Records Generated by the Total Progeny of Branching Stochastic Processes

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Abstract

Let $\{X_k(t), k \geq 1\}$ be a sequence of independent discrete-time branching processes, having the same offspring distribution and $\{Y_k(t), k \geq 1\}$ be the sequence of total progenies of these processes up to time t. In the paper we study relationship between variables $\nu(t) = \min\{k : X_k(t) > 0\}$ and $l_0(t) = \min\{k : Y_k(t) > 1\}$. We prove that in the critical and subcritical cases these variables are asymptotically not correlated. We also obtain a limit distribution for the nth record value generated by the sequence of total progenies.

Key words: branching process, sequence, total progeny, records, critical process ,
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1 Introduction

Let $X(t), X(0) = 1, t \in N_0 = \{0, 1, ...\}$, be a discrete time Markov branching process (i.e. so called Bienayme-Galton-Watson process) with the distribution of direct descendants of one individual $\{P_k, k \in N_0\}$. In the literature properties of the process $X(t)$ have been studied widely. There are many generalizations of the model allowing age-dependence of the number of offspring or containing several type of individuals. These models can well be used in order to describe many population processes.

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We consider family of branching processes $X_n(t), n \in N = \{1, 2, ...\}$ which can be interpreted as sizes of different populations existing in different regions of an area. We introdice the following assumptions:

(a) $X_n(t), n \in N$, are independent for any fixed $t \in N_0$ and $X_n(0) = 1$ for all $n \in N$;

(b)$P\{X_n(1) = k | X_n(0) = 1\} = P_k$ for all $k \in N_0$.

Assumptions (a) and (b) mean that life of individuals in different populations are independent and the distribution of the number of the offspring is the same for all populations.

We define processes $\nu(t), L_n(t), t \in N_0, n \in N$ as following:

$$\nu(t) = \min\{k : X_k(t) > 0\},$$

$$L_1(t) = \min\{k : k > \nu(t), X_k(t) \geq X_{\nu(t)}(t)\}$$
$$L_n(t) = \min\{k : k > L_{n-1}(t), X_k(t) \geq X_{L_{n-1}(t)}(t)\}$$

It is not difficult to see that $\nu(t)$ is the number of the first population which is not degenerated up to time t, and $L_n(t)$ is the number of the population which gives nth "record" of the family. Therefore one should call $X_{L_n(t)}(t)$ as "nth record value" of the family.

The processes introduced above have been considered in Rahimov(1995). In particular, there some limiting distributions for $X_{L_n(t)}$ were obtained in the critical, subcritical and supercritical cases. In present note we discuss record properties of the family generated by the total progeny of the process.

We denote $Y_n(t) = X_n(0) + X_n(1) + ... + X_n(t)$ and consider family $\{\mathrm{Y}_n(t), n \in N\}$ of the total progenies. We define

$$l_0(t) = \min\{k : Y_k(t) > 1\},$$

$$l_n(t) = \min\{k : k > l_{n-1}(t), Y_k(t) \geq Y_{l_{n-1}(t)}(t)\}, n \in N,$$

It is clear that $l_0(t)$ is the number of the first process which gives a birth up to time t (if we consider the birth of a single individual as the event that the initial ancestor does not change up to time t). The process $l_n(t)$ is the number of the population whose total progeny gives "nth record" of the family and $Y_{l_n}(t)$ is the size of the nth total progeny record.

The population sizes in processes $X_n(t)$ can be measured by random characteristics of individuals, i.e. at time t the individual x is assumed to have some weight χ_x (Jagers (1975)). The random characteristic of an individual can also be considered as size of a product produced by the individual during its life. In this case $l_n(t)$ is the number of the region whose total product gives nth record of the area and $Y_{l_n(t)}(t)$ is the size of the record product.

2 Moments, comparition of $\nu(t)$ and $l_0(t)$.

Using the total probability formula by simple arguments we obtain that

$$ES^{l_0(t)} = \frac{SP\{Y(t) > 1\}}{1 - SP\{Y(t) = 1\}}, \tag{1}$$

where $Y(t) = X(0) + X(1) + ... + X(t)$.

