1 Abstract

It is known that branching stochastic processes with continuous space of states are more difficult to study than usual processes. In this research project a modification of the branching process with continuous space of states and with generation-dependent immigration is considered. Duality theorems allowing to obtain limit theorems for this model from those of usual processes and vice versa are proved. Using these results limit distributions are obtained for critical processes in the case of decreasing and increasing rate of immigration when offspring distribution has finite or infinite variance. Possibility of using of proved duality theorems in processes without immigration is also demonstrated.

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2 Introduction

We consider a modification of the branching stochastic process which has a continuous space of states. It is convenient to define the process as a family of nonnegative random variables describing the amount of a product produced by individuals of some population. The initial state of the process is given by a nonnegative random variable $X(0)$. The amount of the product $X(1)$ of the first generation is defined as the sum of random products produced by $N_1(X(0))$ individuals and the product U_1 of immigrating to the first generation individuals. Similarly the amount $X(2)$ of the product of the second generation is defined as the sum of products produced by $N_2(X(1))$ individuals and U_2 , and so on. Here $N_k(t), k \geq 1, t \in T$, are counting processes with independent stationary increments, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, 2, ...\}$ and $U_k, k \geq 1$, are non-negative random variables. This process allow to model situations, when it is difficult to count the number of individuals in the population, but some non-negative characteristic, such as volume, weight or product produced by the individuals can be measured.This modification of branching processes was introduced by Adke and Gadag (1995), who indicated relationship of this model with problems related to non-Gaussian Markov time series, to single server queue models and to other problems.

Investigation of branching processes with continuous state space has a long history. Fist this kind a process has appeared due to Feller [6] who introduced a class of one dimensional diffusions obtained by a passage to the limit from the Bienaym'e-Galton-Watson processes. At the end of sixties M. Jirina [10], [11] defined a branching stochastic process with continuous-state space as a homogeneous Markov process transition probabilities of which satisfy some "branching condition". The continuous-state branching process with immigration was considered by Kawazu and Watanabe [14]. Since then investigation of various models of the branching process with continuous states have been active area of the research. Papers [9], [13] [15], [16],[18] [20], [22] are just some examples of publication in this direction. We also note most recent publications by Zeng [23], Lambert [17] and Duquesne [4], where genealogical trees associated with continuous-state branching processes are considered. Additional references in this direction can be found in books by Athreya and Ney [2] and by Dynkin [5].

In the case, when $X(0)$ and $U_k, k \geq 1$, are integer-valued, process $X(n)$

can be considered as a special case of a controlled branching process introduced first by Sevastyanov and Zubkov [21] and by Yanev [27] , for random control functions. In fact, if we choose $\varphi_1(k, n) = N_k(n)$ and $\varphi_2(k, n) \equiv 1$ in so called Model 2 of φ- branching process, obtain a discrete-state version of the process $X(n)$. Further investigations of controlled branching processes with random control functions can be found in [28]-[30].

As distinct from the cited above papers, where the process has been given by a special form of the Laplace transform, in the process which is considered in this project the branching property can explicitly be presented using counting process $N_n(t)$. This allowed Adke and Gadag [1] to obtain distributional properties of the process $X(n)$ that are similar to those of classic models. In particular it was shown that $Z(n) = N_n(X(n-1))$ is usual Bienaym'e-Galton-Watson process with immigration. The following question is interesting in connection with this situation. Is it possible to use this similarity in investigation of asymptotic behavior of the process? In particular can we obtain limit distributions of X_n directly from known limit theorems for Bienaym'e-Galton-Watson processes?

In this project we proved certain theorems which establish relationship between these two processes in a sense of asymptotic behavior. These results allow to get limit theorems for $X(n)$ from those of $Z(n)$ and vice versa. We demonstrated possibilities of these theorems in obtaining limit distributions for the critical process with generation-dependent immigration in cases of linear and functional normalization. New limit theorems for critical processes $X(n)$ with finite variance of offspring distribution are proved when immigration rate decreases depending on generation number and also when it satisfies Foster-Williamson condition of weak stability. In the case when offspring distribution has infinite variance a spectrum of limit distributions is obtained for critical processes with decreasing and increasing immigration as well. It can be seen from results of the project that these duality theorems are applicable to subcritical and supercritical processes and to the processes without immigration.

