult case:

$$L_{x_1} := \{1, 2\} \quad \text{and} \quad L_{x_2} := \{2, 3\} \ . \tag{3}$$

In this case we have to find a coloring λ of $G \setminus \{x_1, x_2\}$ with

$$(\lambda_{v_1}, \lambda_{v_5}) \neq (1, 3) \ . \tag{4}$$

If, for example, there is a coloring λ of the path $v_1v_2v_3v_4$ with

$$\lambda_{v_4} \neq \lambda_{v_1} \neq 1 \ , \tag{5}$$

then this partial coloring can be extended to v_6 , then to v_5 and finally to the whole graph G . However, such extendable colorings of the path $v_1v_2v_3v_4$ always exist, except when the lists to v_1 , v_2 , v_3 and v_4 have the following "chain structure":

$$L_{v_1} \times L_{v_2} \times L_{v_3} \times L_{v_4} := \{1, a\} \times \{a, b\} \times \{b, c\} \times \{c, a\} \quad \text{where} \quad a \neq b \neq c \neq a \ . \tag{6}$$

But then we can choose

$$\lambda_{v_4} := a \ , \quad \lambda_{v_1} := 1 \quad \text{and} \quad \lambda_{v_2} := a \ , \tag{7}$$

and this partial coloring is extendable, at first to v_5 , with $\lambda_{v_5} \neq 3$, then to x_1 , x_2 and to v_6 , and finally to v_3 , which still has the two colors $b \neq a$ and $c \neq a$ "available". □

Now, we come to a more recursive formulation of our game, which is more easily accessible for proofs by induction. It is based on the simple observation that – since Mr. Paint uses an extra color for each round – it makes no difference whether one looks for coloring extensions of the partially colored graph G , or whether one cuts off the already colored vertices from the graph and colors the remaining graph. More precisely, we have the following reformulation:

Game 1.6 (Reformulation). *In this reformulation Mr. Paint has just one marker. As this is his only possession some call him Mr. Marker, but that is just a nickname.*

Mrs. Correct has a finite stack S_v *of sandpaper pieces for each vertex* v *in* $G_1 := G$ *. They are lying on the corresponding vertices, ready for use.*

The reformulated game of Mr. Paint and Mrs. Correct works as follows:

$1P$: *Mr. Paint starts, choosing a nonempty set of vertices* $V_{1P} \subseteq V(G_1)$ *and marking them with his marker.*

$1C$: *Mrs. Correct chooses an independent subset* $V_{1C} \subseteq V_{1P}$ *of marked vertices in* G_1 *, i.e.,* $uv \notin E(G_1)$ *for all* $u, v \in V_{1C}$ *. She cuts off the vertices in* V_{1C} *, so that the graph* $G_2 := G_1 \setminus V_{1C}$ *remains. The still marked vertices* $v \in V_{1P} \setminus V_{1C}$ *of* G_2 *have to be cleared. Therefore, Mrs. Correct must use one eraser from each of the corresponding stacks* S_v *. She loses if she runs out of erasers and cannot do that, i.e., if already* $S_v = \varnothing$ *for a still marked vertex* $v \in V_{1P} \setminus V_{1C}$ *.*

$2P$: *Mr. Paint again chooses a nonempty set of vertices* $V_{2P} \subseteq V(G_2)$ *and marks them with his marker.*

$2C$: *Mrs. Correct again cuts off an independent set* $V_{2C} \subseteq V_{2P}$ *, so that a graph* $G_3 := G_2 \setminus V_{2C}$ *remains. She also uses (and uses up) some erasers to clear the remaining marked vertices* $v \in V_{2P} \setminus V_{2C}$ *.*

$\vdots \qquad \vdots$

End: *The game ends when one player cannot move anymore, and hence loses.*

Mrs. Correct cannot move if she does not have enough erasers left to clear the vertices she was not able to cut off.

Mr. Paint loses if there are no more vertices left.

With this reformulation the original Definition 1.2 of paintability can be rewritten. At first, we introduce an appropriate notation for the graphs G_1, G_2, ..., produced in this version of the game, and their corresponding mounted graphs. Using characteristic maps/tuples of subsets $U \subseteq V$ and of elements $u \in V$, namely e_U , e_u

$$e_U \;:=\; (?_{(v=U)})_{v \in V} \;\in\; \{0,1\}^V \qquad \text{and} \qquad e_u \;:=\; e_{\{u\}} \quad , \tag{8}$$

based on the "Kronecker query" $?_{(\mathcal{A})}$, defined for statements $\mathcal{A}$ by $?_{(\mathcal{A})}$

$$?_{(\mathcal{A})} := \begin{cases} 0 & \text{if } \mathcal{A} \text{ is false,} \\ 1 & \text{if } \mathcal{A} \text{ is true,} \end{cases} \tag{9}$$

we provide:

Definition 1.7. Let G^ℓ be a mounted graph. We treat G^ℓ as any usual graph; but, when we change the graph, we adapt the stacks of erasers in the natural way. For example we set for sets U of vertices and edges $G^\ell \setminus U$

$$G^\ell \setminus U \ := \ (G \setminus U)^{\ell|_{V \setminus U}} \quad . \tag{10}$$

We also introduce a new operation $\downharpoonleft$ (*down*) which acts only on the stacks of erasers: $G^\ell \downharpoonleft U$

$$G^\ell \downharpoonleft U \ := \ G^{\ell - e_{(U \cap V)}} \quad . \tag{11}$$

Now, the remaining graph G_2, after Mrs. Correct's first move $1C$, together with the remaining stacks of reduced sizes

$$\ell_v^2 - 1 \ \leq \ \ell_v^1 - 1 \ := \ \ell_v - 1 \quad \text{for all } v \in V \,, \tag{12}$$

can be written as:

$$G_2^{\ell^2} \ = \ G_1^{\ell^1} \setminus V_{1C} \downharpoonleft V_{1P} \quad . \tag{13}$$

Furthermore, we obtain a handy recursive definition for paintability:

Definition 1.8 (Paintability – Reformulation)**.** For $\ell \in \mathbb{N}^V$ the *ℓ-paintability* of G, i.e., the paintability of G^ℓ, can be defined recursively as follows:

(i) $G = \varnothing$ is *ℓ-paintable* (where $V = \varnothing$ so that ℓ is the empty tuple).

(ii) $G \neq \varnothing$ is *ℓ-paintable* if $\ell \geq \mathbf{1}$ and if each nonempty subset $V_P \subseteq V$ of vertices contains a *good* subset $V_C \subseteq V_P$, i.e., an independent set $V_C \subseteq V_P$, such that $G^\ell \setminus V_C \downharpoonleft V_P$ is paintable.

It is obvious, that if $V_C \subseteq U \subseteq V_P$ and V_C is good in V_P, then V_C is also good in U. If, in addition, U is independent, then U is good in V_P. Conversely, in Proposition 2.1 we will learn that, if V_C is good in U, then V_C is also good in $V_P \supseteq U$, but for the price of additional erasers, i.e. if we put one additional eraser on each vertex v of $V_P \setminus U$. This will be important when we generalize theorems, based on coloring extension techniques, to paintability.

Before we come to this, we want to mention that, with slight modifications that do not affect the definition of paintability, our game can be viewed as a game in the sense of Conway's game theory [Co], [SSt]. From this point of view, graphs are not just either ℓ-paintable or not ℓ-paintable, but some graphs may be more ℓ-paintable than others. However, this game is not a „cold“ game, i.e., it is usually no *number*.

2 Coloring Extensions and Cut Lemmas

In this section we generalize coloring extension techniques to paintability. When we try to find list colorings, we may choose a particular vertex enumeration $v_1, v_2, \ldots, v_n$, and color the vertices v_i in turn, with a color not used for any neighbor of v_i among the successors $v_1, v_2, \ldots, v_{i-1}$. This technique cannot be used in the frame of paintability, but the following lemmas can provide a replacement. These replacements are then used at the end of the section to prove a sharpening of Thomassen's Theorem. Note that the corresponding list coloring versions of the used lemmas are almost trivial.

The proofs of the lemmas are based on a technique that we call *pre-use of additional erasers*. It means that additional erasers can be used before one has to look after a winning move. More exactly:

Proposition 2.1 (Pre-Usage Argument)**.** *Let G^ℓ be a mounted graph, and assume that Mr. Paint has marked a subset $V_P \subseteq V$, in which Mrs. Correct should find a good subset $V_C \subseteq V_P$. If we put additional erasers on the vertices of a subset $U \subseteq V_P$, then Mrs. Correct may use the additional erasers at first:*

If V_C is good in the remaining set $V_P \setminus U$, with respect to ℓ,
then V_C is also good in V_P, but with respect to $\ell + e_U$.

More general, for arbitrary subsets $U, V_C, V_P \subseteq V$, the following equality holds:

$$G^{\ell + e_{(U \cap V_P)}} \setminus V_C \downharpoonleft V_P \quad = \quad G^\ell \setminus V_C \downharpoonleft (V_P \setminus U) \quad . \tag{14}$$

We start our sequence of lemmas with the following very simple one, which we will use only in the simplest case $\ell_w = 1$:

Lemma 2.2 (Edge Lemma)**.** *Let two vertices w and $u \neq w$ be given. The ℓ-paintability of G implies the $(\ell + \ell_w e_u)$-paintability of $G \cup wu := (V, E \cup \{wu\})$.* $G \cup wu$

Proof. Let a nonempty subset $V_P \subseteq V$ be given. If $w \in V_P$, we pre-use one additional eraser, and choose V\u

$$V_C \quad \text{good in} \quad V_P \backslash u := V_P \setminus \{u\} \tag{15}$$

with respect to ℓ and G. Using Preposition 2.1, we know that

$$V_C \quad \text{is also good in} \quad V_P \tag{16}$$

but with respect to $\ell + e_u$ and G.

If now $w \notin V_C$, then we apply an induction argument to

$$G'^{\ell'} \ := \ G^{\ell + e_u} \setminus V_C \downharpoonleft V_P \quad , \tag{17}$$

which has one eraser viewer at $w \in V_P$, i.e.,

$$\ell'_w \ = \ \ell_w - 1 \quad . \tag{18}$$

It follows the paintability of

$$(G' \cup wu)^{\ell' + \ell'_w e_u} \overset{(17)}{=} (G^{\ell + e_u + \ell'_w e_u} \setminus V_C \downharpoonleft V_P) \cup wu \;=\; (G \cup wu)^{\ell + \ell_w e_u} \setminus V_C \downharpoonleft V_P \quad , \tag{19}$$

so that the recursive Definition 1.8 applies and accomplishes this case.

If $w \in V_C$ then exactly one end of wu lies in V_C (since we chose $V_C \subseteq V_P \backslash u$),

$$(G \cup wu) \setminus V_C \;=\; G \setminus V_C \quad , \tag{20}$$

and

$$(G \cup wu)^{\ell + e_u} \setminus V_C \downharpoonleft V_P \;=\; G^{\ell + e_u} \setminus V_C \downharpoonleft V_P \tag{21}$$

is still paintable, so that

$$V_C \quad \text{is good in} \quad V_P \tag{22}$$

even with respect to $G \cup wu$ and $\ell + e_u \;\leq\; \ell + \ell_w e_u$.

If $w \notin V_P$ things are even simpler, we choose

$$V_C \quad \text{good in} \quad V_P \tag{23}$$

with respect to ℓ and G ; i.e., $G^{\ell} \setminus V_C \downharpoonleft V_P$ is paintable. If, now, $u \in V_C$ then again exactly one end of wu lies in V_C and we can argue as above. In the other case we use an induction argument to prove the paintability of the mounted graph $(G \cup wu)^{\ell + \ell_w e_u} \setminus V_C \downharpoonleft V_P$, and apply Definition 1.8. □

Later on in this paper we will need the following simple lemma, which can also be applied to single vertices (the case $|U| = 1$ as well as the case $|W| = 1$):

Lemma 2.3 (Cut Lemma)**.** *Let $V = U \uplus W$ (disjoined union) be a partition of the vertex set of G , and let $\eta_u := |N(u) \cap W|$ be the number of neighbors of $u \in U$ in W .* $\uplus$

If $G[U]$ is ℓ_U-paintable and $G[W]$ is ℓ_W-paintable then G is $(\ell_U + \ell_W + \eta)$-paintable; where $\eta := (\eta_u)_{u \in U}$, and where this η , as well as ℓ_U and ℓ_W , is "filled up" with zeros, in order to view it as a tuple over V .