Since $P\{Y(t) = 1\} = P\{X(1) = 0\} = P_0$, we have from here

$$El_0(t) = (1 - P_0)^{-1}, varl_0(t) = P_0(1 - P_0)^{-2}, t = 1, 2, ... \tag{2}$$

It is interesting to compare processes $\nu(t)$ and $l_0(t)$. First from the implication $\{X(t) > 0\} \subset \{Y(t) > 1\}$ we obtain that $P\{\nu(t) < l_0(t)\} = 0$ for all $t = 1, 2, ...$ and since $P\{\nu(t) = k, l_0(t) = k\} = P_0^{k-1}Q(t)$, where $Q(t) = P\{X(t) > 0\}$ is the probability of non-extinction, $P\{\nu(t) = l_0(t)\} = Q(t)(1 - P_0)^{-1}$. Therefore, in the supercritical case with probability $(1 - q)(1 - P_0)^{-1}$ processes $\nu(t)$ and $l_0(t)$ are asymptotically equal (here $0 \leq q < 1$ is the extinction probability). In the critical and subcritical cases this probability is zero and, therefore, asymptotically $\nu(t) > l_0(t)$ with the probability approaching 1.

What is the correlation between $\nu(t)$ and $l_0(t)$? Using definitions of $\nu(t)$ and $l_0(t)$ we obtain for $k > m$

$$P\{\nu(t) = k, l_0(t) = m\} = P\{\bigcap_{i=1}^{k-1}\{X_i(t) = 0\}, X_k(t) > 0, \bigcap_{i=1}^{m-1}\{Y_i(t) = 1\},$$

$$Y_m(t) > 1\} = Q(t)P_0^{m-1}P_0^{k-m-1}(t)P\{X(t) = 0, Y(t) > 1\}, \tag{3}$$

where we denote $P_i(t) = P\{X(t) = i\}, i \in N_0$. If we denote the joint probability generating function of $\nu(t)$ and $l_0(t)$ by $H(t, S_1, S_2), 0 \leq S_i \leq 1$, we have from (3) that

$$H(t, S_1, S_2) = Q(t)\left\{\frac{S_1S_2}{1 - P_0S_1S_2} + \frac{S_1^2S_2P\{X(t) = 0, Y(t) > 1\}}{(1 - P_0(t)S_1)(1 - P_0S_1S_2)}\right\} \tag{4}$$

Using relation (4) we obtain that

$$E\nu(t)l_0(t) = \frac{1 - P_0 P_0(t)}{(1 - P_0(t))(1 - P_0)^2} \tag{5}$$

On the other hand it is known Rahimov (1995), that $E\nu(t) = 1/Q(t)$ and $var\nu(t) = P_0(t)/Q^2(t)$. Thus

$$cov(\nu(t), l_0(t)) = \frac{P_0}{(1 - P_0)^2}$$

Consequently we can formulate the following theorem.

Theorem 1 *If $A = EX(1) > 1$ i.e. the process is supercritical, then*

$$\lim_{t\to\infty} \rho(\nu(t), l_0(t)) = \sqrt{\frac{P_0}{q}\frac{1 - q}{1 - P_0}};$$

if $A \leq 1$, i.e. the process critical or subcritical, then

$$\rho(\nu(t), l_0(t)) \sim \frac{Q(t)}{1 - P_0}\sqrt{P_0}, t \to \infty.$$

It is well-known that, if $A \leq 1$, then $Q(t) \to 0$ and $Q(t) \sim constant.A^t$ as $t \to \infty$ if $A < 1, EX(1)logX(1)$ is finite and it can tend to zero linearly or as a regularly varying function if $A = 1$ (see Sevastyanov (1971), Slack (1968)). Thus, if the processes are critical or subcritical, variables $\nu(t)$ and $l_0(t)$ are asymptotically not correlated.