Hence considered here continuous-state process can be treated by traditional for the theory of branching processes technique, while it may serve to model continuously varying branching populations as the more complicated Jirina or Kawazu-Watanabe processes.

3 Results and discussion

3.1 Duality theorems

Let Let $\{W_{in}, i, n \geq 1\}$ be a double array of independent and identically distributed non-negative random variables, $\{N_n(t), t \in T, n \geq 1\}$ be a family of nonnegative, integer-valued independent processes with independent stationary increments, with $N_n(0) = 0$ almost surely, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, ...\}$.

We define process $X(n), n \geq 0$, as following. Let the initial state of the process be $X(0)$ which is an arbitrary non-negative random variable and for $n \geq 0$

$$X(n+1) = \sum_{i=1}^{N_{n+1}(X(n))} W_{in+1} + U_{n+1}, \tag{1}$$

where $\{U_n, n \geq 1\}$ is a sequence of independent non-negative random variables. Assume that families of random variables $\{W_{in}, i, n \geq 1\}, \{U_n, n \geq 1\}$, sequence of stochastic processes $\{N_n(t), t \in T, n \geq 1\}$ and random variable $X(0)$ are independent.

As it was mentioned before $Z(n) = N_n(X(n-1))$ is a Bienaym'e-Galton-Watson process with an immigration component. Now we provide first result establishing relationship between processes $X(n)$ and $Z(n)$ in a sense of limiting behavior. In order to do that we use the following Laplace transforms

$$G(\lambda) = Ee^{-\lambda W_{ni}}, H_n(\lambda) = Ee^{-\lambda U_n}.$$

We also denote

$$\Delta(n) = \frac{P\{Z(n) > 0\}}{P\{X(n) > 0\}}, \delta(n, \lambda) = \frac{1 - H_n(\lambda)}{P\{Z(n) > 0\}}.$$

Let the sequences of positive numbers $\{k(n), n \geq 1\}$ and $\{a(n), n \geq 1\}$ be such that $k(n), a(n) \to \infty$ and for each $\lambda > 0$ there exists

$$\lim_{n \to \infty} k(n)(1 - G(\frac{\lambda}{a(n)})) = b(\lambda) \in (0, \infty). \tag{2}$$

Existence of these sequences follows from monotonicity of the Laplace transform $G(\lambda)$. In fact one may choose

$$a(n) = \frac{\lambda}{G^{-1}(1 - \frac{b(\lambda)}{k(n)})}$$

for a given sequence $k(n)$, where G^{-1} stands for the inverse of $G(\lambda)$.

Theorem 1.1. *Let $\Delta(n) \to 1, n \to \infty$ and $\delta(n, \lambda/a(n)) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then as $n \to \infty$*

$$E[e^{-\lambda X(n)/a(n)} | X(n) > 0] \to \varphi(b(\lambda)) \tag{3}$$

for $\lambda > 0$, if and only if as $n \to \infty$ for each $\lambda > 0$

$$E[e^{\lambda Z(n)/k(n)} | Z(n) > 0] \to \varphi(\lambda). \tag{4}$$

Next theorem relates to the situation when the limit distribution of $Z(n)$ is discrete.

Theorem 1.2. *Let $\Delta(n) \to 1$ and $\delta(n, \lambda) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then as $n \to \infty$*

$$E[e^{-\lambda X(n)} | X(n) > 0] \to \varphi(-\log(G(\lambda))) \tag{5}$$

for each $\lambda > 0$, if and only if as $n \to \infty$ for each $u > 0$

$$E[e^{-uZ(n)} | Z(n) > 0] \to \varphi(u). \tag{6}$$

Now we provide a similar duality result for unconditional distributions of processes $Z(n)$ and $X(n)$. It will also be formulated it terms of Laplace transforms.

