1 Abstract

It is known that asymptotic behavior of a branching stochastic process with immigration is very sensitive to any changes of the immigration process in time. Therefore, from theoretical point of view, providing of functional limit theorems for step functions constructed on sample paths of the process is difficult but important task. On the other hand this kind limit theorems are extremely useful in study of various functionals of the process and in statistical inference such as estimation of parameters of the process, testing some hypothesis on those parameters and other problems. In the project we considered a critical discrete time branching process with generation dependent immigration. In the case when the mean number of immigrating individuals tends to infinity with the generation number, functional limit theorems for centered and normalized process have been proved. It turned out that the limiting processes are deterministically time-changed Wiener with three different covariance functions depending on the behavior of the mean and variance of the number of immigrants.

Applications of these theorems in the statistical problems are also investigated. It is known that in the critical case the conditional least squares estimator (CLSE) of the offspring mean of a discrete time branching process with immigration is not asymptotically normal. If the offspring variance tends to zero, it is normal with normalization factor $n^{2/3}$. The functional theorems obtained in the project allowed to study a situation of its asymptotic normality in the case of non-degenerate offspring distribution for the process with time-dependent immigration, whose mean and variance vary regularly with nonnegative exponents α and β, respectively. It is proved that if $\beta < 1 + 2\alpha$, the CLSE is asymptotically normal with two different normalization factors and if $\beta > 1 + 2\alpha$, its limit distribution is not normal but can be expressed in terms of the distribution of certain functionals of the time changed Wiener process. When $\beta = 1 + 2\alpha$ the limit distribution depends on the behavior of the slowly varying parts of the mean and variance.

Key Words: branching process, immigration, functional, martingale limit theorem, Skorokhod space, least squares estimator.

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2 Introduction

We consider a discrete time branching stochastic process $Z(n), n \geq 0, Z(0) = 0$. It can be defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, n, i \geq 1\}$ and $\{\xi_k, k \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1}$$

Assume that X_{ni} have a common distribution for all n and i, families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. Variables X_{ki} will be interpreted as the number of offspring of the ith individual in the $(k-1)$th generation and ξ_k is the number of immigrating individuals to the kth generation. Then $Z(n)$ can be considered as the size of nth generation of the population.

In this interpretation $A = EX_{ni}$ is the mean number of offspring of a single individual. Process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $A < 1, A = 1$ or $A > 1$ respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, in contrast of classical models, we do not assume that $\xi_n, n \geq 1$ are identically distributed, i. e. immigration rate may depend on the time of immigration.

Investigations show that asymptotic behavior of the process with immigration is very sensitive to any changes of the immigration process in time. For instance, in critical case change of the mean number of immigrating individuals in time leads to such fluctuations of the process, that one needs to use various functional normalization of the process to obtain non degenerate limit distribution for the process (see [17], Ch. III and references therein). Therefore description of processes which can be used as approximating in this situation is of interest. On the other hand this kind of functional limit theorems are useful in estimating parameters and in study of various functionals of the process.

In the project we proved functional limit theorems for critical processes in the case when the immigration mean tends to infinity. It turned out that suitably normalized process may be approximated by a Gaussian process with independent increments and with three different covariance functions depending on behavior of the mean and variance of the number of immigrants. The limiting Gaussian process can be obtained from the Wiener process by a deterministic time-change.

First approximation theorems of branching stochastic processes have appeared due to W. Feller [6], who demonstrated that branching stochastic process without immigration can be approximated by a diffusion process. Lamperti [12], [13] proved convergence of finite dimensional distributions of the process with large number of initial individuals to those of some diffusion processes with two different normalization. These results were extended to the functional form by T. Lindvall [14], [15]. Convergence of finite dimensional distributions of a sequence of Galton-Watson branching processes with stationary immigration has been investigated by Kawazu and Watanabe [11] and Aliev [1]. Wey and Winnicki [19] have shown that random step functions of a critical branching process with immigration converges in Skorohod metric to a nonnegative diffusion process. Fluctuation theorems for the sequence of nearly critical branching processes have been proved by Sriram [18], who obtained a diffusion approximation. In papers by Ispàny, Pap and Van Zuijlen [7], [8] the authors demonstrated that Sriram's result is also valid when offspring variance tends to zero and centralized process can be approximated by Ornstein-Uhlenbeck type processes. Note that in the latter case reproduction process will approach to deterministic multiplication of individuals. The paper [9] of the same authors is also devoted to the critical branching process with varying offspring and immigration distributions. However, in contrast to our situation, in that paper the offspring variance tends to zero.