Proof. Let a nonempty subset $V_P \subseteq V$ be given, and choose

$$W_C \quad \text{good in} \quad W_P \;:=\; V_P \cap W \tag{24}$$

with respect to ℓ_W and $G[W]$. Now, let $N(W_C)$ be the set of all neighbors of vertices in W_C . We pre-use the erasers in the subset

$$\Delta \;:=\; V_P \cap U \cap N(W_C) \;\subseteq\; V_P \cap U \tag{25}$$

and choose

$$U_C \quad \text{good in} \quad U_P \;:=\; V_P \cap U \setminus N(W_C) \tag{26}$$

with respect to ℓ_U and $G[U]$; i.e., using Preposition 2.1, we know that

$$U_C \quad \text{is also good in} \quad V_P \cap U \;=\; U_P \uplus \Delta \tag{27}$$

but with respect to $\ell_U + e_\Delta$ and $G[U]$. In other words, if we bring in the set

$$V_C \;:=\; U_C \uplus W_C \quad , \tag{28}$$

the mounted graphs

$$G[W]^{\ell_W} \setminus W_C \downharpoonleft (V_P \cap W) \;=\; (G^{\ell_W} \setminus V_C \downharpoonleft V_P)[W \setminus W_C] \tag{29}$$

and

$$G[U]^{\ell_U + e_\Delta} \setminus U_C \downharpoonleft (V_P \cap U) \;=\; (G^{\ell_U + e_\Delta} \setminus V_C \downharpoonleft V_P)[U \setminus U_C] \tag{30}$$

are paintable, and an induction argument implies that

$$(G^{\ell_W + \ell_U + e_\Delta + \eta'} \setminus V_C \downharpoonleft V_P)[V \setminus V_C] \;=\; G^{\ell_W + \ell_U + e_\Delta + \eta'} \setminus V_C \downharpoonleft V_P \tag{31}$$

is paintable as well, where

$$\eta'_u \;:=\; |N(u) \cap W \setminus W_C| \quad \text{for all } u \in U\,. \tag{32}$$

Since neighbors u of elements $w \in W_C$ have fewer neighbors in $W \setminus W_C$ than in W

$$\eta'_u \;<\; \eta_u \quad \text{for all } u \in N(W_C)\,, \tag{33}$$

and

$$\eta' + e_\Delta \;\leq\; \eta \quad . \tag{34}$$

It follows that

$$G^{\ell_W + \ell_U + \eta} \setminus U_C \downharpoonleft V_P \tag{35}$$

is paintable, so that the recursive Definition 1.8 applies. □

Lemma 2.3 does not suffice to prove Thomassen's Theorem 2.6. We will need the following version of its $|W| = 1$ case, which requires more additional erasers, but also saves one at one distinguished neighbor u_0 of w :

Lemma 2.4 (Vertex Lemma)**.** *Let $wu_0 \in E$ be given and set $\eta_w := 2$, $\eta_{u_0} := 0$, $\eta_u = 2$ for all other neighbors u of w , and $\eta_v = 0$ for the remaining vertices v of G .*

If $G \backslash w$ is ℓ-paintable then G is $(\ell + \eta)$-paintable; where $\eta := (\eta_v)_{v \in V}$, and where $\ell \in \mathbb{N}^{V \backslash w}$ is "filled up" with one zero ($\ell_w := 0$), in order to view it as tuple over V .

Proof. Let a nonempty subset $V_P \subseteq V$ be given. Using an induction argument, as in the last part of the proof of Lemma 2.2, we may suppose that $w \in V_P$. Let

$$N \;:=\; \{u \neq u_0 \mid \operatorname{dist}(u, w) \leq 1\} \tag{36}$$

and choose

$$V'_C \quad \text{good in} \quad V'_P \;:=\; V_P \setminus N \tag{37}$$

with respect to ℓ and $G\backslash w$; i.e.,

$$(G\backslash w)^{\ell} \setminus V'_C \downharpoonleft V'_P \tag{38}$$

is paintable. Of course, we want to apply a pre-usage argument to the difference

$$V_P \setminus V'_P \;=\; V_P \cap N \quad . \tag{39}$$

We distinguish two cases:

If $u_0 \in V'_C$ we apply Lemma 2.3 to $G^{\ell+e_w} \setminus V'_C \downharpoonleft V'_P$, where we choose $W := \{w\}$, $U := (V\backslash w) \setminus V'_C$ and use the inherited stacks, e.g., $\ell_W := e_w$. It follows that

$$G^{\ell+\eta'} \setminus V'_C \downharpoonleft V'_P \;=\; G^{\ell+\eta'+e_{(V_P \cap N)}} \setminus V'_C \downharpoonleft V_P \tag{40}$$

is paintable; where $\eta'_w := 1$, $\eta'_u := 1$ for all neighbors u of w in $G \setminus V'_C$, and $\eta'_v := 0$ for the remaining vertices v of G . As we assumed $u_0 \in V'_C$ this means that $\eta'_{u_0} = 0$ and hence

$$\eta' + e_{(V_P \cap N)} \;\leq\; \eta \quad , \tag{41}$$

so that

$$V'_C \quad \text{is good in} \quad V_P \tag{42}$$

with respect to $\ell + \eta$ and G .

If $u_0 \notin V'_C$ then, on one hand, w has no neighbor in V'_C , and $V'_C \cup \{w\}$ is independent in G , on the other hand, as we have seen above,

$$G^{\ell+e_{(V_P \cap N)}} \setminus (V'_C \cup \{w\}) \downharpoonleft V_P \;=\; (G\backslash w)^{\ell} \setminus V'_C \downharpoonleft V'_P \tag{43}$$

is paintable. Hence,

$$V'_C \cup \{w\} \quad \text{is good in} \quad V_P \tag{44}$$

with respect to G and $\ell + \eta \,\geq\, \ell + e_{(V_P \cap N)}$. □

We will also need the following lemma that, together with the Edge Lemma 2.2, could be used in another proof of the Cut Lemma 2.3:

Lemma 2.5 (Merge Lemma)**.** *Let $G^{\ell} := G'^{\ell'} \cup G''^{\ell''}$ be the union $G' \cup G''$ of two graphs G' and G'', together with the inherited erasers, i.e.,* $G'^{\ell'} \cup G''^{\ell''}$

$$\ell - \mathbb{1} \;:=\; (\ell' - \mathbb{1}) \,+\, (\ell'' - \mathbb{1}) \quad ; \tag{45}$$

where $\ell' - \mathbb{1}$ and $\ell'' - \mathbb{1}$ are "filled up" with zeros, in order to view them as tuples over the set V. Suppose further that in G'' there are no erasers at the vertices of the intersection, i.e.,

$$\ell''|_U \;\equiv\; 1 \;, \qquad \textit{where} \quad U := V(G') \cap V(G'') \quad . \tag{46}$$

If $G'^{\ell'}$ and $G''^{\ell''}$ are paintable, then $G^{\ell} := G'^{\ell'} \cup G''^{\ell''}$ is paintable as well.

Proof. In order to prove the paintability of G^ℓ, we have to find a good subset V_C in each fixed given nonempty subset $V_P \subseteq V$. To this end, we choose

$$V'_C \quad \text{good in} \quad V'_P := V_P \cap V(G') \tag{47}$$

with respect to $G'^{\ell'}$, and we choose

$$V''_C \quad \text{good in} \quad V''_P := (V_P \setminus V(G')) \uplus (U \cap V'_C) \tag{48}$$

with respect to $G''^{\ell''}$. Since no erasers lie at the vertices $u \in U \cap V''_P$ of G'', they have to be cut off, i.e.,

$$U \cap V''_P \;\subseteq\; V''_C \;\subseteq\; V''_P \ . \tag{49}$$

Moreover, intersecting these sets with U, we see that

$$U \cap V''_C \;=\; U \cap V''_P \overset{(48)}{=} U \cap V'_C \ . \tag{50}$$

Hence, if we define

$$V_C \;:=\; V'_C \cup V''_C \ , \tag{51}$$

then

$$V'_P \cap V''_P \overset{(48)}{=} U \cap V'_C \;=\; U \cap V_C \;=\; U \cap V''_C \ , \tag{52}$$

and it follows that

$$G' \setminus V_C \;=\; G' \setminus V'_C \ , \qquad G'' \setminus V_C \;=\; G'' \setminus V''_C \tag{53}$$

and

$$V_P \setminus V_C \;=\; (V'_P \setminus V_C) \uplus (V''_P \setminus V_C) \;=\; (V'_P \setminus V'_C) \uplus (V''_P \setminus V''_C) \ . \tag{54}$$

Therefore,

$$\begin{aligned} G^\ell \setminus V_C \downharpoonleft V_P \;&=\; \big((G'^{\ell'}) \cup (G''^{\ell''})\big) \setminus V_C \downharpoonleft (V_P \setminus V_C) \\ &\overset{(54)}{=} \big((G'^{\ell'} \setminus V_C) \cup (G''^{\ell''} \setminus V_C)\big) \downharpoonleft \big((V'_P \setminus V'_C) \uplus (V''_P \setminus V''_C)\big) \\ &\overset{(53)}{=} \big((G'^{\ell'} \setminus V'_C) \cup (G''^{\ell''} \setminus V''_C)\big) \downharpoonleft (V'_P \setminus V'_C) \downharpoonleft (V''_P \setminus V''_C) \\ &=\; \big(G'^{\ell'} \setminus V'_C \downharpoonleft (V'_P \setminus V'_C)\big) \;\cup\; \big(G''^{\ell''} \setminus V''_C \downharpoonleft (V''_P \setminus V''_C)\big) \\ &=\; (G'^{\ell'} \setminus V'_C \downharpoonleft V'_P) \;\cup\; (G''^{\ell''} \setminus V''_C \downharpoonleft V''_P) \ , \end{aligned} \tag{55}$$

and, based on an induction argument, the last obtained term indicates the paintability of $G^\ell \setminus V_C \downharpoonleft V_P$. However, this means that V_C is good in V_P with respect to the examined graph G^ℓ. □

Now, we are prepared to sharpen Thomassen's Theorem [Th], [Di, p. 122] about the 5-list colorability of planar graphs:

Theorem 2.6. *Planar graphs are* 5*-paintable.*

Proof. The proof works almost exactly the same as the original one, but the coloring extension arguments have to be replaced. We start with a slightly modified induction hypothesis, and will prove by induction the following assertion for all plane graphs G with at least 3 vertices. In connection with Lemma 2.2 (which allows us to reinsert the removed edge v_1v_2) this assures the 5-paintability of plan triangulations, and hence all planar graphs:

> Suppose that every inner face of G^ℓ is bounded by a triangle and its outer face by a cycle $C = v_1 \dots v_k v_1$. Suppose further that there is no eraser at v_1 and at v_2 ($\ell_{v_1} = \ell_{v_2} := 1$), that there are 2 erasers at each other vertex v_i of the boundary C ($\ell_{v_i} := 3$), and that there are 4 at each inner vertex u ($\ell_u := 5$). Then Mrs. Correct can enforce a proper coloring of $G^\ell \backslash v_1v_2$.