Theorem 1.3. *Let for sequences $\{a(n), n \geq 1\}$ and $\{k(n), n \geq 1\}$ condition (2) be satisfied. Then*

$$Ee^{-\lambda X(n)/a(n)} \to \varphi(b(\lambda)) \tag{7}$$

if and only if for each $\lambda > 0$ as $n \to \infty$

$$Ee^{-\lambda Z(n)/k(n)} \to \varphi(\lambda). \tag{8}$$

3.2 Moments of the process $Z(n)$

As it was indicated before process $Z(n) = N_n(X(n-1))$ is a Bienaym'e-Galton - Watson process with immigration. The offspring distribution and the distribution of the number of immigrating "individuals" have Laplace transforms $G(f(\lambda)) = Ee^{-\lambda\xi_n}$ and $H_n(f(\lambda)) = Ee^{-\lambda\eta_n}$, respectively (see [1]). Here $\xi_n = N_n(W_{n-1}), \eta_n = N_n(U_{n-1})$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

We obtained the moments of offspring and immigration distributions by standard arguments. It is easy to see that

$$m = E\xi_n = -\frac{d}{d\lambda}G(f(\lambda))_{\lambda=0} = EWEN,$$

where $N = N_1(1), W = W_1$. Similarly

$$\alpha(n) = E\eta_n = -\frac{d}{d\lambda}H_n(f(\lambda))_{\lambda=0} = EU_nEN.$$

Since

$$\frac{d^2}{df^2}G(f(\lambda)) = \frac{d^2}{df^2}G(f(\lambda))\left\{\frac{df(\lambda)}{d\lambda}\right\}^2 + \frac{d}{df}G(f(\lambda))\frac{d^2f(\lambda)}{d\lambda^2},$$

we obtain

$$E\xi_n^2 = \frac{d^2G(f(\lambda))}{d\lambda^2}\bigg|_{\lambda=0} = EW^2(EN)^2 + EWvarN$$

One of the important parameters in the theory of usual branching processes is the factorial moment of the offspring distribution $B = E\xi_n(\xi_n - 1)$. We obtain from the above that

$$B = EW[varN - EN] + EW^2(EN)^2.$$

In particular when $E\xi_n = 1$ (the critical case) we have

$$B = EWvarN + (EN)^2varW.$$

By similar arguments we obtain that

$$E\eta_n^2 = EU_nvarN + EU_n^2(EN)^2$$

and for the factorial moment $\beta(n) = E\eta_n(\eta_n - 1)$ we have

$$\beta(n) = (EN)^2 EU_n(U_n - 1) + EN(N-1)EU_n.$$

3.3 A Foster-Williamson type theorem

Now we demonstrate applicability of Theorem 1.3 to obtain a version of well known result by Foster and Williamson (1971). They assume convergence in distribution of the normalized immigration process (the partial sum of the number of immigrating individuals) to a random variable ξ. Since ξ is nonnegative and has an infinitely divisible distribution its Laplace transform has the form (see Feller [7], page 426)

$$Ee^{-\lambda\xi} = \exp\left\{ -\int_0^\infty \frac{1 - e^{-\lambda x}}{x} dP(x) \right\},$$

where $P(x)$ is a measure such that $\int_0^\infty x^{-1} dP(x) < \infty$. First we state the theorem for the process $Z(n)$ from [8].

Theorem A. *If $m = 1, B \in (0, \infty)$ and*

$$\frac{1}{n} \sum_{k=1}^{n} N_k(U_{k-1}) \xrightarrow{\mathrm{D}} \xi, \tag{9}$$

then $Z(n)/n \xrightarrow{\mathrm{D}} W$, with

$$Ee^{\lambda W} = \exp\left\{ -\int_0^\infty \frac{1 - e^{-\lambda x}}{x} dQ(x) \right\},$$

where $Q(x) = R * P(x), R(x) = 1 - \exp\{-2x/B\}$.

Now we formulate Foster-Williamson type result for process $X(n)$. It is natural that the condition on immigration must be given in terms of $\{U_k, k \geq 1\}$ the "immigrating mass".

Theorem 3.1. *If $m = 1, B \in (0, \infty)$ and*

$$\frac{EN}{n} \sum_{k=1}^{n} U_k \xrightarrow{\mathrm{D}} \xi, \tag{10}$$

then $X(n)/n \xrightarrow{\mathrm{D}} X$, *with*

$$Ee^{-\lambda X} = \exp\left\{ -\int_0^\infty \frac{1-e^{-\lambda x EW}}{x} dQ(x) \right\},$$

and $Q(x)$ *is the same as in Theorem A.*

Example. Let the immigration process be stationary, i. e. $\{U_k, k \geq 1\}$ have a common distribution and $a = EU_k$ is finite. Then, due to weak law of large numbers, condition (10) is satisfied with $\xi = aEN$. Thus the Laplace transform of ξ is $e^{-\lambda aEN}$. From equality