It is known that (see Alzaid, Al-Osh [2], Dion, at all [5] and Franke, Seligmann [4]) in the case of Bernoulli offspring distribution process defined in equation (1) can be considered as an integer-valued, first order autoregressive (INAR(1)) time series model with noise ξ_k. In this framework considered here process $Z(n)$ can be related to INAR(1) model with non stationary (rising) noise.

As an example of application of obtained functional limit theorems we proved that conditional least squares estimator (CLSE) of offspring mean A is asymptotically normal. Estimation of the offspring and immigration parameters in the branching process with a stationary immigration has been an active area of the research for a long time. As a result of this activity, it has been established that a maximum likelihood approach leads to useful results, if the number of immigrating individuals ξ_n and all offspring sizes X_{ni} are observable. The first estimation results based on observation of the population sizes are due to Heyde and Seneta [23-26]. In the supercritical case it was shown that the Lotka-Nagaev and Harris type ratio estimators can be used to estimate the offspring mean [23, 24]. In subcritical case, using an analogy between immigration-branching process and the first order autoregressive process, the same authors derived asymptotically normal estimators for offspring and immigration means [25, 26]. However, it was shown later that in the critical or nearly critical case the conditional least squares estimator (CLSE) is not asymptotically normal (see Sriram [18], Wei and Winnicki [20, 21]). Results of [7] and [8] have shown that when the process is nearly critical and the offspring variance tends to zero, it has a normal limit distribution with normalization factor $n^{3/2}$. Assuming that the offspring variance tends to zero means that in the long run the reproduction process approaches a deterministic multiplication process. The results of [21] have been extended to the controlled branching process with a random control function (see [19]). For estimation problems in non-classical models of branching processes we also refer to [28, 29] and references therein.

In the project we describe the situations of asymptotic normality of the CLSE of the offspring mean in the case when the offspring variance does not tend to zero under the assumption $A = 1$. More precisely, we prove that if the immigration mean tends to infinity depending on the time of immigration, it is possible to estimate the offspring mean by an asymptotically normal CLSE. Assuming that the immigration mean and variance vary regularly with nonnegative exponents α and β respectively, we establish that if $\beta < 1 + 2\alpha$, the CLSE is asymptotically normal. If $\beta > 1 + 2\alpha$, its limit distribution is not normal but can be expressed in terms of the distribution of certain functionals of the time changed Wiener process. When $\beta = 1 + 2\alpha$, the limit distribution depends on the behavior of the slowly varying parts of $\alpha(n)$ and $\beta(n)$. The normalization factor depends on the mean number of immigrants and on relationship between mean and variance of the immigration. We also demonstrate that in important particular cases of the Poisson and geometric immigration distribution the CLSE is asymptotically normal.

A natural approach to the problem of obtaining asymptotic distributions for estimators of parameters in a branching process, when the estimators are given explicitly, is analyzing the explicit expression using results from the limit theory of the branching processes. Depending on the explicit expression of the estimator, standard or functional limit theorems for the branching process may be used. For example, proofs of results in [18] and [20,21] are based on diffusion approximation of the normalized process. In [7] and [8] the authors first prove that the process can be approximated by an Ornstein-Uhlenbeck type process, and using this approximation, obtain asymptotic normality of the estimator.

Since the CLSE has a form of a ratio of certain functions of the process, to derive its asymptotic distribution, one needs a central limit theorem (CLT) for the process in the functional form. Therefore, application of the general limit theorems on convergence to a mixture of infinitely divisible distributions ([17], Ch 2) does not allow to obtain the limit distributions in this model. In proofs of our results we use the the functional limit theorems from first part of this project. We note that the approximation theorems in the project are obtained using a functional CLT for martingales. Thus, in our proofs the martingale CLT have been used through functional limit theorems for the branching process. This scheme allows to determine the threshold for the asymptotic normality of the CLSE and to analyze the situations, where the limit distribution is not normal. We also note that in the time homogenous models a direct use of martingale CLT is sometimes possible (see [26]).