If $|G| = 3$, then $G = C$ and the assertion is trivial. We may thus assume that there are edges inside C, and we can distinguish between the following two cases:

Case 1. If C has a chord v_iv_j, then v_iv_j lies on two unique cycles

$$C', C'' \;\subseteq\; C + v_iv_j \tag{56}$$

with

$$v_1v_2 \in C' \quad \text{and} \quad v_1v_2 \notin C'' \quad . \tag{57}$$

Let G' resp. G'' denote the subgraph of G induced by the vertices lying on or inside C' resp. C''. Using an induction argument, we know that the assertion holds for $G'^{\ell'}$, with the inherited pieces of sandpaper ($\ell' := \ell|_{V(G'))}$). Similarly, it also holds for G'', but with v_i and v_j in the place of v_1 and v_2, i.e., $G''\backslash v_iv_j$ is ℓ''-paintable when all erasers at v_i and at v_j are removed ($\ell''_{v_i} = \ell''_{v_j} := 1$ and $\ell''_u := \ell_u$ for the other vertices u in G''). Now Lemma 2.5 applies and proves the paintability of

$$G^\ell \backslash v_1v_2 \;=\; G'^{\ell'} \backslash v_1v_2 \,\cup\, G''^{\ell''} \backslash v_iv_j \tag{58}$$

Case 2. If C has no chord, let $v_1, u_1, \dots, u_m, v_{k-1}$ be the neighbors of v_k in their natural cyclic order around v_k. By definition of C, all these neighbors u_i lie in the inner face of C. Since the inner faces of G are bounded by triangles, and there are no multiple edges,

$$P \;:=\; v_1u_1 \dots u_m v_{k-1} \tag{59}$$

is a path in G. Since C is chordless,

$$\tilde{C} \;:=\; P \cup (C \backslash v_k) \tag{60}$$

is a cycle – the boundary cycle of $G\backslash v_k$. By induction we know that $G\backslash v_k \backslash v_1v_2$ is paintable, where at the new boundary vertices u_i two erasers suffice.

We now extend the paintability of $G\backslash v_k \backslash v_1v_2$ to $G\backslash v_1v_2 \backslash v_kv_1$ and finally to $G\backslash v_1v_2$. To this end we apply Lemma 2.4 to $G\backslash v_1v_2\backslash v_kv_1$, with v_k in the role of w and v_{k-1} in the role of u_0. Afterwards, we apply Lemma 2.2, with v_k in the role of w and v_1 in the role of u. Altogether, we had to add 2 erasers at each of the u_i and on the new vertex v_k; the sizes of the other stacks remained unchanged. □

3 Kernels and Edge Paintability

In this section we generalize some results about edge list colorability to edge paintability; where a graph G is called edge ℓ-paintable if its line graph is ℓ-paintable. Two further edge paintability results, concerning the complete graph K_n and regular planar graphs, are presented at the end of Section 5. All results of this section are based on the existence of kernels (Lemma 3.1) and the examination of orientations. We use the following notations for these kind of investigations: $\rightarrow, e^{\rightarrow}$

$\rightarrow: E \longrightarrow V$, $e \longmapsto e^{\rightarrow}$ denotes a fixed *orientation* of G. Therefore, $e^{\rightarrow}$ is always one end of e, and $e^{\leftarrow}$ denotes the other one ($\{e^{\rightarrow}, e^{\leftarrow}\} = e$). $\vec{G} := (V, E, \rightarrow)$ is the corresponding *oriented graph*. $D = D(G) = D(\vec{G})$ denotes the set of all orientations $\varphi\colon E \ni e \longmapsto e^{\varphi} \in e$ of G. We write $u \rightarrow v$ (resp. $u \overset{\varphi}{\rightarrow} v$) if we want to say that $uv \in E$ and that $(uv)^{\rightarrow} = v$ (resp. $(uv)^{\varphi} = v$). $N^+_{\varphi}(v) := \{\, w \in V \mid v \overset{\varphi}{\rightarrow} w \,\}$ denotes the set of *φ-successors* of $v \in V$, $d^+_{\varphi}(v) := |N^+_{\varphi}(v)|$ its *φ-outdegree*, and $d^+_{\varphi} := \big(d^+_{\varphi}(v)\big)_{v \in V}$ the *outdegree tuple*. We abbreviate $N^+(v) := N^+_{\rightarrow}(v)$ and $d^+ := d^+_{\rightarrow}$. Similarly, we define $N(v) = N_G(v) := \{\, w \in V \mid vw \in E \,\}$ and $d_G := \big(d(v)\big)_{v \in V}$. As usual, $\Delta(G)$ is the maximal degree, and $\Delta^+(\varphi)$ is the maximal outdegree of the vertices in G.

$\vec{G}$ D $u \rightarrow v$ $N^+_{\varphi}(v)$ d^+_{φ} $N^+(v)$, d^+ $N_G(v)$, d_G $\Delta(G)$ $\Delta^+(\varphi)$

Now, the following paintability version of Bondy, Boppana and Siegel's Lemma, in [Ga, Lemma 2.1] or [Di, Lemma 5.4.3], follows easily with a simple induction argument from Definition 1.8:

Lemma 3.1 (Kernel Lemma). *Let $\vec{G}$ be a directed graph, such that each induced subgraph $G[V_P]$ of $\vec{G}$ has a* kernel *– i.e., an independent subset $V_C \subseteq V_P$ such that, for each vertex $u \in V_P \setminus V_C$ there is a $\bar{u} \in V_C$ with $u \rightarrow \bar{u}$ – then G is $(d^+ + 1)$-paintable.*

Proof. We may assume $G \neq \varnothing$. Let V_C be a kernel of a fixed given nonempty subset $V_P \subseteq V$. As necessarily $V_C \neq \varnothing$, and as $G \setminus V_C$ fulfills the preconditions of the Lemma, we may apply an induction argument, and see that $G \setminus V_C$ is $(d^+_{G \setminus V_C} + 1)$-paintable, i.e.,

$$(G \setminus V_C)^{d^+_{G \setminus V_C} + 1 + e_{(V_P \setminus V_C)}} \downharpoonleft V_P \;=\; (G \setminus V_C)^{d^+_{G \setminus V_C} + 1} \tag{61}$$

is paintable. Now, because of

$$d^+_G(v) \;>\; d^+_{G \setminus V_C}(v) \quad \text{for all } v \in V_P \setminus V_C \quad , \tag{62}$$

the paintability of

$$G^{d^+_G + 1} \setminus V_C \downharpoonleft V_P \tag{63}$$

follows; so that the recursive Definition 1.8 applies. □

Galvin used in [Ga] Bondy, Boppana and Siegel's Lemma to prove the list coloring conjecture for bipartite graphs (see also [Di, Theorem 5.4.4]). Using our version this an be sharpened to paintability (without further modifications in the proof). Together with

König's classical calculation [Di, Proposition 5.3.1] of the chromatic index of bipartite graphs we obtain:

Theorem 3.2. *Bipartite graphs G are edge $\Delta(G)$-paintable.*

Galvin's result also implies the existence of certain generalized Latin Squares, which was conjectured by Dinitz. With the sharper Theorem 3.2 this existence result can be generalized further, leading to a version with stacks of erasers on a „chess board".

Borodin, Kostochka and Woodall exploited in [BKW] Galvin's remarkable new method to prove further sharpenings and applications. We sharpen their main result [BKW, Theorem 3], and our Theorem 3.2, as follows:

Theorem 3.3. *Bipartite multigraphs G are edge ℓ-paintable, when for each edge $e = uw$ we set*

$$\ell_e := \max\{d(u), d(w)\} \quad .$$

Proof. We refer to Galvin's original proof as it was printed in Diestel's book [Di]. Borodin, Kostochka and Woodall's proof use a terminology different from those in [Di, Theorem 5.4.4 & Corollary 5.4.5], and does not explicitly work with orientations. However, the only real difference to the proof in [Di] is that the authors have chosen the underlying coloring $c\colon E \longrightarrow \mathbb{Z}$ more carefully (see the remark after [BKW, Corollary 1.1]). Based on the construction of c in the proof of [BKW, Theorem 3], and using our sharpened Kernel Lemma 3.1 instead of [Di, Lemma 5.4.3], the proof in [Di] yields the stated theorem. □

They also provide a proof for a sharpening of Shannon's bound of the chromatic index of multigraphs. This proof is based on the following interesting lemma, which we state for paintability:

Lemma 3.4. *If G, H and B are multigraphs, where B is bipartite and $G = H \cup B$, and if*

$$\ell_e := \max\{d_G(u) + d_H(w),\, d_H(u) + d_G(w)\} \quad \text{for each edge } e = uw \,,$$

then G is edge ℓ-paintable.

Proof. The proof is based on Theorem 3.3, and works almost exactly as in [BKW, Lemma 4.1]: We may assume

$$E(H) \cap E(B) = \varnothing \quad . \tag{64}$$

Since $d_G(v) \geq d_H(v)$ for each $v \in V$, it follows that

$$\ell_e > d_{LH}(e) \quad \text{for all } e \in E \,, \tag{65}$$

where LH is the line graph of H. Hence, in view of Theorem 6.1, H is edge paintable using the inherited erasers.

Using Theorem 3.3, we see that the other part B is edge ℓ'-paintable, where

$$\ell'_{uw} := \max\{d_B(u), d_B(w)\} \qquad \text{for all } uw \in E(B)\,. \tag{66}$$

Since each edge uw of B (as a vertex of the line graph LG) has

$$\eta_{uw} := |N_{LG}(uw) \cap E(H)| = d_H(u) + d_H(w) \tag{67}$$

neighbors in $E(H)$, so that

$$\begin{aligned} \ell_{uw} &= \max\{\, d_G(u) + d_H(w)\,,\ d_H(u) + d_G(w)\,\} \\ &= \max\{\, d_B(u) + (d_H(u) + d_H(w))\,,\ (d_H(u) + d_H(w)) + d_B(w)\,\} \\ &= \max\{\, d_B(u)\,,\ d_B(w)\,\} + (d_H(u) + d_H(w)) \\ &= \ell'_{uw} + \eta_{uw}\ , \end{aligned} \tag{68}$$

the Cut Lemma 2.3 (with $LG,\ E(B),\ E(H)$ in the place of $G,\ U,\ W$) to prove the ℓ-paintability of G. □

With this lemma we obtain the following sharpening of Shannon's bound:

Theorem 3.5. *Multigraphs G are edge ℓ-paintable, where*

$$\ell_{uw} := \max\{d(u), d(w)\} + \lfloor \frac{1}{2} \min\{d(u), d(w)\} \rfloor \qquad \textit{for all } uw \in E\,.$$

In particular, G is edge $\lfloor \frac{3}{2}\Delta(G) \rfloor$-paintable.

Proof. As in [BKW, Theorem 4] one can apply Lemma 3.4 to a maximal cut $E(U,W)$

$$B = (V, E(U,W))\,, \quad V = U \uplus W \tag{69}$$

in G, and to

$$H := G \setminus E(B)\ ; \tag{70}$$

which fulfills

$$d_H(v) \le \tfrac{1}{2} d_G(v) \quad \text{for all } v \in V\,, \tag{71}$$

since otherwise we could move a vertex v to the other side of the partition, and would obtain a contradiction to the maximality of $|E(U,W)|$. □

The figure $\lfloor \frac{3}{2}\Delta(G) \rfloor$ in this theorem is best possible. The so-called "thick triangle" with $\lfloor \frac{1}{2}\Delta \rfloor$, $\lfloor \frac{1}{2}\Delta \rfloor$ and $\lceil \frac{1}{2}\Delta \rceil$ edges between the vertices shows this; it has chromatic index $\lfloor \frac{3}{2}\Delta \rfloor$.

Clearly, it would be interesting to find a paintability version of Vizing's Theorem. This is an open problem, even for list colorings. The recoloring techniques (Kempe-chains) used in the known proofs of the original edge coloring theorem do not work with list colorings. In [Ko] Kostochka needed the additional assumption that G has girth at least $8\Delta(G)\big(\ln(\Delta(G)) + 1.1\big)$, in order to prove that simple graphs G are edge $(\Delta(G){+}1)$-list colorable. However, if the list color conjecture is true, this holds without further assumptions about the girth as well.