3 The limit theorem for record values

Now we consider the record value $Y_{l_n(t)}(t)$. We shall prove a limit theorem for the conditional distribution of $Y_{l_n(t)}(t)$ given $B(t, l_n(t)) = \{X_i(t) > 0, 1 \leq i \leq l_n(t)\}$. In this connection we shall use the following limit theorem for the total progeny of the process obtained by A. Pakes (1971). Denote $B = 2\sum_{k=2}^{\infty} k(k-1)P_k$.

If $A = 1, B \in (0, \infty)$, then

$$\lim_{t\to\infty} P\{t^{-2}Y(t) \leq x | X(t) > 0\} = F(x),$$

where $F(x)$ is the distribution function having the Laplace transform

$$\int_0^\infty e^{-\lambda x} dF(x) = \sqrt{B\lambda} cosech(\sqrt{B\lambda}). \qquad (6)$$

Let W be a random variable having Laplace transform of (6) What are moments of W? To find the moments of the random variable W we use the following series representation of the Laplace transform (6) (see Gradshteyn and Ryzik (1980), p. 35):

$$\sqrt{B\lambda} cosech(\sqrt{B\lambda}) = 1 - \frac{B\lambda}{6} + \frac{7(B\lambda)^2}{360} - ... + (-1)^n 2(2^{2n-1} - 1)\frac{B_{2n}(B\lambda)^n}{(2n)!} + ...,$$

for $|B\lambda| < \pi$, where B_n are the Bernoulli numbers given by the relation

$$\frac{Z}{e^Z - 1} = \sum_{k=0}^{\infty} \frac{B_k Z^k}{k!}, Z < 2\pi.$$

It is known that

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_n = \sum_{j=0}^{n=1} \frac{n!}{j!(n-j)!}, n > 2.$$

For the even index Bernoulli numbers the formula is as following

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=1}^{\infty} \frac{1}{j^{2n}}.$$

Thus we can write from the above representation that

$$E(W^m) = (-1)^{m-1} \frac{2(2^{2m-1} - 1)m! B^m B_{2m}}{(2m)!} = \frac{4(2^{2m-1} - 1)m! B^m}{(2\pi)^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}}.$$

As examples, we obtain from the last equality :

$$EW = \frac{B}{6}, EW^2 = \frac{7B^2}{2\pi^4} \sum_{j=1}^{\infty} \frac{1}{j^4}.$$

If we use the infinte product representation of the Laplace transform from (6) (see Gradshteyn and Ryzik (1980)) we have

$$\sqrt{B\lambda} cosech(\sqrt{B\lambda}) = \prod_{k=1}^{\infty} (1 + \frac{\lambda B}{k^2 \pi^2})^{-1}.$$

Taking inverse Laplace transform from the last function we get that W has the same distrubution as

$$\sum_{k=1}^{\infty} \frac{BE_k}{k^2\pi^2},$$

where $E_k, k \geq 1$, are independent and identically disributed random variables having the exponential density function $f_E(x) = e^{-x}, x \geq 0$.

To find the distribution function of W we first consider random variable V having the Laplace transform $\sqrt{\lambda} cosech(\sqrt{\lambda})$. The Laplace transform of probability density function (pdf) of V equals $\sqrt{\lambda} cosech(\sqrt{\lambda})$. If we denote the distribution function of V by $F_V(x)$, then the Laplace transform of $F_V(x)$ equals λ^{-1}(LT of of the pdf of V) $= cosech(\sqrt{\lambda})/\sqrt{\lambda}$. Thus taking the inverse LT of $cosech(\sqrt{\lambda})/\sqrt{\lambda}$ (see Murray (1968), formula (32.151)), we get

$$F_V(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-k^2\pi^2 x}.$$

Consequently the distribution of W is as following

$$F(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k exp\{-\frac{k^2\pi^2 x}{B}\}.$$

Now we state our main theorem.