$$\lambda aEN = \int_0^\infty \frac{1-e^{-\lambda x}}{x} dP(x)$$

we obtain that measure $P(x)$ has only one atom of mass aEN at $x = 0$. Therefore $Q(x) = P * R(x) = a(1 - e^{-2x/B})$. From here denoting $\psi(\lambda) = -\log Ee^{-\lambda X}$ we have

$$\psi(\lambda) = \frac{2aEN}{B} \int_0^\infty \frac{1-e^{-\lambda x EN}}{x} e^{-2x/B} dx,$$

consequently

$$\frac{d}{d\lambda}\psi(\lambda) = \frac{aENEW}{1 + BEW\lambda/2}.$$

By integration we obtain from the last equation that $\psi(\lambda) = \frac{2aEN}{B}\log(1 + \lambda BEW/2)$. We can see that in this case the limit distribution in Theorem 3.1 is gamma.

Corollary. *If* $m = 1, B \in (0, \infty)$ *and immigration is stationary with* $a = EU_k < \infty$, *then* $X(n)/n$ *as* $n \to \infty$ *has a gamma limit distribution with density function*

$$\frac{1}{\Gamma(\frac{2aE(N)}{B})} \left(\frac{2}{E(W)B} \right)^{\frac{2aE(N)}{B}} x^{\frac{2aE(N)}{B}-1} e^{-\frac{2x}{E(W)B}}.$$

3.4 The probability of non extinction

In the case of stationary immigration $P\{X(n) \neq 0\}$ approaches 1 as $n \to \infty$. However, if the immigration rate depends on the environment, this probability may approach to any number between 0 and 1 inclusively. Moreover, it turned out that the asymptotic behavior of the process strongly depends on the behavior of this probability. Here we provide some results for $P\{X(n) \neq 0\}$ in the case when the immigration rate approaches to zero as $n \to \infty$.

Let $\gamma(n) = EU_n < \infty$ for each $n \geq 1$, regularly varies when $n \to \infty$ and $EW, EN, \alpha(n)$ and $\beta(n)$ are finite for each $n \geq 1$.From now on we also assume that

$$P\{U_n > 0\} = O(\gamma(n)), n \to \infty.$$

Theorem 4.1. *Let* $m = 1, B \in (0, \infty)$ *and* $\gamma(n) \to 0, n \to \infty$. *Then*
a) If $\gamma(n) \log n \to \infty$, *then* $P\{X(n) \neq 0\} \to 1$;
b) If $\gamma(n) \log n \to 0, \beta(n) \to 0$, *then* $P\{X(n) \neq 0\} \to 0$;
c) If $\gamma(n) \log n \to C \in (0, \infty)$, *then* $P\{X(n) \neq 0\} \to 1 - \exp(-2CEN/B)$.

It is clear that when $\gamma(n)$ approaches zero "faster" than $(\log n)^{-1}$, the probability of non extinction may tend to zero arbitrarily. Next theorem gives the asymptotic behavior of that probability, which essentially determine the form of limit distribution of the process. We introduce two functions which are important in further considerations. Let

$$Q_1(n) = \frac{2EN}{B}\gamma(n) \log n, \quad Q_2(n) = \frac{2EN}{Bn} \sum_{k=1}^{n} \gamma(k).$$

Theorem 4.2. *If* $m = 1, B \in (0, \infty), \gamma(n) \log n \to 0$ *and* $\beta(n) = o(Q_1(n) + Q_2(n))$, *then as* $n \to \infty$

$$P\{X(n) \neq 0\} \sim Q_1(n) + Q_2(n).$$

Examples. We consider some examples of possible asymptotic behavior of $P\{X(n) \neq 0\}$. Let $\gamma(n) = C_1/n^{\theta}$.

a) If $\theta < 1$, then $\sum_{k=1}^{n} \gamma(k) \sim const\, n^{1-\theta}$ and $P\{X(n) \neq 0\} \sim Q_1(n)$.
b) If $\theta > 1$, then $\sum_{k=1}^{n} \gamma(k) < \infty$ and $P\{X(n) \neq 0\} \sim Q_2(n)$.
c) If $\theta = 1$, then $Q_1(n) \sim Q_2(n)$ and $P\{X(n) \neq 0\} \sim 2Q_1(n)$.