4 Alon and Tarsi's Theorem

In this section we discus a surprising connection between colorings and orientations of graphs. Let $\vec{G}$ be an oriented graph, and suppose we have an ℓ-product L

$$L := \prod_{v \in V} L_v \tag{72}$$

of lists L_v of sizes ℓ

$$\ell_v := |L_v| > d^+(v) \quad . \tag{73}$$

Is there an L-coloring of G ?

One could conjecture that there is one since each list L_v (to each fixed vertex $v \in V$) contains so many colors that – if all successors $u \in N^+(v)$ are already colored – there is at least one color in L_v that differs from the colors of the neighbors $u \in N^+(v)$. If we now use this "evasion color" to color the vertex v , and do the same for all other vertices of V , then we obtain a proper coloring of G , since in each edge uv one end "takes care" of the other end (either $u \in N^+(v)$ or $v \in N^+(u)$).

However, this train of thought runs on nonexisting rails. We cannot just assume that "all successors $u \in N^+(v)$ are already colored". An example which shows the validity of the desired conclusion is the directed circuit of length 3, which is not colorable with 2 colors. Nevertheless, our consideration contains some plausibility, and one could ask for an additional condition that makes it work. Alon and Tarsi found such a condition in [AlTa]. They proved that ℓ-list colorings exist, if the sets of *even* and *odd Eulerian (spanning) subgraphs* EE and EO of $\vec{G}$ do not have the same size, i.e., EE, EO

$$|EE| \neq |EO| \quad ; \tag{74}$$

where a directed graph $\vec{G}$ is *even/odd Eulerian* if it has even/odd many edges, and if the indegree of each single vertex $v \in V$ equals its outdegree. In their paper they work with the set $D_\alpha = D_\alpha(G) = D_\alpha(\vec{G})$ of all orientations φ with $d^+_\varphi = \alpha \in \mathbb{Z}^V$, and with D_α $DE_\alpha = DE_\alpha(\vec{G})$, resp. $DO_\alpha = DO_\alpha(\vec{G})$, the sets of *even* resp. *odd* ones, i.e., those which DE_α, DO_α differ from $\to$ ($e^\varphi \neq e^{\to}$) on even resp. odd many edges $e \in E$. At the end they used the fact that

$$|DE_{d^+}| = |EE| \qquad \text{and} \qquad |DO_{d^+}| = |EO| \quad . \tag{75}$$

This is not hard to see (see also [Scha, Lemma 2.6]). In this paper we state our theorems using DE_α and DO_α instead of EO and EE . Of course,

$$DE_\alpha = DO_\alpha = \varnothing \tag{76}$$

if there are no $\varphi \in D(G)$ with $d^+_\varphi = \alpha$, i.e., no *realizations* of α . This is for example the case if $\alpha \ngeq 0$ or if

$$\sum_{v \in V} \alpha_v \neq |E| \quad , \tag{77}$$

since

$$\sum_{v\in V} d_\varphi^+ \;=\; |E| \quad \text{for all orientations } \varphi\in D(G)\,. \tag{78}$$

Alon and Tarsi's work was a forerunner to the Combinatorial Nullstellensatz [Al2], which has many applications. In [Scha2] we proved a quantitative sharpening of this Nullstellensatz, which also led to a (weighted) qualitative version of the Alon-Tarsi-Theorem. The difference $|DE_\alpha| - |DO_\alpha|$ (which can also be written as *permanent* of an incidence matrix, as in the last section or in [Scha2, Corrolary 5.5]) equals a weighted sum over certain colorings. Here, we present a paintability sharpening of the Alon-Tarsi-Result. Our proof can be generalized to polynomials, as described in the last section, leading to a paintability version of the Combinatorial Nullstellensatz. This version of the Nullstellensatz is more general than the following sharpening of Alon and Tarsi's Theorem. However, Alon and Tarsi have already asked in the original paper [AlTa] for a combinatorial proof of their result. Therefore, at first we will work in the purely combinatorial frame of orientations of graphs in order to shed some light on the surprising connection between colorings and orientations of graphs. The more abstract and algebraic generalization to polynomials follows later, in Section 7. We have:

Theorem 4.1. *Let $\vec{G}$ be a directed graph and $\alpha\in\mathbb{N}^V$, then*

$$\boxed{|DE_\alpha(\vec{G})| \neq |DO_\alpha(\vec{G})| \implies \vec{G} \text{ is } (\alpha+1)\text{-paintable.}}$$

The proof of this theorem contains an explicit winning strategy. It is a proof by induction, and uses the notations in the reformulated Game 1.6. We will examine the orientation sets $DE_{\alpha+\mathbb{N}^U}$

$$D_S := \bigcup_{\alpha\in S} D_\alpha \;\;,\qquad DE_S := \bigcup_{\alpha\in S} DE_\alpha \quad\text{and}\quad DO_S := \bigcup_{\alpha\in S} DO_\alpha \;\;, \tag{79}$$

where for $S\subseteq\mathbb{N}^V$ we will use the following type of set $\alpha+\mathbb{N}^U$

$$\alpha+\mathbb{N}^U \;:=\; \{\, \alpha'\geq\alpha \;\not|\; \alpha'(v)=\alpha(v) \;\text{ for all }\; v\notin U \,\} \;\;, \tag{80}$$

with $\alpha\in\mathbb{Z}^V$ and $U\subseteq V$.

One single induction step in this proof will be partitioned into four parts. In the first part we have to modify the induction hypothesis a little bit. The second part describes the winning strategy of Mrs. Correct; it is mainly contained in the following lemma. In the third part we have to understand why this strategy singles out an independent set. This is also contained in the following lemma (in its very last sentence). The finally step is contained in the second lemma below, and will show that the induction hypothesis remains true when we cut of the independent set. Figure 2 below illustrates our first lemma:

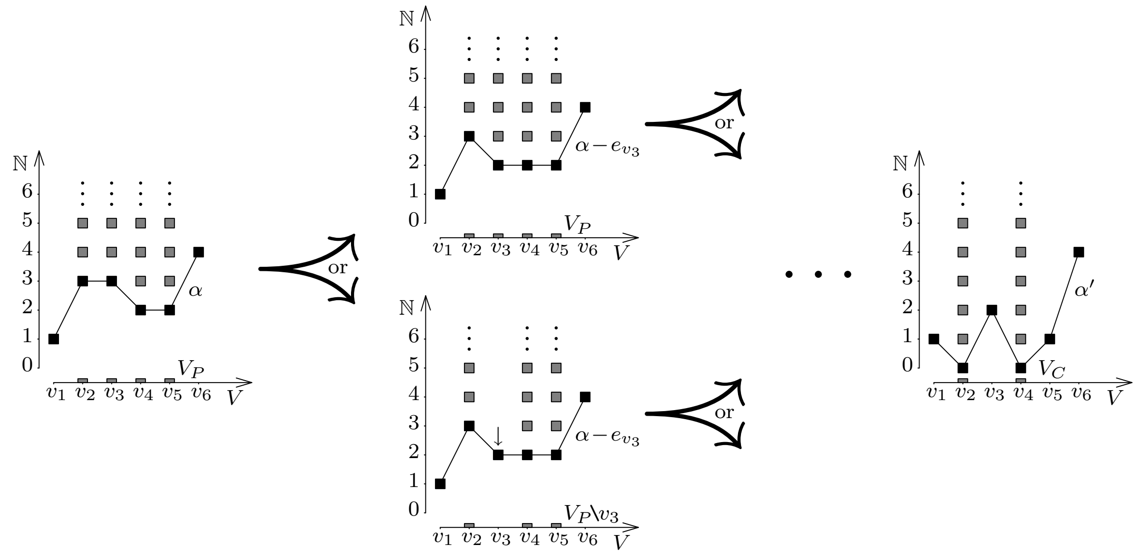


Figure 2: $v \longmapsto \alpha_v$ *and* $\alpha + \mathbb{N}^{V_P}$ *in Lemma 4.2.*

Lemma 4.2. *Let* $\vec{G} = (V, E, \to)$ *be a directed graph,* $\alpha \in \mathbb{N}^V$, $V_P \subseteq V$ *nonempty and* $u \in V_P$, *then:*

(i) $(\alpha - e_u) + \mathbb{N}^{V_P} \;=\; \alpha + \mathbb{N}^{V_P} \;\uplus\; (\alpha - e_u) + \mathbb{N}^{V_P \setminus u}$.

(ii) $DE_{(\alpha-e_u)+\mathbb{N}^{V_P}} \;=\; DE_{\alpha+\mathbb{N}^{V_P}} \uplus DE_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}$ *and*
$DO_{(\alpha-e_u)+\mathbb{N}^{V_P}} \;=\; DO_{\alpha+\mathbb{N}^{V_P}} \uplus DO_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}$.

(iii) $|DE_{\alpha+\mathbb{N}^{V_P}}| \neq |DO_{\alpha+\mathbb{N}^{V_P}}|$ *implies that*
$|DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}| \neq |DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}|$ *or*
$|DE_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}| \neq |DO_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}|$.

(iv) $|DE_{\alpha+\mathbb{N}^{V_P}}| \neq |DO_{\alpha+\mathbb{N}^{V_P}}|$ *implies that there is a* $V_C \subseteq V_P$ *and an* $0 \le \alpha' \le \alpha$ *s.t.*
$|DE_{\alpha'+\mathbb{N}^{V_C}}| \neq |DO_{\alpha'+\mathbb{N}^{V_C}}|$, $\alpha'|_{V_C} \equiv 0$ *and* $\alpha'_v < \alpha_v$ *for all* $v \in V_P \setminus V_C$.
Furthermore, each such set V_C *is independent in* $\vec{G}$.

Proof. The tuples $\sigma \in (\alpha - e_u) + \mathbb{N}^{V_P}$ in the set on the left side of Equation (i) fulfill $\sigma_u \ge \alpha_u - 1$. On the right side we simply distinguish between those with $\sigma_u > \alpha_u - 1$ and those with $\sigma_u = \alpha_u - 1$.

In order to obtain part (ii), we just have to take the preimages of the sets in (i) under the map $\varphi \longrightarrow d^+_\varphi$, which we viewed, either as a map defined on the set DE of all even *DE*
orientations, or as a map defined on the set DO of all odd orientations. *DO*

Now, we take the absolute value of the sets in part (ii) and obtain

$$\begin{aligned} |DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}| &= |DE_{\alpha+\mathbb{N}^{V_P}}| + |DE_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}| \quad \text{and} & (81)\\ |DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}| &= |DO_{\alpha+\mathbb{N}^{V_P}}| + |DO_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}| \quad . & (82)\end{aligned}$$

If we extend this system of linear equations with

$$\begin{aligned} |DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}| &= |DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}| \quad \text{and} & (83)\\ |DE_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}| &= |DO_{(\alpha-e_u)+\mathbb{N}^{V_P\setminus u}}| \quad , & (84)\end{aligned}$$

it follows that:

$$|DE_{\alpha+\mathbb{N}^{V_P}}| = |DO_{\alpha+\mathbb{N}^{V_P}}| \quad . \tag{85}$$

Part (iii) is the contraposition to this conclusion.