As it was mentioned before, our results are obtained under the assumption $A = 1$. However, the scheme of proofs can be used in subcritical and supercritical cases and in an array of branching processes (nearly critical case). Of course, first one needs to establish functional limit theorems for the nonclassical processes and then apply them in obtaining of asymptotic properties of an estimator of the offspring mean. Further, as in the classical case, one may establish results for the estimator without any assumption of the criticality of the reproduction process. One more possible application of the scheme of our proofs is deriving asymptotic distributions for a weighted CLSE, which minimizes a standardized sum of squared errors. More detailed discussion on this matter we provide in concluding remarks.

3 Results and discussion

3.1 The functional limit theorems

From now on we assume that $A = EX_{ni}$ and $B = varX_{ni}$ are finite. We also assume that $\alpha(n) = E\xi_n < \infty$, $\beta(n) = var\xi_n < \infty$ for each $n \geq 1$ and regularly varying when $n \to \infty$ functions, i. e. have the following form

$$\alpha(n) = n^\alpha L_\alpha(n), \quad \beta(n) = n^\beta L_\beta(n), \tag{2}$$

where $\alpha, \beta \geq 0, L_\alpha(n)$ and $L_\beta(n)$ are slowly varying as $n \to \infty$ functions. Then $A(n) = EZ(n)$ and $B^2(n) = varZ(n)$ are finite for each $n \geq 1$ and, when $A = 1$,

$$A(n) = \sum_{k=1}^{n} \alpha(k), \quad B^2(n) = \Delta^2(n) + \sigma^2(n), \tag{3}$$

where

$$\Delta^2(n) = B \sum_{k=1}^{n} \alpha(k)(n-k), \quad \sigma^2(n) = \sum_{k=1}^{n} \beta(k).$$

For each $t \in \mathbb{R}_+ = [0, \infty)$ we define sequence of step functions

$$Y_n(t) = \frac{Z([nt]) - A([nt])}{B(n)}.$$

Everywhere from now D, d and P denote convergence of random functions in Skorohkod topology and convergence of random variables in distribution and in probability respectively.

Theorem 1. *If $A = 1$, $B \in (0, \infty)$, $\alpha(n) \to \infty$ and $\beta(n) = o(n\alpha(n))$, then $Y_n(t) \xrightarrow{D} W(t^{2+\alpha})$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, where $(W(t), t \in \mathbb{R}_+)$ is standard Brownian motion.*

Note that in Theorem 1 we do not require a Lindeberg type condition on offspring or immigration distribution. In fact, in this case it is satisfied for the immigration process due to normalization by $B^2(n)$. However in opposite case we need one on the sequence $\{\xi_n, n \geq 1\}$ which seems natural for the process with inhomogeneous immigration. Thus we denote for each $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{\sigma^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2; |\xi_k - \alpha(k)| > \varepsilon\sigma(n)]. \tag{4}$$

Theorem 2. *If* $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \alpha(n) = o(n^{-1}\beta(n))$ *and* $\delta_n(\varepsilon) \to 0$ *as* $n \to \infty$ *for each* $\varepsilon > 0$ *, then* $Y_n(t) \xrightarrow{D} W(t^{1+\beta})$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$.

Next theorem is related to the case when $n\alpha(n)$ and $\beta(n)$ have the same rate.

Theorem 3. *If* $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \beta(n) \sim cn\alpha(n),\ c \in (0,\infty)$ *and* $\delta_n(\varepsilon) \to 0$ *as* $n \to \infty$ *for each* $\varepsilon > 0$ *, then* $Y_n(t) \xrightarrow{D} W(t^{1+\beta}) = W(t^{2+\alpha})$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$.

Remarks. 1. Using Lemma 1 in Section 3, one can see that condition $\beta(n) = o(n\alpha(n))$ is equivalent to $\sigma^2(n) = o(\Delta^2(n))$, condition $\alpha(n) = o(n^{-1}\beta(n))$ is equivalent to $\Delta^2(n) = o(\sigma^2(n))$ and $\beta(n) \sim cn\alpha(n)$ as $n \to \infty$, if and only if $\sigma^2(n) \sim \theta B^2(n)$ with $\theta = d/(d + (1+\beta)B)$, where $d = c(1+\alpha)(2+\alpha)$.