Theorem 2 *If* $A = 1, B \in (0, \infty)$, *then*

$$\lim_{t\to\infty} P\{\frac{1}{t^2} Y_{l_n(t)}(t) \leq x | B(t, l_n(t))\} = F^{n+1}(x).$$

Remark. Let us consider n+1 independent branching processes $\{X_1(t), ..., X_{n+1}(t)\}$ and their total progenies $\{Y_1(t), ..., Y_{n+1}(t)\}$. If we denote $Z_n(t) = \max\{Y_1(t), ..., Y_{n+1}(t)\}$, then it follows from Theorem 2 that

$$\sup_x \left| P\left\{ Z_n(t) \leq t^2 x \right| B(t, n+1)\} - P\left\{ Y_{l_n(t)}(t) \leq t^2 x | B(t, l_n(t)) \right\} \right| \to 0,$$

as $\to \infty$. Thus the asymptotic behavior of nth record value is the same as the behavior of the maximum of $n+1$ total progenies.

The proof of the theorem is based on two lemmas. First we obtain exact formulas for the probability of $B_n = B(t, l_n), l_n = l_n(t)$. Let

$$A(t, x) = \frac{P\{Y(t) = x, X(t) > 0\}}{1 - P\{Y(t) < x, X(t) > 0\}},$$

$$\bar{A}(t,x) = A(t,x)P\{Y(t) \geq x, X(t) > 0\}.$$

Lemma 1*For any $n \geq 1$ and $t \in N_0$*

$$P\{B_n\} = \sum_{i_n=t+1}^{\infty} A(t,i_n) \sum_{i_{n-1}=i_n}^{\infty} A(t,i_{n-1}) ... \sum_{i_2=i_3}^{\infty} A(t,i_2) \sum_{i_1=i_2}^{\infty} \bar{A}(t,i_1). \quad (7)$$

Proof. Let us consider the case $n = 1$. It follows from the total probability formula that

$$P\{B_1\} = \sum_{i_1=2}^{\infty} P\{B(t,i_1), l_0 = 1, l_1 = i_1\}$$

It is not difficult to see that

$$\{l_0 = 1, l_1 = i_1\} = \{\bigcap_{i=2}^{i_1-1} \{Y_i(t) < Y_1(t)\}, Y_{i_1} \geq Y_1(t)\}.$$

Therefore, since $\{X(t) > 0, Y(t) \leq t\} = \emptyset$ for $t > 0$, we have

$$P\{B_1\} = \sum_{i_1=2}^{\infty} \sum_{j=t+1}^{\infty} P\{B(t,i_1, \bigcap_{i=2}^{i_1-1} \{Y_i(t) < j\}, Y_{i_1} \geq j, Y_1(t) = j\}. \quad (8)$$

Taking into account independence of pairs $(X_i(t), Y_i(t)), i = 1, 2, ...,$ we obtain from (8) that

$$P\{B_1\} = \sum_{j=t+1}^{\infty} \bar{A}(t,j).$$

Let now $n = 2$. In this case it is clear that

$$P\{B_2\} = \sum_{i_1=2}^{\infty} \sum_{i_2=i_1+1}^{\infty} P\{B(t,i_2), l_0 = 1, l_1 = i_1, l_2 = i_2\}.$$

Here, if we denote

$$C_1(x) = \bigcap_{i=2}^{i_1-1} \{Y_i(t) < x\}, C_2(x) = \bigcap_{i=i_1+1}^{i_2-1} \{Y_i(t) < x\},$$

then we obtain

$$\{l_0 = 1, l_1 = i_1, l_2 = i_2\} = \{C_1(Y_1(t)), C_2(Y_{i_1}(t)), Y_{i_1}(t) \geq Y_1(t), Y_{i_2}(t) \geq Y_{i_1}(t)\}.$$