In order to prove part (iv), we may use part (iii), as illustrated in Figure 2, to produce sequences

$$\alpha =: \alpha^0 \gneq \alpha^1 \gneq \cdots \gneq \alpha^t \geq 0 \qquad \text{and} \qquad V_C =: V_C^0 \supseteq V_C^1 \supseteq \cdots \supseteq V_C^t \tag{86}$$

with the property

$$|DE_{\alpha^i+\mathbb{N}^{V_C^i}}| \neq |DO_{\alpha^i+\mathbb{N}^{V_C^i}}| \qquad \text{for } i = 0, 1, \ldots, t\,. \tag{87}$$

Note that

$$\alpha^t|_{V_C^t} \equiv 0 \tag{88}$$

if and only if the sequence of componentwise nonnegative α^i in (86) can no longer be extended through application of part (iii); hence, in this case part (iv) holds, if we set

$$\alpha' := \alpha^t \qquad \text{and} \qquad V_C := V_C^t \quad . \tag{89}$$

It remains to be shown that the existence of an edge uv with both ends in V_C would lead to a contradiction: Suppose there is one, then turning around this edge uv gives rise to a fixpoint free involution

$$\Theta_{uv}: \ D(G) \xrightarrow{\cong} D(G) \quad . \tag{90}$$

This involution can be restricted to an involution

$$D_{\alpha'+\mathbb{N}^{V_C}} \xrightarrow{\cong} D_{\alpha'+\mathbb{N}^{V_C}} \quad , \tag{91}$$

since – if we apply Θ_{uv} to an orientation $\varphi \in D_{\alpha'+\mathbb{N}^{V_C}}$ – the two changing outdegrees $d_\varphi^+(u)$ and $d_\varphi^+(v)$ are irrelevant for its membership to $D_{\alpha'+\mathbb{N}^{V_C}}$. That is because

$$\alpha'_u = 0 \quad \text{and} \quad \alpha'_v = 0 \quad , \tag{92}$$

by Equation (88), and because if $\sigma := d_\varphi^+$ belongs to $\alpha' + \mathbb{N}^{V_C}$ then each $\sigma' \geq 0$, which differs from σ only on vertices $w \in V_C$ with $\alpha'_w = 0$, belongs to $\alpha' + \mathbb{N}^{V_C}$ as well. Altogether, as Θ_{uv} maps even orientations to odd orientations and *vice versa*, we see that

$$|DE_{\alpha'+\mathbb{N}^{V_C}}| \;=\; |DO_{\alpha'+\mathbb{N}^{V_C}}| \;, \tag{93}$$

a contradiction. □

Now we come to our second lemma which allows us to cut off independent sets $V_C \subseteq V$. For our main theorem we will need only the case $V_P = V_C$:

Lemma 4.3. *Let $\vec{G} = (V, E, \to)$ be a directed graph, $\alpha \in \mathbb{N}^V$, $V_P \subseteq V$, $uv \in E$, $u{\to}v$, $E' \subseteq E$ and let $V_C \subseteq V$ be an independent set in $\vec{G}$, then:*

(i) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;=\; |DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;+\; |DO_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$ *and*

$|DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;=\; |DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;+\; |DE_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$.

(ii) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;\neq\; |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that*

$|DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;\neq\; |DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$ *or*

$|DE_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;\neq\; |DO_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$.

(iii) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;\neq\; |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that there is an* $0 \leq \alpha' \leq \alpha$ *such that*

$|DE_{\alpha'+\mathbb{N}^{V_P}}(\vec{G} \setminus E')| \;\neq\; |DO_{\alpha'+\mathbb{N}^{V_P}}(\vec{G} \setminus E')|$.

(iv) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;\neq\; |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that there is an* $0 \leq \alpha'' \leq \alpha|_{V\backslash V_C}$ *s.t.*

$|DE_{\alpha''+\mathbb{N}^{V_P\backslash V_C}}(\vec{G} \setminus V_C)| \;\neq\; |DO_{\alpha''+\mathbb{N}^{V_P\backslash V_C}}(\vec{G} \setminus V_C)|$.

Proof. When we restrict an orientation φ of G to $E\backslash uv$, we obtain an orientation of the smaller graph $G\backslash uv$. This restricted orientation $\varphi|_{E\backslash uv}$ has the same parity (either even or odd) as φ if $u \overset{\varphi}{\to} v$, and the opposite parity in the other case. Conversely, each orientation φ' of the smaller graph $G\backslash uv$ extends to one orientation of G with the same parity as φ', and to one orientation with the opposite orientation as φ'. Restriction of orientations leads to bijections

$$DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G}) \xrightarrow{\;\cong\;} DE_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \;\uplus\; DO_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \quad \text{and} \tag{94}$$

$$DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G}) \xrightarrow{\;\cong\;} DO_{(\alpha-e_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \;\uplus\; DE_{(\alpha-e_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \quad , \tag{95}$$

and part (i) follows.

As in the proof of Lemma 4.2(iii), we deduce part (ii) from part (i). Likewise, iteration of part (ii) yields part (iii), we just have to use that in inequalities of the form

$$|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;\neq\; |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \tag{96}$$

negative values of α may be replaced by zeros, as

$$DE_\alpha(\vec{G}) \;=\; \varnothing \;=\; DO_\alpha(\vec{G}) \quad \text{for } \alpha \not\geq 0\,. \tag{97}$$

In order to prove part (iv), at first, we remove the set $E(U,W)$

$$E' \;:=\; E(V_C\,,\, V\setminus V_C) \tag{98}$$

of all edges between V_C and $V\setminus V_C$. Let $0 \leq \alpha' \leq \alpha$ be as in part (iii). As V_C is independent, the vertices of V_C are isolated in $\vec{G}\setminus E'$, so that

$$d^+_\varphi(v) \;=\; 0 \tag{99}$$

and hence

$$d^+_\varphi(v) \;\in\; \alpha'_v + \mathbb{N} \qquad \Longleftrightarrow \qquad d^+_\varphi(v) \;=\; \alpha'_v \tag{100}$$

for all orientations $\varphi\colon E\setminus E' \to V$ and all $v \in V_C$. It follows that

$$D_{\alpha'+\mathbb{N}^{V_P}}(\vec{G}\setminus E') \;\;=\;\; D_{\alpha'+\mathbb{N}^{V_P\setminus V_C}}(\vec{G}\setminus E') \;\;, \tag{101}$$

and if we set

$$\alpha'' \;:=\; \alpha'|_{V\setminus V_C} \;\;, \tag{102}$$

this extends to

$$D_{\alpha'+\mathbb{N}^{V_P}}(\vec{G}\setminus E') \;\;=\;\; D_{\alpha'+\mathbb{N}^{V_P\setminus V_C}}(\vec{G}\setminus E') \;\;=\;\; D_{\alpha''+\mathbb{N}^{V_P\setminus V_C}}(\vec{G}\setminus V_C) \;\;, \tag{103}$$

where we have used that

$$E(\vec{G}\setminus E') \;=\; E(\vec{G}\setminus V_C) \;\;. \tag{104}$$

Moreover, these equalities also hold when we replace D with DE or DO, so that the inequality in part (iv) follows from those in part (iii). □

With this we are prepared for the winning strategy and the main proof:

Proof of Theorem 4.1. We present a winning strategy for Mrs. Correct, described in the language of the reformulation 1.6. We suppose that, when the game has reached the i^{th} round, Mrs. Correct has (at least) α^i_v erasers left at each vertex v of $\vec{G}_i$, and that she has managed to ensure

$$|DE_{\alpha^i}(\vec{G}_i)| \;\;\neq\;\; |DO_{\alpha^i}(\vec{G}_i)| \;\;, \tag{105}$$

where $\alpha^i = (\alpha^i_v)_{v\in V(\vec{G}_i)} \in \mathbb{N}^{V(\vec{G}_i)}$. (For $i=1$, $\vec{G}_1 := \vec{G}$ and $\alpha^1 := \alpha$ this holds.)

Now Mr. Paint makes his i^{th} move:

iP: Mr. Paint chooses a nonempty subset $V_{iP} \subseteq V(\vec{G}_i)$, and marks the vertices in V_{iP} with his marker. If already $V(\vec{G}_i) = \varnothing$, then the game ends here, Mr. Paint is defeated and Mrs. Correct wins.

Now, after Mr. Paint's preselection, Mrs. Correct makes her i^{th} move in the following way, which is always possible, so that the game does not stop when it is her turn and she indeed does not lose:

iC: Mrs. Correct knows from the induction hypothesis (105) that

$$D_{\alpha^i}(\vec{G}_i) \;\neq\; \varnothing \quad , \tag{106}$$

and, using double counting, she concludes that

$$\sum_{v\in V(\vec{G}_i)} \alpha^i_v \;=\; |E(\vec{G}_i)| \tag{107}$$

With the same reasoning she then sees, that

$$D_{\alpha^i}(\vec{G}_i) \;=\; D_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i) \tag{108}$$

so that the induction hypothesis (105) can be rewritten as

$$|DE_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i)| \;\neq\; |DO_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i)| \quad . \tag{109}$$

Now, she applies the algorithm used in the proof of Lemma 4.2 (iv) to $\vec{G}_i$, α^i and V_{iP} in place of $\vec{G}$, α and V_P , and obtains an independent set $V_{iC} := V_C$ and a tuple $\alpha'^i := \alpha'$.

Mrs. Correct knows from 4.2 (iv) that V_{iC} is independent, and she cuts it off.

She further knows that for all still marked vertices $v \in V_{iP} \setminus V_{iC}$:

$$\alpha^i_v \;>\; \alpha'^i_v \geq 0 \quad , \tag{110}$$

so that there are enough erasers to clear the remaining markings. Moreover, at least α'^i_v erasers remain at each vertex v of $\vec{G}_i$, and this will be enough to establish the induction hypothesis for

$$\vec{G}_{i+1} := \vec{G}_i \setminus V_C \quad : \tag{111}$$

As Mrs. Correct knows from 4.2 (iv),

$$|DE_{\alpha'^i+\mathbb{N}^{V_{iC}}}(\vec{G}_i)| \;\neq\; |DO_{\alpha'^i+\mathbb{N}^{V_{iC}}}(\vec{G}_i)| \quad . \tag{112}$$

Therefore, she can apply the algorithm behind 4.3 (iv) to $\vec{G}_i$, V_{iC} , again V_{iC} and α'^i in place of $\vec{G}$, V_P , V_C and α . She obtains a tuple $\alpha^{i+1} := \alpha'' \in \mathbb{N}^{V(\vec{G}_{i+1})}$ such that

$$|DE_{\alpha^{i+1}}(\vec{G}_{i+1})| \;\neq\; |DO_{\alpha^{i+1}}(\vec{G}_{i+1})| \quad . \tag{113}$$

This is exactly the induction hypothesis required for the next round, and since

$$\alpha^{i+1}_v \;\leq\; \alpha'^i_v \qquad \text{for all } v \in V(\vec{G}_{i+1})\,, \tag{114}$$

the values α^{i+1}_v in this hypothesis are actually covered by the numbers of erasers in the remaining stacks S_v .

The graph $\vec{G}_{i+1}$ and the reduced stacks S_v of size (at least) α^{i+1}_v will be passed to the next round. After some finite time $t \in \mathbb{N}$, the graph $\vec{G}_t$ will be empty, Mr. Paint cannot move any more, and Mrs. Correct's strategy succeeds. □

5 Applications of Alon and Tarsi's Theorem

There are several "classical" applications of Alon and Tarsi's Theorem. The proofs in these applications lead, without further modifications, to paintability statements, if we use our Theorem 4.1 instead of the original version from Alon and Tarsi.

The first two applications are already obtained in [AlTa], and are based on the following definition:

Definition 5.1. $L(G)$

$$L(G) \;:=\; \max_{H \le G} \frac{|E(H)|}{|V(H)|} \quad .$$

In other words, if G is oriented, so that

$$|E(H)| \;=\; \sum_{v \in V(H)} d_H^+(v) \quad , \tag{115}$$

then $L(G)$ is simply the maximum value of the average outdegree of a subgraph of G. Hence, there is no orientation φ with maximal outdegree $\Delta^+(\varphi)$ strictly smaller than $L(G)$. However, the next natural number $\lceil L(G) \rceil$ is exactly the lowest possible maximal outdegree, as, e.g., shown in [AlTa, Lemma 3.1]: $\Delta^+(\varphi)$ $\lceil \ldots \rceil$

Lemma 5.2. *Each graph G has an orientation $\varphi\colon E \longrightarrow V$ with*

$$\Delta^+(\varphi) \;=\; \lceil L(G) \rceil \quad \textit{for all } v \in V \;.$$

Proof. Subdividing each edge $e \in E$ with a new vertex $\bar{e}$ yields a bipartite graph B with vertex set $V(B) = V \uplus \bar{E}$. Replacing the original vertices $v \in V \subseteq V(B)$ with $L := \lceil L(G) \rceil$ copies $(v,1)$, $(v,2)$, ..., (v,L) of v we obtain a bipartite graph B^L, in which the inserted vertices $\bar{e} \in \bar{E}$ have degree $2L$.