Theorems 4.1 and 4.2 will be used in next section, where various limit distributions for process $X(n)$ will be provided. However these results are of independent interest as well. In particular Theorem 5 shows that event $\{X(n) \neq 0\}$ may occur, roughly speaking, either because of descendants of "recent immigrants" or because of the individuals immigrated in the beginning of the process. For explanation of this phenomenon for usual processes we refer to [19].

3.5 Limit distributions for $X(n)$

In this section we provide limit distributions for process $X(n)$, when the immigration mean approaches to zero from generation to generation obtained in the project. We denote

$$a = \frac{2EN}{B}, \nabla(n) = \frac{2\alpha(n)}{B}.$$

Theorem 5.1. *If* $m = 1, B \in (0, \infty), \beta(n) \to 0$ *and* $\gamma(n) \to 0$ *such that* $\gamma(n) \log n \to \infty$*, then*

$$\lim_{n\to\infty} P[(\frac{X(n)}{n})^{\gamma(n)} \leq x] = x^a, 0 \leq x \leq 1.$$

If $\gamma(n) \log n \to C$, it follows from Theorem 4.1 that process $X(n)$ may extinct with positive probability. Therefore in this case we considered conditional process $X(n)$, given $X(n) > 0$.

Theorem 5.2. *If* $m = 1, B \in (0, \infty)$ *and* $\gamma(n) \log n \to C \in (0, \infty)$*, then*

$$\lim_{n\to\infty} P[(X(n))^{\gamma(n)} \leq x | X(n) > 0] = \frac{x^a - 1}{e^{aC} - 1}, 1 \leq x \leq e^C.$$

When $\gamma(n) \log n \to 0$, the form of the limit distribution depends on the behavior of function $\theta(n) = Q_1(n)/Q_2(n)$.

Theorem 5.3. *If* $m = 1, B \in (0,\infty), \gamma(n)\log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to \infty$, *then*

$$\lim_{n\to\infty} P[\frac{\log X(n)}{\log n} \leq x | X(n) > 0] = x, 0 \leq x \leq 1.$$

Theorem 5.4. *If* $m = 1, B \in (0,\infty), \gamma(n)\log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to 0$, *then*

$$\lim_{n\to\infty} P[\frac{2X(n)}{Bn} \leq x | X(n) > 0] = 1 - e^{-x}, x \geq 0.$$

When $\theta(n)$ has a positive finite limit two essentially different limit distributions having atoms are obtained.

Theorem 5.5. *If* $m = 1, B \in (0,\infty), \gamma(n)\log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to \theta \in (0,\infty)$, *then*

$$a) \lim_{n\to\infty} P[\frac{\log X(n)}{\log n} \leq x | X(n) > 0] = \frac{x\theta}{1+\theta}, 0 \leq x \leq 1.$$

$$b) \lim_{n\to\infty} P[\frac{2X(n)}{Bn} \leq x | X(n) > 0] = \frac{\theta + 1 - e^{-x}}{1+\theta}, x \geq 0.$$

It is not difficult to see that limit distribution in part a) of last theorem has an atom of the mass $(1-\theta)^{-1}$ at point $x = 1$ and limit distribution in part b) has an atom of the mass $\theta(1+\theta)^{-1}$ at point $x = 0$.

3.6 Infinite offspring variance and regularly varying tails

As it was indicated before process $Z(n) = N_n(X(n-1))$ is a Bienaym'e-Galton - Watson process with immigration. The offspring distribution and the distribution of the number of immigrating "individuals" have Laplace transforms $G(f(\lambda)) = Ee^{-\lambda\xi_n}$ and $H_n(f(\lambda)) = Ee^{-\lambda\eta_n}$, respectively Adke, Gadag (1995). Here $\xi_n = N_n(W_{n-1}), \eta_n = N_n(U_{n-1})$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

We assume that Laplace transforms of random variables W and N can be represented in the form

$$Ee^{-\lambda W} = e^{-a\lambda} + (1-e^{-a\lambda})^{1+\alpha} L_\alpha(1-e^{-\lambda}), \tag{11}$$

and

$$Ee^{-\lambda N} = e^{-b\lambda} + (1 - e^{-b\lambda})^{1+\beta} L_\beta(1 - e^{-\lambda}), \tag{12}$$

where a, b are fixed positive numbers $0 < \alpha, \beta \leq 1, L_\alpha(s)$ and $L_\beta(s)$ are slowly varying functions as $s \uparrow 1$. It is not difficult to see that in this case $EW = a$ and $EN = b$ are finite but second moments may not be finite. Note that in the case of finite variances relations (11) and (12) are satisfied with $\alpha = \beta = 1$ and $L_\alpha(s)$ and $L_\beta(s)$ having finite limits. The following proposition was essential in this part of the project.