Consequently the above probability can be written as following

$$P\{B_2\} = \sum_{i_1=2}^{\infty} \sum_{i_2=i_1+1}^{\infty} \sum_{l=t+1}^{\infty} \sum_{m=l+1}^{\infty} P\{B(t,i_2), C_1(l), C_2(m), D_{1i_1}(i,m), R_{i_1,i_2}(l,m)\},$$

where $D_{ij}(l,m) = \{Y_i(t) = l, Y_j(t) = m\}, R_{ij}(l,m) = \{Y_i(t) \geq l, Y_j(t) \geq m\}$. Again taking into account independence of pairs consisting of branching processes and their total progenies, we obtain

$$P\{B_2\} = \sum_{l=t+1}^{\infty} A(t,l) \sum_{m=l+1}^{\infty} \bar{A}(t,m),$$

i.e. we have formula (7) with $n = 2$. In general case the formula can be proved by similar arguments.

Lemma 2. *For any $n \geq 1, t \in N_0$ and $k \geq t+1$*

$$P\{Y_{l_n}(t) = k, B_n\} = \rho(k,t) \sum_{i_1=t+1}^{k} A(t,i_1) \sum_{i_2=i_1}^{k} A(t,i_2) ... \sum_{i_n=i_{n-1}}^{k} A(t,i_n). \quad (9)$$

where $\rho(k,t) = P\{Y(t) = k, X(t) > 0\}$.

The proof of Lemma 2 is based on the same sort of arguments as the proof of Lemma 1. Therefore we omit the proof.

Proof of Theorem 2. Let us first focus our attention on the "integral" probability

$$Q(t,x) = P\{Y_{l_n(t)}(t) \leq t^2 x, B_n\} = \sum_{t+1 \leq k \leq xt^2} P(k,t), \quad (10)$$

where $P(k,t)$ is the probability on the left side of (9). Now we analyse the right side of (9). If we denote $\tilde{F}(t,x) = P\{Y(t) < x, X(t) > 0\}$, then it is not difficult to see that

$$\sum_{i_n=i_{n-1}}^{k} A(t,i_n) = \sum_{i_n=i_{n-1}}^{k} \frac{\tilde{F}(t,i_n+1,) - \tilde{F}(t,i_n)}{1-\tilde{F}(t,i_n)} =$$

$$= \int_{i_{n-1}}^{k+1} \frac{d\tilde{F}(t,x)}{1-\tilde{F}(t,x)} = \ln \frac{1-\tilde{F}(t,i_{n-1})}{1-\tilde{F}(t,k+1)}.$$

By similar arguments we obtain that

$$\sum_{i_{n-1}=i_{n-2}}^{k} A(t,i_{n-1}) = \ln \frac{1-\tilde{F}(t,i_{n-1})}{1-\tilde{F}(t,k+1)} = \frac{1}{2}\ln^2 \frac{1-\tilde{F}(t,i_{n-2})}{1-\tilde{F}(t,k+1)}.$$

If we repeat these arguments n times and take into account that $\tilde{F}(t,t+1) = P\{Y(t) < t+1, X(t) > 0\} = 0$, we have

$$P(k,t) = P\{Y(t)=k, X(t)>0\}\frac{(-1)^n}{n!}\ln^n[1-\tilde{F}(t,k+1)], \qquad (11)$$

for $k \geq t+2$, and

$$P(t+1,t) = P^{n+1}\{Y(t)=t+1, X(t)>0\}. \qquad (12)$$

Using relations (11) and (12) we obtain from (10) that

$$Q(t,x) = \frac{(-1)^n}{n!}\int_{\beta(t)}^{\alpha(t)} \ln^n y dy + P(t+1,t), \qquad (13)$$

where $\alpha = \alpha(t) = 1-\tilde{F}(t,t+2), \beta = \beta(t) = 1-\tilde{F}(t,t^2x+1)$. If we use the formula

$$\int (\ln y)^n dy = (-1)^n n! y \sum_{k=0}^{n} \frac{(-\ln y)^k}{k!} + C \qquad (14)$$

for the antiderivative of $(\ln y)^n$, we find that the first term on the right side of (13) equals

$$\sum_{k=0}^{n}\frac{1}{k!}[\alpha(-\ln\alpha)^k - \beta(-\ln\beta)^k] = \beta\frac{(-\ln\beta)^{n+1}}{(n+1)!} - \alpha\frac{(-\ln\alpha)^{n+1}}{(n+1)!} + \beta R_2(t) - \alpha R_1(t), \qquad (15)$$

where

$$0 \leq R_1(t) = \sum_{k=n+2}^{\infty}\frac{(-\ln\alpha)^k}{k!}, 0 \leq R_2(t) = \sum_{k=n+2}^{\infty}\frac{(-\ln\beta)^k}{k!}$$