Now, it is sufficient to find a matching of $\bar{E}$ in B^L. Such a matching $\bar{e} \longmapsto (v_{\bar{e}}, i_{\bar{e}})$ induces an orientation $\varphi\colon e \longmapsto \bar{e} \longmapsto (v_{\bar{e}}, i_{\bar{e}}) \longmapsto v_{\bar{e}}$ of G with maximal indegree L, so that the opposite orientation is as required. However, each subset $F \subseteq E$ of edges in G "meets" at least $|F|/L$ vertices of G, and this means that each subset $\bar{F} \subseteq \bar{E}$ has at least $|\bar{F}|/L$ neighbors in B, and at least $|\bar{F}|$ neighbors in B^L, so that Hall's Theorem guarantees the existence of such a matching. □

It follows:

Theorem 5.3. *Every bipartite graph G is $(\lceil L(G) \rceil + 1)$-paintable.*

Proof. Bipartite directed graphs $\vec{G}$ do not contain odd Eulerian subgraphs, so that

$$|DO_{d^+}(\vec{G})| \overset{(75)}{=} |EO(\vec{G})| \;=\; 0 \;<\; |\{\varnothing\}| \;\le\; |EE(\vec{G})| \overset{(75)}{=} |DE_{d^+}(\vec{G})| \quad , \tag{116}$$

and Theorem 4.1 applies. □

In particular we have:

Corollary 5.4. *Every k-regular bipartite graph is $(\lceil \frac{k}{2} \rceil + 1)$-paintable.*

As in [AlTa, Corollary 3.4] we obtain, as second corollary:

Corollary 5.5. *Every bipartite planar graph G is 3-paintable.*

Proof. G is contained in a triangulation with $3|V| - 6$ edges, and we have to remove at least $1/3$ of the edges (at least one edge from each triangular face) to obtain the original bipartite graph G. Hence, G contains at most $2|V| - 4$ edges, and it follows that $L(G) < 2$ (since each subgraph $H \leq G$ is bipartite and planar as well). □

Fleischner and Stiebitz examined in [FlSt] 4-regular Hamiltonian graphs, and solved a coloration problem of Erdős. They made the following observation about Eulerian subgraphs, which is connected to the stated 3-paintability by Theorem 4.1 and (75):

Theorem 5.6. *If a directed graph $\vec{G}$ is the edge-disjoint union of a Hamiltonian circuit and some mutually vertex-disjoint, cyclically oriented triangles, then*

$$|EE(\vec{G})| - |EO(\vec{G})| \equiv 2 \pmod 4 ,$$

and, consequently, $\vec{G}$ is 3-paintable.

Häggkvist and Janssen found in [HäJa, Theorem 3.1] a bound for the list chromatic index of the complete graph K_n, which is sharp for at least all odd n. Using Theorem 4.1 instead of Alon and Tarsi's classical version (which they use at the end of the proof of [HäJa, Preposition 2.4]) we get:

Theorem 5.7. *K_n is edge n-paintable.*

Ellingham and Goddyn's confirmation of the list coloring conjecture for planar r-regular edge r-colorable multigraphs G (see [ElGo] or the end of Section 5 in [Scha2]), also can be generalized. In the original proof they show that the difference

$$|DE_{r-1}(\overrightarrow{LG})| - |DO_{r-1}(\overrightarrow{LG})| , \tag{117}$$

where $\overrightarrow{LG}$ is the arbitrarily oriented line graph of G, equals the number of edge r-colorings of G (up to a constant factor). Thus, the existence of a edge r-coloring implies the assumptions of Theorem 4.1, and hence the r-paintability. For arbitrarily graphs this trick does not work. This is because the corresponding difference of even and odd orientations usually equals just a weighted sum over certain colorings [Scha2, Corollary 5.5(i)], so that the contributions of the different colorings may cancel each other. We have:

Theorem 5.8. *Planar r-regular edge r-colorable multigraphs are edge r-paintable.*

6 A Sharpening of Brooks' Theorem

In this section we prove a sharpening of Brooks' Theorem. We start with the following slightly weaker version of this sharpening, which holds for all connected graphs. It could be proven using the simple Cut Lemma 2.3 (with $|U| = 1$) instead of Theorem 4.1. However, the presented proof shall demonstrate the main idea for proving the sharper version 6.5:

Theorem 6.1. *Each connected graph G is ℓ-paintable for each $\ell = (\ell_v) \gneq d_G$.*

Proof. Let $u \in V$ be such that $\ell_u > d(u)$. Then choose an acyclic orientation of G with u as the only vertex with all its edges directed outwards. Now

$$EO = \varnothing \quad \text{but} \quad EE = \{\varnothing\} \neq \varnothing \tag{118}$$

and Theorem 4.1 (in combination with the equations in (75)) applies. □

In what follows we want to replace the "$\gneq$" in Theorem 6.1 with a "$\geq$". This is not possible for all graphs, but for almost all. The proof of this sharpening works as above, except that we use not only acyclic orientations. We will allow one directed circuit of even length, with at most one chord, i.e., one "shortcut". Such orientations exist in all graphs, except the so called Gallai Trees, which are defined as follows:

Definition 6.2 (BG, ABC)**.** G is called a *Brooks Graph* (BG) if it is an odd circuit or a complete graph. It is called a *Gallai Tree* (GT) if it is connected and if its blocks are BGs. BG GT

A circuit C in a graph G is called a *Brooks Circuit* (BC) in G if the induced graph $\bar{C} := G[C] = G[V(C)]$ is a BG, otherwise it is called an *Anti-Brooks Circuit* (ABC). BC $\bar{C}$ ABC

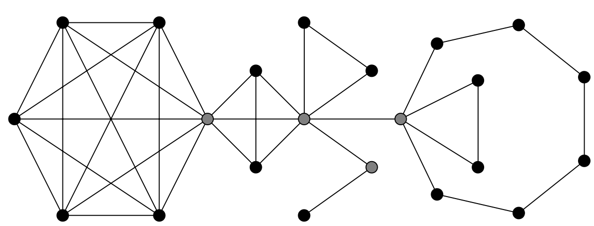


Figure 3: A Gallai Tree with 8 blocks and 4 articulation points.

Lemma 6.3. *The* GT*s are exactly those connected graphs which do not have an* ABC*.*

Proof. We start with GTs. Each circuits C in a GT G lie inside of a block. If this block is a odd circuit, then C coincides with this circuit and is a BC. If the block is a complete graph, then C induces a complete graph and again is a BC. Hence, in any case, C is no ABC.

Conversely, each non-GT contains a block G which is no BG, and it is sufficient to find an ABC in this 2-connected graph G:

Let C be a circuit of maximal length in G. We may suppose that the induced graph $\bar{C}$ is a BG, so that $\bar{C} \neq G$ and there is a neighbor $n \notin C$ of some vertex $s \in C$. There is a shortest path P from n to $C\backslash s$ in $G\backslash s$. Let t be its endpoint in $C\backslash s$. Together with the edge ns and the two halves of C we then have three disjoint paths from s to t, which give rise to three circuits. One of these must have an even length. This even circuit is not C. Otherwise, $\bar{C}$ would be complete, and hence there would be a circuit through sn, P and all vertices of $\bar{C}$, a contradiction to the maximality of C. Thus, the even circuit runs through P and does not induce an edge from n to a neighbor of s in C, as C was chosen maximal. Therefore, it does not induce a complete graph and is an ABC. □

Lemma 6.4. *A minimal* ABC C *in* G *– with respect to inclusion* $\subseteq$ *on the corresponding sets of vertices* $V(C)$ *– has even length and induces at most one additional edge. This induced chord (if present) splits* C *into two odd circuits.*

Proof. If a minimal ABC C has no chord in G, then $\bar{C}$ has to be a circuit of even length (since $\bar{C}$ is no BG), and we are done. Therefore, let

$$v_0 v_s \quad \text{be a chord of} \quad C = v_0 v_1 \cdots v_\ell \quad (v_\ell = v_0) \tag{119}$$

in G, i.e., $1 < s < \ell - 1$. By the minimality of C,

$$A := v_0 v_1 \cdots v_s\, v_0 \quad \text{and} \quad B := v_s v_{s+1} \cdots v_\ell\, v_s \tag{120}$$

are no ABCs and thus $\bar{A}$ and $\bar{B}$ are BGs.

Suppose now, that either $\bar{A}$ or $\bar{B}$ is not an odd circuit. Then $\bar{A}$ or $\bar{B}$ (say $\bar{B}$) is complete with more than three vertices ($\ell > s + 2$). Due to the minimality of C the smaller circuits

$$C_{s+1} := v_0 v_1 \cdots v_s\, v_{s+1}\, v_0 \ , \quad C_{s+2} := v_0 v_1 \cdots v_s\, v_{s+2}\, v_0 \ , \ \ldots, \quad C_{\ell-1} := v_0 v_1 \cdots v_s\, v_{\ell-1}\, v_0 \tag{121}$$

are no ABCs, and hence induce complete graphs (as $v_0 v_s$ is already a chord of them). However, $\bar{C}$ was incomplete, a contradiction.

Thus, we know that each chord splits C into two odd induced circuits, and we only have to prove that there are not two of them. Suppose therefore, that $v_0 v_s$ and $v_a v_b$ are two of them. Then $0 < a < s < b < \ell$, and, as $\bar{C} \neq K_4$, there is a further vertex v_c on the circuit, say $b < c < \ell$. The "eight-graph"

$$\infty := v_0 v_1 \cdots v_a\, v_b v_{b-1} \cdots v_s\, v_0 \tag{122}$$

is then smaller than C and hence no ABC. Additionally, as $\mathbb{F}_2$-sum of odd circuits, it is an even circuit. It follows that our "eight-graph" ∞ induces a complete graph and, in particular, the edge $v_b v_\ell$. However,

$$B \;=\; v_s v_{s+1}\cdots v_b\cdots v_c\cdots v_\ell\, v_s \tag{123}$$

was an induced circuit, a contradiction. □

Theorem 6.5. *Connected non-*GT*s G are d_G-paintable.*

Proof. Due to Lemma 6.3 there is a minimal ABC C in G . By Lemma 6.4, C is an even circuit which we orient cyclicly. The additional edge in $\bar{C}$ (if present) can be oriented arbitrarily. It is easy to extend this orientation of $\bar{C}$ step by step to an orientation of G in such a way that each vertex has indegree at least 1, and the only directed circles are C and possibly (if there is a chord) one half of $\bar{C}$. In any case,

$$EE \;=\; \{C,\varnothing\} \quad \text{but} \quad |EO| \;\le\; 1 \;\;, \tag{124}$$

and the statement follows from Theorem 4.1 (and the equations in (75)). □

The list colorability version of this theorem is well known. One speaks of *degree colorability*, and we could call it *degree paintability*. It was independently proven by Borodin [Bo] and Erdős, Rubin and Taylor [ERT], and can also be found in the comprehensive article [FKS] of Fiala, Král and Škrekovski about T-colorings. They examine list colorings λ which avoid given sets T_e of distances ($|\lambda_u-\lambda_v|\notin T_e$) on the edges $e=uv$ of G , where $0\in T_e$ for all $e\in E$. The list coloring specialization of our result is the case $T_e=\{0\}$ for all $e\in E$. The more general case of arbitrary sets $T_e=\{t_e\}$ of size one, but with a directed kind of distance ($\lambda_{e^\rightarrow}-\lambda_{e^\leftarrow}\neq t_e$), can be covered using our version [Scha2, Theorem 5.4(ii) & Corollary 5.5(i)] of Alon and Tarsi's theorem. In [FKS] list colorings of Gallai Trees are also examined in detail, and it is shown for which types of lists L_v with $|L_v|=d(v)$ no colorings exist.