Proposition. *If (11) and (12) are satisfied and* $ab = 1$*, then* $Z(n)$ *is critical and the offspring distribution has Laplace transform*

$$G(f(\lambda)) = e^{-\lambda} + (1 - e^{-\lambda})^{1+\theta} L(1 - e^{-\lambda}), \tag{13}$$

where $\theta = \min(\alpha, \beta)$ *and* $L(x)$ *is slowly varying function such that*

$$L(x) \sim \begin{cases} L_\alpha(x), & \text{if } \alpha < \beta \\ L_\beta(x) b^\beta, & \text{if } \alpha > \beta \\ L_\alpha(x) + L_\beta(x) b^\beta, & \text{if } \alpha = \beta \end{cases}$$

for $0 < \alpha, \beta < 1$ *and*

$$L(x) \sim L_\alpha(x) + bL_\beta(x) + \frac{b-1}{2}$$

for $\alpha = \beta = 1$.

The proposition allows just to assume throughout that (13) is satisfied with $0 < \theta \leq 1$ and with some slowly varying function $L(x)$. We define by $V(n)$ usual Bienaym'e-Galton-Watson process with offspring distribution defined by Laplace transform $G(f(\lambda))$. It is known Harris (1966) that, if $0 < G(f(\infty)) < 1$, then process $V(n)$ has a stationary measure $\{\mu_k, k \geq 1\}$ whose generating function $U(s)$ is analytic in the disk $|s| < q$, where q is the extinction probability, and satisfies Abel's equation

$$U(G(f(-\log s))) = 1 + U(s) \tag{14}$$

with initial condition $U(G(f(\infty))) = 1, U(0) = 0, U(1) = \infty$.

If $G(f(\lambda))$ satisfies (13), then it is not difficult to see Slack(1968), that

$$U(s) = \frac{1 + o(1)}{\theta(1-s)^{\theta} L(1-s)}, s \uparrow 1 \tag{15}$$

solves equation (14). On the other hand $U(1-s)$ is invertible and its inverse $g(x), x > 0$, has the form

$$g(x) = \frac{M(x)}{x^{1/\theta}}, \tag{16}$$

where $M(x)$ varies slowly at infinity and $\theta M^{\theta}(x) L(g(x)) \to 1$ as $x \to \infty$.

We also assume that $\alpha(n) < \infty, \beta(n) < \infty$ for each $n \geq 1, \alpha(n)$ varies regularly at infinity and as $n \to \infty$

$$P\{U_n > 0\} = O(EU_n) \tag{17}$$

Remark. If $U_n, n \geq 1$ takes nonnegative integer values, condition (17) is obviously satisfied. In general (17) may hold, for instance, if distribution of U_n has an atom at zero which seems natural in the case of vanishing immigration. Let, for example, $U_n, n \geq 1$ has the following cumulative distribution function

$$P\{U_n \leq x\} = \frac{a_n + 1 - e^{-x/b_n}}{1 + a_n}, x \geq 0,$$

where a_n and b_n are some positive numbers. We see that in this case $P\{U_n > 0\} = (1 + a_n)^{-1}$ and $EU_n = b_n(1 + a_n)^{-1}$ and condition (25) is satisfied, if $\liminf_{n \to \infty} b_n > 0$.