Since $\tilde{F}(x) \leq Q(t)$, it follows from definitions $\alpha(t)$ and $\beta(t)$ that $\alpha(t), \beta(t) \to 1$ as $t \to \infty$, therefore $0 < -\ln\alpha < 1, 0 < \ln\beta < 1$ for sufficiently large t. Thus we have

$$R_1(t) = o((-\ln\alpha)^{n+1}), R_2(t) = o((-\ln\beta)^{n+1}), t \to \infty.$$

On the other hand according to Pakes's theorem

$$-\ln\beta \sim \tilde{F}(t, xt^2+1) \sim Q(t)F(x), t\to\infty,$$

where $F(x)$ is defined by (6) and $Q(t) \sim 2/Bt, t\to\infty$. Taking into account the relation

$$-\ln\alpha \sim \tilde{F}(t, t+2) = P\{Y(t)=t+1, X(t)>0\} = P_1^t,$$

in (13) and (15) we conclude that

$$Q(t,x) \sim \frac{Q^{n+1}(t)F^{n+1}(x)}{(n+1)!}, t\to\infty. \tag{16}$$

Now we consider the probability $P\{B_n\}$. It is not difficult to see that formula (7) can be rewritten in the following form:

$$P\{B_n\} = \int_0^{Q(t)} \frac{1}{1-y_1} \int_{y_1}^{Q(t)} \frac{1}{1-y_2} ... \int_{y_{n-2}}^{Q(t)} \frac{1}{1-y_{n-1}} \int_{y_{n-1}}^{Q(t)} \frac{Q(t)-y_n}{1-y_n} dy_n...dy_1 \tag{17}$$

The consequent integration in (17) gives

$$\int_{y_{n-1}}^{Q} \frac{Q-y_n}{1-y_n} dy_n = (1-Q)\ln\frac{1-Q}{1-y_{n-1}} + Q - y_{n-1} \equiv A_1(Y_{n-1}),$$

$$\int_{y_{n-2}}^{Q} \frac{A_1(y_{n-1})}{1-y_{n-1}} dy_{n-1} = -(1-Q)\frac{1}{2!}\ln^2\frac{1-y_{n-2}}{1-Q} + A_1(y_{n-2}) \equiv A_2(y_{n-2})$$

and so on after nth integration we obtain

$$P\{B_n\} = 1-(1-Q(t))\sum_{k=0}^{n}\frac{1}{k!}\ln^k\gamma(t),$$

where $\gamma(t) = (1-Q(t))^{-1}$. Using the series representation of e^x we have

$$P\{B_n\} = \frac{1-Q(t)}{(n+1)!}\ln^{n+1}\gamma(t) + R(t),$$

where

$$0 < R(t) \le \sum_{n+2}^{\infty}\frac{\ln^k\gamma(t)}{k!} = \frac{\ln^{n+2}\gamma(t)}{1-\ln\gamma(t)}.$$

Since $Q(t) \to 0$ as $t \to \infty$, we conclude from the last relation that

$$P\{B_n\} \sim \frac{Q^{n+1}(t)}{(n+1)!}, t \to \infty. \tag{18}$$

The assertion of the theorem follows strightforward from relations (16) and (18). Theorem 2 is proved.

In the conclusion we note that, using methods of present work, one could study the behavior of nth record value generated by total progenies of noncritical processes. Problems of investigation of inter-record and record "times", study of the asymptotic behavior of expected record values are also open.

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