As a corollary of Theorem 6.5 and the less sharp but more general Theorem 6.1 we obtain the following sharpening of Brook's Theorem [Di, Theorem 5.2.4]:

Corollary 6.6. *Connected non-*BG*s G are $\Delta(G)$-paintable.*

Proof. If G has more than one block, then its blocks build the so-called block tree (see, e.g., [Di, Proposition 3.1.2]), and one can find a vertex v with $d(v)<\Delta(G)$. In this case Theorem 6.1 suffices. If there is just one block then G is no GT and Theorem 6.5 applies. □

7 Paintability of Polynomials and the Combinatorial Nullstellensatz

In this section we view polynomials $P \in \mathcal{R}[X_V] := \mathcal{R}[\,X_v \mid v \in V\,]$ over integral domains R as generalizations of graphs. If R is factorial (i.e. a unique factorization ring) of characteristic different from 2, then the function that maps directed graphs $\vec{G}$ to their graph polynomial $P_{\vec{G}}$ is an embedding. More precisely, for directed graphs $\vec{G}$ on a fixed finite vertex set V, the map $\mathcal{R}[X_V]$ $P_{\vec{G}}$

$$\vec{G} \;\longmapsto\; P_{\vec{G}} \;:=\; \prod_{e \in E} \big(X_{e^{\mapsto}} - X_{e^{\leftarrow}} \big) \tag{125}$$

is injective; if $\mathcal{R}$ (and hence $\mathcal{R}[X_V]$) is factorial, and if $+1 \neq -1$ in $\mathcal{R}$.

It turns out that the generalization of Alon and Tarsi's theorem (the list colorability version of our Theorem 4.1) to polynomials is nothing else than the well-known Combinatorial Nullstellensatz [Al2, Theorem 1.2], [Scha2, Theorem 3.3(ii)] (see [Scha2, Theorem 3.3(i)] for a (weighted) quantitative sharpening). In this section, we want to generalize this Nullstellensatz to paintability. However, this only works for colorings of polynomials, and not for arbitrary nonzeros. We define, with the aim of generalizing the corresponding graph-theoretic terms:

Definition 7.1 (Colors)**.** We call symbolic variables $T, T_1, T_2, \ldots$ *colors* (they are algebraically independent over $\mathcal{R}[X_V]$). Each point $x \in \{T_1, T_2, \ldots\}^V$ with $P(x) \neq 0$ is a *(correct) coloration* of $P \in \mathcal{R}[X_V]$. ℓ-products $L = \prod_{v \in V} L_v$ that are made up of colors, i.e., $L_v \subseteq \{T_1, T_2, \ldots\}$ for all $v \in V$, are called *color ℓ-products.* P is *ℓ-list colorable* if there exists a coloration x of P in each color ℓ-product L . $T_1, T_2, \ldots$

Definition 7.2. Let $\ell \in \mathbb{N}^{\bar{V}}$. A *mounted polynomial* P^ℓ is a polynomial $P \in \mathcal{R}[X_V]$ together with $\ell_v - 1$ erasers at each index $v \in V$. We treat P^ℓ as any usual polynomial; but – when we change the polynomial – we adapt the stacks of erasers in the natural way. For example we define, for sets $U \subseteq V$ of indices and symbolic variables $T \notin \mathcal{R}$, P^ℓ

$$P^\ell \setminus U \;=\; P^\ell \setminus_T U \;:=\; (P \setminus_T U)^{\ell|_{V \setminus U}} \quad , \tag{126}$$

where $P \setminus U$

$$P \setminus U \;=\; P \setminus_T U \;:=\; P\Big|_{\substack{X_v = T \\ v \in U}} \;\in\; \mathcal{R}'[X_{V \setminus U}] \;:=\; (\mathcal{R}[T])[X_{V \setminus U}] \tag{127}$$

is the polynomial over $\mathcal{R}' := \mathcal{R}[T]$ obtained by substituting the color T for the variables X_v with $v \in U$. Which symbolic variable we choose does not play a role, but it has to be chosen outside the current ring of coefficients. In particular, $(P \setminus U_1) \setminus U_2$ has to be red as $(P \setminus_{T_1} U_1) \setminus_{T_2} U_2$ with $T_1 \notin \mathcal{R}$ and $T_2 \notin \mathcal{R}' := \mathcal{R}[T_1]$.

We also introduce a new operation $\downharpoonleft$ (*down*) which acts only on the stacks of erasers: $G^\ell \downharpoonleft U$

$$G^\ell \downharpoonleft U \;:=\; G^{\ell - e_U} \quad . \tag{128}$$

Now, we define in generalization of Definition 1.8:

Definition 7.3 (Paintability)**.** For $\ell \in \mathbb{N}^V$ the *ℓ-paintability* of $P \in \mathcal{R}[\,X_v \mid v \in V\,]$, i.e., the paintability of P^ℓ, can be defined recursively as follows:

(i) If $V = \varnothing$ then P is *ℓ-paintable* if and only if $P \neq 0$ (where ℓ is the empty tuple).

(ii) If $V \neq \varnothing$ then P is *ℓ-paintable* if $\ell \geq \mathbf{1}$ and if each nonempty subset $V_P \subseteq V$ of indices contains a *good* subset $V_C \subseteq V_P$, i.e., a subset $V_C \subseteq V_P$ such that $P^\ell \setminus V_C \downharpoonleft V_P$ is paintable.

Proposition 7.5 below shows that this indeed generalizes the recursive Definition 1.8. The other nonrecursive definition of paintability and the game descriptions of Section 1 can be generalized in a similar way, but a simple example may be more illustrative:

Example 7.4. The polynomial

$$P \;:=\; X_1 - X_2 \;\in\; \mathbb{Z}[X_1, X_2] \tag{129}$$

is not 1-list colorable (and not 1-paintable), since

$$P(T_1, T_1) \;=\; T_1 - T_1 \;=\; 0 \;\;. \tag{130}$$

However, one additional eraser, e.g., at X_2 (at $v = 2$), fixes the problem. If we “clear” X_2 then

$$P(T_1, X_2) \;=\; T_1 - X_2 \;\neq\; 0 \;\;, \tag{131}$$

and this univariate polynomial over the ring $\mathbb{Z}[T_1]$ is 1-paintable.

Proposition 7.5. *Let G be a graph, $\ell \in \mathbb{N}^V$ and $\rightarrow : E \longrightarrow V$ an arbitrarily orientation, then*

$$\boxed{G \text{ is } \ell\text{-paintable.} \iff P_{\vec{G}} \text{ is } \ell\text{-paintable.}}$$

Proof. For $G = (\varnothing, \varnothing)$ the proposition holds, since in this case $P_{\vec{G}} = 1 \neq 0$. Now, let a nonempty set $V_P \subseteq V$ and a subset $V_C \subseteq V_P$ be given, suppose $\ell \geq \mathbf{1}$, and define $\ell' \in \mathbb{N}^{V \setminus V_C}$ by

$$(G \setminus V_C)^{\ell'} \;:=\; G^\ell \setminus V_C \downharpoonleft V_P \;\;. \tag{132}$$

It suffices to prove:

$$P_{\vec{G}} \setminus V_C \text{ is } \ell'\text{-paintable.} \iff G \setminus V_C \text{ is } \ell'\text{-paintable and } V_C \text{ is independent.} \tag{133}$$

This is not so hard to see. The set of edges E of G can be partitioned into the set of edges with both ends outside of V_C, those with one end inside and one end outside, and those with both ends inside of V_C, so that

$$P_{\vec{G}} \setminus_T V_C \;=\; P_{\vec{G}\setminus V_C} \prod_{\substack{(u,v)\in V_C\times V\setminus V_C \\ uv\in E}} \pm(T - X_v) \prod_{e\in E(G[V_C])} (T - T) \;. \tag{134}$$

When is the right side of this equation ℓ'-paintable? The last factor on the right side vanishes, and avoids the paintability, if and only if $E(G[V_C]) \neq \varnothing$, i.e., if and only if V_C is not independent. In the "independent case" it is equal to 1, and does not affect the paintability. The middle factor is different from zero, no mater what symbolic variables $T_i \notin \mathcal{R}[T]$ we substitute for the remaining variables X_v, and does not play a role. Finally, using an induction argument, we may assume that:

$$P_{\vec{G}\setminus V_C} \text{ is } \ell'\text{-paintable.} \iff G \setminus V_C \text{ is } \ell'\text{-paintable.} \tag{135}$$

Altogether, these observations prove the required Equivalence (133). □

As in Proposition 1.4 (with the same explanation) we have:

Proposition 7.6. *Let* $P \in \mathcal{R}[X_V]$ *be given and* $\ell \in \mathbb{N}^V$, *then*

$$\boxed{P \text{ is } \ell\text{-paintable.} \implies P \text{ is } \ell\text{-list colorable.}}$$

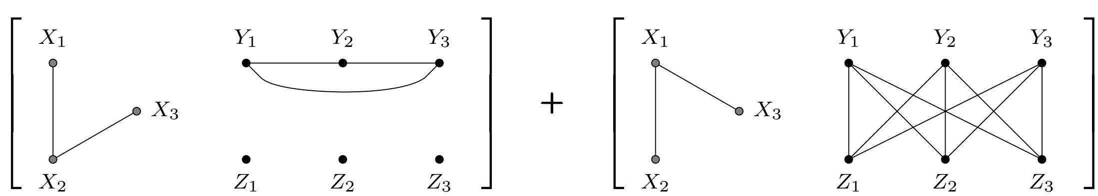
When we are interested in graph colorings, then we only have to look at homogeneous polynomial, since graph polynomials are homogeneous. However, in view of degree considerations, we always can restrict ourselves to the homogeneous case:

Proposition 7.7. *As the* T_i *and* X_v *are algebraically independent over* $\mathcal{R}$, *we may focus on one homogeneous component* H *of* $P \in \mathcal{R}[X]$. *If* $x \in \{T_j, X_v \mid j \in \mathbb{N}, v \in V\}^n$, *then*

$$\boxed{H(x) \neq 0 \implies P(x) \neq 0} \;.$$

This makes the verification of the following example easier than that of the purely graph theoretic Example 1.5:

Example 7.8. The following polynomial – a sum of two graph polynomials – is 2-list colorable but not 2-paintable (the signs of the graph polynomials do not matter):



Proof. The two graph polynomials P_{left} and P_{right} in the posed sum $P = P_{\text{left}} + P_{\text{right}}$ have different degrees (5 and 11). Therefor, a partial coloring of P is, in view of Proposition 7.7, correct if and only if it is correct for P_{left} or for P_{right}. It is, in view of Proposition 7.5, correct for P_{left} respectively P_{right} if and only if it is correct for the underlying graph of P_{left} respectively P_{right}. Finally, it is correct for a graph if and only if its restrictions to the components of the graph are correct.

Now, we are able to prove the unpaintability of P: In order to prevail, Mr. Paint colors in his first two rounds the variables/vertices X_1, X_2 and X_3, i.e.,

$$V_{1P} := \{X_1, X_2, X_3\} \quad \text{and} \quad V_{2P} := \{X_1, X_2, X_3\} \setminus V_{1C} \tag{136}$$

(where we identified variables, vertices and indices). Note that Mrs. Correct has to "cut off" all three "vertices" during these first two rounds, because there is only one eraser at each vertex. Afterwards, at most one of the two paths is correctly colored – either the path in the left graph or the path in the right graph. Both paths cannot have been correctly colored, since this would mean that their union, a triangle, would inherit a correct 2-coloring. However, the remaining graphs, a K_3 and a $K_{3,3}$, are not 2-list colorable and hence not 2-paintable. Therefore, Mr. Paint has one strategy to prevent the correct coloring of the K_3, and one strategy to prevent the correct coloring of the $K_{3,3}$. He uses both of them as follows: If the path in the left graph is properly colored, he avoids the correct coloring of the remaining left graph, i.e., the triangle K_3. If the right path is properly colored, he avoids the correct coloring of the remaining right graph, i.e., the $K_{3,3}$. At the end neither the entire left nor the entire right graph are properly colored, and Mr. Paint wins.