3.7 Limit theorems in the case of infinite variance

Here we show how limit theorems for $X(n)$ can be deduced from those of $Z(n)$ in the case of functional normalization. We use the following functions which were introduced in Rahimov (1986) as normalizing:

$$T(x) = \exp\{\int_0^x g(u)du\}, \ \ \Omega(x) = T(U(1 - x^{-1})).$$

As it was noted before, relation (13) is satisfied in the case of finite variance, if $\theta = 1$ and $L(s) \to C_1 > 0, s \uparrow 1$. We exclude here the situation of $C_1 = 0$, as in this case the offspring variance is zero. From here it follows that $M(x)$

has a finite limit $C_2 \geq 0$ as $x \to \infty$. Therefore $xT'(x)/T(x) = M(x)$ also has finite limit, which means that $T(x)$ is a regularly varying function. From here and relation (15) we conclude that $\Omega(x)$ also varies regularly as $x \to \infty$.

Theorem 7.1. *If (13) is satisfied, $\alpha(n) \to 0, \alpha(n)d(n) \to \infty$ and $\beta(n) \to 0$, then*

$$\lim_{n\to\infty} P\left\{ (\frac{\Omega(X(n))}{\Omega(1/g(n))})^{\alpha(n)} \leq x \right\} = x, 0 \leq x \leq 1.$$

Now we provide results concerning the situation when $\alpha(n)$ approaches zero faster.

Theorem 7.2. *If (13) and (17) are satisfied, $\alpha(n) \to 0, \alpha(n)d(n) \to C \in (0, \infty)$, then*

$$\lim_{n\to\infty} P\left\{ \frac{(\Omega(X(n)))^{\alpha(n)} - 1}{(\Omega(1/g(n)))^{\alpha(n)} - 1} \leq x | X(n) > 0 \right\} = x, 0 \leq x \leq 1.$$

Note that when conditions of Theorem 7.2 are fulfilled $\Omega^{\alpha(n)}(1/g(n)) = T^{\alpha(n)}(n) \to e^C$ as $n \to \infty$. When $\alpha(n) \to 0$ faster than $1/d(n)$, the behavior of the process is effected by new parameter $\gamma(n) = Q_1(n)/Q_2(n)$.

Theorem 7.3. *If (13) and (17) are satisfied, $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n))$ and $\gamma(n) \to \infty$, then*

$$\lim_{n\to\infty} P\left\{ \frac{\log \Omega(X(n))}{\log \Omega(1/g(n))} \leq x | X(n) > 0 \right\} = x, 0 \leq x \leq 1.$$

When $\gamma(n) \to 0, n \to \infty$ we eventually come to the situation when process $X(n)$ is not effected by immigration component at all.

Theorem 7.4. *If (13) and (17) are satisfied, $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n))$ and $\gamma(n) \to 0$, then*

$$\lim_{n\to\infty} P\left\{ g(n)X(n) \leq x | X(n) > 0 \right\} = 1 - e^{-x}, x \geq 0.$$

Theorem 7.5. *If (13) and (17) are fulfilled,* $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n) + Q_2(n))$ *and* $\gamma(n) \to \gamma \in (0, \infty)$*,as* $n \to \infty$*, then the following two assertions hold*

$$i) \lim_{n\to\infty} P\left\{\frac{\log\Omega(X(n))}{\log\Omega(1/g(n))} \le x|X(n) > 0\right\} = \frac{x\gamma}{1+\gamma}, 0 \le x \le 1;$$

$$ii) \lim_{n\to\infty} P\left\{g(n)X(n) \le x|X(n) > 0\right\} = \frac{1+\gamma-e^{-x}}{1+\gamma}, x \ge 0.$$

3.8 Increasing immigration

We consider the case $\alpha(n) \to \infty$ as $n \to \infty$. Let $h(n) = ng(n) = M(n)/n^{1/\theta-1}, B(n) = \sum_{k=1}^{n} \beta(k)$.

Theorem 8.1. *If (13) is fulfilled,* $\alpha(n)h(n) \to C \in (0, \infty)$ *and* $B(n)g^2(n) \to 0$ *as* $n \to \infty$*, then* $g(n)X(n)$ *converges in distribution to random variable* $Z(, \theta, C)$ *which has a infinitely divisible distribution with Laplace transform*

$$\Psi(\theta, C, \lambda) = \exp\left\{-C\int_0^1 (\frac{x^{1-\theta}}{1-x+(\lambda a)^{-\theta}})^{1/\theta}dx\right\}, \lambda > 0. \qquad (18)$$

Remark. It is not difficult to see that, if $\theta = 1$ the limit distribution is gamma with density function

$$\frac{a^{-C}}{\Gamma(C)}x^{C-1}e^{-x/a}, x \ge 0.$$

If $\theta = 1/2$, then the Laplace transform in (18) is

$$(1+\sqrt{a\lambda})^{-C}e^{-C\sqrt{a\lambda}}$$

and in general for each natural k

$$\Psi(1/2k, C, \lambda) = \exp\left\{-\frac{C}{2k(1+a\lambda)^{2k}}F_1(2k, 2k, 2k+1, \frac{a\lambda}{1+a\lambda})\right\},$$

where $F_1(a, b, c, y)$ is Gauss' hypergeometric function.