2. Since $\beta(n)/n\alpha(n)$ is regularly varying with exponent $\beta - 1 - \alpha$ and, when $\beta(n) \sim cn\alpha(n)$, as $n \to \infty$ has a positive finite limit, we conclude that $\beta = \alpha + 1$. This explains equality $W(t^{1+\beta}) = W(t^{2+\alpha})$ in Theorem 3.

Now we consider examples of the immigration process which satisfy conditions of provided theorems.

Example 1. Let $\xi_k, k \ge 1$ be Poisson with mean $\lambda(k) \to \infty, k \to \infty$ and regularly varies with exponent α. Then $\Delta^2(n) = B\sum_{k=1}^{n} \lambda(k)(n-k), \sigma^2(n) = \sum_{k=1}^{n} \lambda(k)$ and clearly $\sigma^2(n) = o(\Delta^2(n))$. In this case we obtain the following result from Theorem 1.

Corollary 1. If $A = 1, B \in (0,\infty)$ and $\xi_k, k \ge 1$ are Poisson with mean $\lambda(k) \to \infty, k \to \infty$ and $(\lambda(k))_{k=1}^{\infty}$ is regularly varying with exponent α, then $Y_n(t) \xrightarrow{D} W(t^{2+\alpha})$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$.

Example 2. Let now $\xi_k, k \ge 1$ have positive geometric distributions with parameter $p_k = k^{-1}$ i. e. $P\{\xi_k = i\} = q_k^{i-1}p_k, i = 1, 2, ..., q_k = 1 - p_k$. In this case $\alpha(k) = k, \beta(k) = q_k p_k^{-2} = k^2(1 - k^{-1})$. Consequently we have $\Delta^2(n) \sim Bn^3/6$ and $\sigma^2(n) \sim n^3/3$. Therefore $\sigma^2(n) \sim 2B^2(n)/(B+2)$. Now we show fulfilment of the Lindeberg condition. Since $ES^{\xi_k} = (p_k S)(1 - q_k S)^{-1}$, we find that $(ES^{\xi_k})''' = 6p_k q_k^2 (1 - q_k S)^{-4}$. Therefore $E\xi_k(\xi_k - 1)(\xi_k - 2) = 6q_k^2 p_k^{-3}$.

From this we conclude that $E|\xi_k - \alpha(k)|^3 = O(k^3), k \to \infty$ which leads to relation

$$C_n^3 =: \sum_{k=1}^{n} E|\xi_k - \alpha(k)|^3 = O(n^4), n \to \infty.$$

Thus $C_n^3/\sigma^3(n) = O(n^{-1/2}), n \to \infty$, i. e. Lyapunov condition is satisfied for $\xi_k, k \geq 1$. Now we obtain the following result from Theorem 3.

Corollary 2. If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are geometric with parameter $p_k = k^{-1}$, then $Y_n(t) \xrightarrow{D} W(t^3)$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$.

Example 3. Let $\xi_k, k \geq 1$ be such that $p = P\{\xi_k = k^2\} = 1 - P\{\xi_k = 0\}, q = 1 - p$. Then simple calculations give that $\Delta^2(n) \sim Bpn^4/12, \sigma^2(n) \sim pqn^5/5$ as $n \to \infty$ and consequently $\Delta^2(n) = o(\sigma^2(n)$. Since in this case $C_n^3 \sim pq(p^2 + q^2)n^7/7$ and $\sigma^3(n) \sim (pq/5)^{3/2} n^{15/2}$ the Lyapunov condition is again satisfied. Thus we obtain from Theorem 2 that $Y_n(t)$ converges as $n \to \infty$ to $W(t^5)$ weakly in Skorokhod space $D(R+, R)$.