We come now to the 2-list colorability, and have to examine all possible 2-products L: If

$$L_{Y_1} = L_{Y_2} = L_{Y_3} \tag{137}$$

then the $K_{3,3}$ in the right graph is colorable from the corresponding lists (just choose the same color for Y_1, Y_2 and Y_3, and extend this partial coloring). Since the right path is 2-list colorable these partial coloring extends to the whole right graph. In view of Proposition 7.7, the obtained coloring is also a coloring of P.

In the other case the K_3 in the left graph is properly colorable. Since the left path is 2-list colorable this partial coloring extends to the whole left graph. In view of Proposition 7.7, the obtained coloring again is a coloring of P. □

If we only consider colorations and not arbitrary nonzeros the important Combinatorial Nullstellensatz (see [Al2, Theorem 1.2] & [Scha2, Theorem 3.3(ii)]) can easily be formulated without degree restrictions (such as those in (146)). This immediately follows from the observation in Proposition 7.7. However, this formulation for colorations can also be proven with the ideas behind Theorem 4.1, and leads to the following sharper paintability version:

Theorem 7.9. *Let* $P = \sum_{\delta \in \mathbb{N}^V} P_\delta X^\delta \in \mathcal{R}[X_V]$ *and* $\alpha \in \mathbb{N}^V$, *then*

$$\boxed{P_\alpha \neq 0 \implies P \text{ is } (\alpha+1)\text{-paintable.}}$$

In order to prove Theorem 7.9, we will need the following generalization of Lemma 4.2:

Lemma 7.10. *Let* $P = \sum_{\delta \in \mathbb{N}^V} P_\delta X^\delta \in \mathcal{R}[X_V]$ *be a polynomial,* $\alpha \in \mathbb{N}^V$, $V_P \subseteq V$ *nonempty and* $u \in V_P$, *then:*

(i) $(\alpha - e_u) + \mathbb{N}^{V_P} = \alpha + \mathbb{N}^{V_P} \uplus (\alpha - e_u) + \mathbb{N}^{V_P \setminus u}$.

(ii) $\displaystyle\sum_{\delta \in (\alpha - e_u) + \mathbb{N}^{V_P}} P_\delta = \sum_{\delta \in \alpha + \mathbb{N}^{V_P}} P_\delta + \sum_{\delta \in (\alpha - e_u) + \mathbb{N}^{V_P \setminus u}} P_\delta$.

(iii) $\displaystyle\sum_{\delta \in \alpha + \mathbb{N}^{V_P}} P_\delta \neq 0 \implies \sum_{\delta \in (\alpha - e_u) + \mathbb{N}^{V_P}} P_\delta \neq 0 \ \vee \sum_{\delta \in (\alpha - e_u) + \mathbb{N}^{V_P \setminus u}} P_\delta \neq 0$.

(iv) $\displaystyle\sum_{\delta \in \alpha + \mathbb{N}^{V_P}} P_\delta \neq 0 \implies \begin{cases} \text{There is an } \alpha' \leq \alpha \text{ and a } V_C \subseteq V_P \text{ such that: } \alpha'|_{V_C} \equiv 0, \\ \alpha'_v < \alpha_v \text{ for all } v \in V_P \setminus V_C, \text{ and } \displaystyle\sum_{\delta \in \alpha' + \mathbb{N}^{V_C}} P_\delta \neq 0 . \end{cases}$

Proof. The proof works exactly as in Lemma 4.2. □

With this, the proof of Theorem 7.9 works almost like the one of Theorem 4.1:

Proof of Theorem 7.9. Let a nonempty subset $V_P \subseteq V$ be given. In view of Proposition 7.7 we may assume that P is homogeneous of degree

$$\deg(P) = \deg(X^\alpha) , \tag{138}$$

so that

$$\sum_{\delta \in \alpha + \mathbb{N}^{V_P}} P_\delta = P_\alpha \neq 0 , \tag{139}$$

and we can apply Lemma 7.10 (iv). This yields a potentially good subset $V_C \subseteq V_P$ and a tuple $\alpha' \leq \alpha$. We substitute T for all variables X_v with $v \in V_C$ in P, and obtain the polynomial

$$P \setminus V_C \in \mathcal{R}'[X_{V \setminus V_C}] \quad \text{with} \quad \mathcal{R}' := \mathcal{R}[T] . \tag{140}$$

We know that

$$(P \setminus V_C)_{\alpha''} \neq 0 \quad \text{for} \quad \alpha'' := \alpha'|_{V \setminus V_C} , \tag{141}$$

since even

$$(P \setminus V_C)_{\alpha''}|_{T=1} \overset{7.2}{=} \Big(P\Big|_{\substack{X_v = 1 \\ v \in V_C}}\Big)_{\alpha''} = \sum_{\delta \in \alpha' + \mathbb{N}^{V_C}} P_\delta \overset{7.10}{\neq} 0 , \tag{142}$$

as

$$\alpha'|_{V_C} \overset{7.10}{\equiv} 0 \quad . \tag{143}$$

Using an induction argument, it follows that $P \setminus V_C$ is $(\alpha''+1)$-paintable. Hence,

$$P^{(\alpha'+1)} \setminus V_C \;=\; (P \setminus V_C)^{(\alpha''+1)} \tag{144}$$

and even more

$$P^{(\alpha+1)} \setminus V_C \downharpoonleft V_P \tag{145}$$

is paintable; which means, in view of Definition 7.3, that P is $(\alpha+1)$-paintable. □

The algorithm behind this last proof has polynomial running time. When applied to the graph polynomial $P_{\vec{G}}$ (defined in (125)) of an arbitrarily oriented graph G, it produces graph colorings. However, computing the graph polynomial ($\vec{G} \mapsto P_{\vec{G}}$) generally requires exponential time.

Note further that it was necessary to use symbolic variables in Theorem 7.9, a similar version,where we allow Mr. Paint to use elements of the ground ring $\mathcal{R}$, does not hold, not even under the usual degree restriction (146). The polynomial $P := X_1 + X_2 - 2 \in \mathbb{Z}[X_1, X_2]$ with one eraser at X_1 ($\alpha := (1,0)$) is a counterexample: Mr. Paint may use the pseudo color $0 \in \mathbb{Z}$ for X_1 in his first move ($1P:\ X_1 := 0$). If then Mrs. Correct does not use the eraser, then Mr. Paint uses $2 \in \mathbb{Z}$ as color for X_2 ($2P:\ X_2 := 2$), and wins. If Mrs. Correct uses the eraser, then Mr. Paint uses $1 \in \mathbb{Z}$ as color for X_2 and for the "emptied" X_1 ($2P:\ X_1 = X_2 := 1$), and wins.

We also want to mention that in [Scha2, Corollary 3.4] we proved a very useful corollary to the Combinatorial Nullstellensatz:

If

$$\deg(P) \;\leq\; \sum_{v\in V} \alpha_v \;=:\; \deg(X^\alpha) \tag{146}$$

and

$$P_\alpha \;=\; 0 \quad , \tag{147}$$

e.g., if

$$\deg(P) \;<\; \sum_{v\in V} \alpha_v \quad , \tag{148}$$

then the polynomial map $x \mapsto P(x)$ does not have exactly one nonzero over each fixed given $(\alpha+1)$-product $L \subseteq \mathcal{R}^n$:

$$\bigl|\{\, x \in L \;\vdots\; P(x) \neq 0 \,\}\bigr| \;\neq\; 1 \tag{149}$$

This can frequently be used to prove that certain problems do not have exactly one solution, which is particularly interesting if such problems do have exactly one trivial/known solution. In this case the trivial/known solution obviously cannot be the only one, and we have very elegantly been convinced that there must be nontrivial/unknown ones.

One could expect that Theorem 7.9 leads to a version of this “Not-Exactly-One Theorem” (149) without degree restriction, over color $(\alpha+1)$-products. However, this is not the case, as the trivial example $P(X_1) := X_1$, $L_1 := \{T_1\}$ (i.e., $V=\{1\}$, $\alpha=(0)$ and $P_\alpha = 0$) shows.

8 Hypergraphs

In this last section we want to demonstrate how our version of the Combinatorial Nullstellensatz 7.9 can be applied to hypergraphs $G=(V,E)$. It is obvious how our game have to be generalized to hypergraphs. Again, it is Mrs. Correct's job to avoid incorrect colorings, which are colorings with monochromatic edges, i.e., edge e with all its vertices $v\in e$ equally colored. Paintability and list colorability are then defined as in the definitions 1.2, 1.3 and 1.8, where a set of vertices is independent in G if it does not contain all vertices of an edge. Of course, we can restrict ourselves, without loose of generality, to symbolic variables $T_1, T_2, \ldots$ as colors, as in Definition 7.1.

Now, let $A=(a_{ev})\in R^{E\times V}$ be a matrix with

$$a_{ev}\neq 0 \iff v\in e \tag{150}$$

and with vanishing rowsums, i.e., with

$$\sum_{v\in e} a_{ev} \;=\; 0 \quad \text{for all } e\in E\,. \tag{151}$$

Let further P_A

$$P_A \;:=\; \prod_{e\in E}\sum_{v\in V} a_{ev}X_v \;\in\; \mathcal{R}[X_V] \tag{152}$$

be the *matrix polynomial* of A . Then it is easy to see, that the colorings of P_A (see Definition 7.1) are exactly the colorings of G (with symbolic variables). Therefore, we obtain the following specialization of Theorem 7.9, which may be seen as a generalization of Theorem 4.1. (See [RaWe] for a list coloring version, which uses generalized orientations and generalized Eulerian subgraphs.):

Theorem 8.1. *Let G and A be as above. For $\alpha\in\mathbb{N}^V$ holds*

$$\boxed{\operatorname{per}_\alpha(A)\neq 0 \implies G \text{ is } (\alpha+1)\text{-paintable.}}$$

where $\operatorname{per}_\alpha(A)$

$$\operatorname{per}_\alpha(A) \;:=\; \sum_{\substack{\sigma:\,E\to V\\ |\sigma^{-1}(v)|=\alpha_v}}\;\prod_{e\in E} a_{e,\sigma(e)} \quad .$$

Proof. It is easy to see that $(P_A)_\alpha = \mathrm{per}_\alpha(A)$, so that Theorem 7.9 applies. □

Note, that, by definition,

$$\mathrm{per}_\alpha(A) \;=\; 0 \quad \text{if} \quad \sum_{v\in V} \alpha_v \;\neq\; |E| \;\;. \tag{153}$$

If $\vec{G}$ is a directed graph and A its incidence matrix, then the graph polynomial equals the matrix polynomial,

$$P_{\vec{G}} \;=\; P_A \;\;, \tag{154}$$

and

$$(P_{\vec{G}})_\alpha \;=\; \mathrm{per}(A) \;=\; |DE_\alpha(\vec{G})| - |DO_\alpha(\vec{G})| \;\;. \tag{155}$$

This is easy to see, and shows that Theorem 7.9 can be seen as a generalization of Theorem 8.1, which on its own generalizes Theorem 4.1.

Note also that, if $\sum_{v\in V} \alpha_v = |E|$,

$$\Big(\prod_{v\in V} \alpha_v!\Big)\; \mathrm{per}_\alpha(A) \;=\; \mathrm{per}(A\langle|\alpha\rangle) \;\;, \tag{156}$$

where $\mathrm{per} := \mathrm{per}_{\mathbf{1}}$ is the usual permanent (see [Minc]), and where $A\langle|\alpha\rangle$ is a matrix that contains the v^{th} column of A exactly α_v times (see also [Scha2, Definition 5.2]). per $A\langle|\alpha\rangle$

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