Now we consider the case $\alpha(n)h(n) \to 0$. In this case there exists positive sequence $m(n), n \geq 1$ such that $\alpha(n)h(m(n))$ has a finite limit as $n \to \infty$ and we have the following result.

Theorem 8.2. *If (13) is fulfilled with* $0 < \theta < 1, \alpha(n)h(n) \to 0, \alpha(n)h(m(n)) \to C \in (0, \infty)$ *and* $B(n)g^2(m(n)) \to 0$ *as* $n \to \infty$, *then* $X(n)g(m(n))$ *converges in distribution to random variable* $W(\theta, C)$ *which has a stable distribution with Laplace transform*

$$Ee^{-\lambda W(\theta,C)} = \exp\left\{-\frac{a^{1-\theta}C\theta}{1-\theta}\lambda^{1-\theta}\right\}, \lambda > 0.$$

Example. Let in relation (13) $0 < \theta < 1$ and $L(s) \to C_0 \in (0, \infty), s \uparrow 1$. Then it is clear that in (24) $M(x) \to C_1 = (C_0\theta)^{-1/\theta}$ as $x \to \infty$. If we take $m(n) = (\alpha(n))^{r\theta}$ where $r = 1/(1-\theta)$, then $\alpha(n)h(m(n)) \to C_1$ and $g(m(n)) \sim C_1/(\alpha(n))^r$. Hence we obtain the following result from Theorem 7.2.

Corollary. If conditions of Theorem 8.2 are satisfied and $L(s) \to C_0, S \uparrow 1$, then $X(n)(\alpha(n))^{-r}$ as $n \to \infty$ converges in distribution to random variable $W(\theta, C_0)$ such that

$$Ee^{-\lambda W(\theta,C_0)} = \exp\left\{-\frac{a^{1-\theta}}{C_0(1-\theta)}\lambda^{1-\theta}\right\}.$$

4 Conclusions and Recommendations

In the project we achieved the results we were planning to obtain.

1. Duality Theorems allowing to obtain limit theorems for the process with continuous states from those of usual processes are proved.

2. Using duality theorems limit distributions are obtained for critical processes
a) with decreasing immigration;
b) with increasing immigration;
c) when Foster-Williamson condition is satisfied;
d) when offspring variance is finite and infinite.

3. Applicability of the duality theorems to processes without immigration is also demonstrated.

We recommend for future research work the following:

1. Applying duality theorems obtained in this project study subcritical and supercritical processes.

2. Applying duality theorems obtain limit theorems for processes without immigration in all cases of criticality.

3. Construction of multi type processes with continuous states and (if possible) obtain duality theorems for them.

4. Developing of an estimation theory for parameters of this process is another direction for further investigation.

5 Activities related to the project

1. Seminars: Two seminars in the Department of Mathematical Sciences: one in the beginning of the project and another one by the end.
2. Conferences: The problems related to the project were discussed in the following conferences.
a) MODELLING 2005, Third IMACS Conference on Mathematical Modelling. July 4-8, 2005, Pilsen, Szech Republic.
b) In the City Seminar on Probability Theory, Department of Probability and Statistics, Tashkent State University, Uzbekistan, June, 2006.

6 List of Publications

1. Rahimov I., Al-Sabah W. Reduction of a continuous state branching stochastic process to usual processes. Journal: "STATISTICS AND PROBABILITY LETTERS" (USA). (submitted)
2. Rahimov I., Al- Sabah W. Duality Theorems For a Branching Process with Continuous States and Applications. "JOURNAL OF APPLIED PROBABILITY" (UK). (submitted).
3. Rahimov I., Al-Sabah W. Branching Processes with Continuous Space of States. Technical Report No 355, KFUPM, May, 2006.

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