3.2 The statistical applications

Now we consider applications of our theorems related to conditional least-squares estimator of offspring mean. Let $\Im(n)$ for each $n \geq 0$ be the σ-algebra containing all the history of the process up to nth generation, i.e. it is generated by $\{Z(k), k = 0, 1, ..., n\}$. We obtain from (1) that

$$E[Z(n)|\Im(n-1)] = AZ(n-1) + \alpha(n), \quad n \geq 1. \tag{5}$$

If we assume that the immigration mean $\alpha(n)$ is known, then non weighted CLSE $\hat{A}_n$ of A minimizes sum of squared errors

$$\sum_{k=1}^{n} (Z(k) - AZ(k-1) - \alpha(k))^2.$$

By usual arguments we obtain

$$\hat{A}_n = \frac{\sum_{k=1}^{n} (Z(k) - \alpha(k))Z(k-1)}{\sum_{k=1}^{n} Z^2(k-1)}. \tag{6}$$

We also assume that there exists $C \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha(n)} = C. \tag{7}$$

We note that if $\beta < \alpha + 1$, then $C = 0$ and if $\beta > \alpha + 1$, then $C = \infty$. When $\beta = \alpha + 1$ the value of C depends on the relative rate of variation of the slowly varying parts of $\alpha(n)$ and $\beta(n)$.

The proofs of the limit theorems for $\hat{A}_n$ are based on the following scheme. First we write the centered $\hat{A}_n$ as a ratio of some functions of the process $Z(n)$. Then we express numerator and denominator of the ratio in terms of certain functionals of the process $Y_n(t)$ and some additional terms. Next we study the asymptotic behavior of each term using the functional CLT for $Y_n(t)$ and properties of the regularly varying functions. If we apply the continuous mapping and Slutsky's theorems [23], we obtain one or another limit distribution depending on which term predominates in the ratio.

Now we provide the first result related to the case of asymptotical normality of the CLSE.

Theorem 4. *If $A = 1,\ B \in (0, \infty),\ C \in [0, \infty),\ \alpha(n) \to \infty$ and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then*

$$n\sqrt{\alpha(n)}(\hat{A}_n - A) \xrightarrow{d} N(0, a^2),$$

where $N(0, a^2)$ is normal random variable with mean 0 and variance

$$a^2 = \frac{(1+\alpha)^2(2\alpha+3)^2}{3\alpha+4}\left(\frac{B}{1+\alpha} + C\right).$$

In the case $C = 0$ we have $\sigma^2(n) = o(B^2(n))$ as $n \to \infty$, and condition $\delta_n(\varepsilon) \to 0, n \to \infty$, is automatically satisfied. In the case $C > 0$ the condition is equivalent to the Lindeberg condition for the family $\{\xi_n, n \geq 1\}$ of the number of immigrating individuals.

Example 4. Let $\xi_k, k \geq 1$ be Poisson with mean $\alpha(k) \to \infty, k \to \infty$, which is a regularly varying function with exponent α. Then $\beta(n) = o(n\alpha(n)), n \to \infty$, therefore, $C = 0$ in Theorem 1 and the Lindeberg type condition is satisfied. In this case we obtain the following result.

Corollary 3. If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are Poisson with mean $\alpha(k) \to \infty, k \to \infty$, and $(\alpha(k))_{k=1}^{\infty}$ is regularly varying with exponent α, then $n\sqrt{\alpha(n)}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance

$$a^2 = \frac{(1+\alpha)(2\alpha+3)^2 B}{3\alpha+4} .$$

Example 5. Let now the random variables $\xi_k, k \geq 1$, have positive geometric distributions with parameters $p_k = k^{-1}$ i. e. $P\{\xi_k = i\} = q_k^{i-1} p_k, i = 1, 2, ..., q_k = 1 - p_k$. In this case $\alpha(k) = k$ and $\beta(k) = q_k p_k^{-2} = k^2(1 - k^{-1})$. Consequently, we have $\Delta^2(n) \sim Bn^3/6$ and $\sigma^2(n) \sim n^3/3$. Therefore, $\sigma^2(n) \sim 2B^2(n)/(B+2)$. Now we show that the Lindeberg condition is fulfilled. Since $Es^{\xi_k} = (p_k s)(1 - q_k s)^{-1}$, we find that $(Es^{\xi_k})''' = 6p_k q_k^2 (1 - q_k s)^{-4}$. Therefore, $E\xi_k(\xi_k - 1)(\xi_k - 2) = 6q_k^2 p_k^{-3}$. From this we conclude that $E|\xi_k - \alpha(k)|^3 = O(k^3), k \to \infty$, which leads to the relation

$$C_n^3 =: \sum_{k=1}^{n} E|\xi_k - \alpha(k)|^3 = O(n^4), n \to \infty.$$

Thus, $C_n^3/\sigma^3(n) = O(n^{-1/2}), n \to \infty$, i.e. Lyapunov condition is satisfied for $\{\xi_k, k \geq 1\}$. In this case we obtain the following result from Theorem 1.

Corollary 4. If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are geometric with parameters $p_k = k^{-1}$, then $n^{3/2}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance $a^2 = 50(B+2)/7$.

Next theorem is related to the case $C = \infty$, but

$$\lim_{n \to \infty} \frac{\beta(n)}{n\alpha^2(n)} = 0. \tag{8}$$

Theorem 5. *If* $A = 1,\ B \in (0, \infty),\ C = \infty,\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ *as* $n \to \infty$ *for any* $\varepsilon > 0$ *and (8) is satisfied, then*

$$\frac{n^{3/2}\alpha(n)}{\sqrt{\beta(n)}}(\hat{A}_n - A) \xrightarrow{d} N(0, b^2),$$

as $n \to \infty$, *where* $N(0, b^2)$ *is a normal random variable with mean* 0 *and variance*

$$b^2 = \frac{(1+\alpha)^2(2\alpha+3)^2}{2\alpha+\beta+3}.$$

It follows from condition (8) that the normalization factor in Theorem 5 tends to infinity faster than n.

Example 6. Let $\xi_k, k \geq 1$ be such that $p = P\{\xi_k = k^2\} = 1 - P\{\xi_k = 0\}, q = 1 - p$. It is obvious that in this case $\alpha(n) = n^2 p,\ \beta(n) = n^4 pq$. Then simple calculations give $\Delta^2(n) \sim Bpn^4/12, \sigma^2(n) \sim pqn^5/5$ as $n \to \infty$ and, consequently, $\Delta^2(n) = o(\sigma^2(n)$. Since in this case $C_n^3 \sim pq(p^2+q^2)n^7/7$ and $\sigma^3(n) \sim (pq/5)^{3/2} n^{15/2}$, the Lyapunov condition is again satisfied. In this case the condition (8) is satisfied and we obtain from Theorem 2 that $(p/q)^{1/2} n^{3/2}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance $b^2 = 441/11$.

Now we consider the case

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = 0. \tag{9}$$

Theorem 6. *If $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and (9) is satisfied, then*

$$n(\hat{A}_n - A) \xrightarrow{d} \frac{W^2(1) - 1}{2\int_0^1 W^2(t^{1+\beta})dt},$$

where $W(t)$ is the standard Wiener process.

The next theorem shows that the limits of the asymptotic normality of CLSE is determined by ratio $n\alpha^2(n)/\beta(n)$. We assume that

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = d_0 \in (0,\infty). \tag{10}$$

Theorem 7. *If $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and (10) is satisfied, then*

$$n(\hat{A}_n - A) \xrightarrow{d} \frac{2^{-1}(W^2(1) - 1) + c_0\eta}{c_0^2/(2\alpha+3) + \zeta},$$

where

$$c_0 = \frac{\sqrt{d_0(1+\beta)}}{1+\alpha},\ \eta = W(1) - (1+\alpha)\int_0^1 W(t^{1+\beta})t^\alpha dt,$$

$$\zeta = 2c_0 \int_0^1 t^{1+\alpha} W(t^{1+\beta}) dt + \int_0^1 W^2(t^{1+\beta}) dt.$$

3.3 The regularly varying moments

In proofs of our theorems we need asymptotic formulas for the mean and variance of the process. Therefore, we first investigated the asymptotic behavior of moments of the process. In this study the assumption that the immigration mean and variance are regularly varying functions plays an important roll. Moreover, we obtain certain new properties of the regularly varying functions, which are of independent interest.

If a sequence $(C_n)_{n=1}^{\infty}$ or function f is regularly varying with exponent ρ, we write $(C_n)_{n=1}^{\infty} \in R_\rho$ and $f \in R_\rho$, [3]. The following result is a discrete form of well known Karamata's theorem on regularly varying functions.

Lemma 1. *If $(C_n)_{n=1}^{\infty} \in R_\rho$, then for any $\theta \in (-\rho - 1, \infty)$*

$$\sum_{k=1}^{n} k^{\theta} C_k \sim \frac{n^{\theta+1} C_n}{\theta + \rho + 1} \tag{11}$$

as $n \to \infty$ and $(\sum_{k=1}^{n} k^{\theta} C_k)_{n=1}^{\infty} \in R_{\theta+\rho+1}$.

We also need a uniform convergence theorem for the regulaly varying sequences.

Lemma 2. *If $A(n)$ is a regularly varying function with exponent $\alpha \geq 0$, then*

$$\sup_{\varepsilon \leq t \leq a} |\frac{A(nt)}{A(n)} - t^{\alpha}| \to 0 \tag{12}$$

as $n \to \infty$ for any $0 < \varepsilon \leq a < \infty$.

The following lemma allows to obtain the asymptotic behavior of the mean and the variance of the procerss

Lemma 3. *If $\alpha(n) \to \infty$ as $n \to \infty$ and regularly varies with exponent α and $A(n) = \alpha(1) + ... + \alpha(n)$, then as $n \to \infty$*

a)

$$\Delta^2(n) \sim \frac{B\alpha(n)n^2}{(\alpha+1)(\alpha+2)}, \quad \sigma^2(n) \sim \frac{n\beta(n)}{\beta+1}. \tag{13}$$

b) For each $\gamma \geq 0$

$$\sum_{k=1}^{n} A^\gamma(k) \sim \frac{n}{\gamma\alpha+\gamma+1} A^\gamma(n). \tag{14}$$

Lemma 3 shows that the man and variance of the process are also regularly varying sequences when so are the mean and variance of the number of immigrating individuals. Beyond this, the proofs of the main theorems use several lemmas of a technical character.

4 Conclusions and Recommendations

In the project we achieved the results we were planning to obtain.

1. Random step functions on trajectory of the process with time inhomogeneous immigration are constructed.

2. Finite dimensional distributions are studied to determine possible approximating processes. This investigation showed that the finite dimensional distributions of the process can not be obtained by a recurrent formula, in contrast to the time homogeneous processes.

3. Convergence of random the step functions, constructed on the trajectory of the process, to approximating processes in Skorokhod topology is proved. It is shown that the Gaussian process with independent increments and with three different covariance functions, depending on behavior of the mean and variance of the number of immigrants, is the approximating process.

4. Applicability of the functional limit theorems in the estimation theory of the parameters of the process is demonstrated. New interesting results on the asymptotic normality of the CLSE of the offspring mean are obtained, using the functional theorems of the project.

The results and the methods developed in this project can be applied in variety of problems, related to non-classical branching processes. As part of this, we recommend for future research work the following:

1. Applying the functional limit theorems obtained in this project to study the improved estimators (such as weighted CLSE) of the offspring mean.

2. Applying methods of the project to obtain the functional limit theorems in subcritical and supercritical processes.

3. Applying methods of the project to obtain the functional limit theorems for sequences of branching processes.

4. The results and the methods of the project, investigate a validity of the bootstrap approach in the non-classical this processes.

5 Activities related to the project

1. Seminars: One seminar in the Department of Mathematics and Statistics: "Functional limit theorems for branching processes with time-dependent immigration", October 3, 2006.

2. Conferences: The problems related to the project were discussed in the following conferences.

a) Rahimov I. Convergence of the immigration-branching process and conditional least-squares. WORKSHOP ON STOCHASTIC MODELLING IN POPULATION DYNAMICS, April 10-14, 2007, Marseille, France.

b) In the City Seminar on Probability Theory, Department of Probability and Statistics, Tashkent State University, Uzbekistan, June, 2007.

6 List of Publications

1. Rahimov I. Functional Limit Theorems for Critical Processes with Immigration. "ADVANCES IN APPLIED PROBABILITY" (ISI), Vol 39, No 4, December 2007, (in press).

2. Rahimov I. Asymptotic distribution of the CLSE in a critical process with Immigration. STOCHASTIC PROCESSES AND THEIR APPLICATIONS, (ISI), (accepted), DOI information: 10.1016/j.spa.2007.11.004.

3. Rahimov I. Convergence of a non-homogeneous immigration-branching process: Technical Report, May 2007, No 376, Dep. Math. And Statistics, KFUPM.

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