LIMIT THEOREMS FOR THE SIZE OF SUBPOPULATION OF PRODUCTIVE INDIVIDUALS

I. Rahimov

Department of Mathematical Sciences, KFUPM, Box. 1339, Dhahran, 31261, Saudi Arabia

ABSTRACT

We consider n-type critical indecomposable Galton-Watson process with types of individuals $1, 2, ..., n$. Let $\nabla \subseteq \{1, 2, ..., n\}, \tau$ and $t, \tau < t$, be two discrete observation times and $\boldsymbol{\theta}(t)$ be n-dimensional vector of nonnegative functions. We consider n-variate process $\mathbf{X}(\tau, t) = (X_1(\tau, t), ..., X_n(\tau, t))$, where $X_i(\tau, t)$ is the number of type i individuals at time τ whose number of descendants at time t of types $j, j \in \nabla$ is greater than corresponding level given by vector $\boldsymbol{\theta}(t-\tau)$. We study asymptotic behavior of $\mathbf{X}(\tau, t)$ when $\tau, t \to \infty$ in different cases of relationship between times τ and t for properly chosen level functions.

AMS 1991 Subject Classification: Primary 60J80, Secondary 60G70.
Key Words: reduced process, large population, multivariate geometric distribution, infinite second moment, productive individuals, random vectors, normal distribution.

1 Introduction

We consider evolution and reproduction of n-type of some individuals (animals, cells, bacteria or particles). Each of $X_i(t), i = 1, ..., n$ existing at time $t \in N_0 = \{0, 1, ...\}$ individuals of type i lives a unit of time and at the end of its life independently of others generates a random number of new individuals of types $1, 2, ..., n$. These new individuals behave similarly. Suppose that the number of offspring of different individuals of the same type are independent and identically distributed random vectors following the probability distribution $\{P_{\mathbf{k}}^{(i)}, \mathbf{k} \in N_0^n\}$, where $N_0^n = N_0 \times N_0 \times ... \times N_0, i = 1, 2, ..., n$. In other words $P_{\mathbf{k}}^{(i)}, \mathbf{k} = (k_1, ..., k_n)$ is the probability that a single individual of type i at the end of its life generates k_j individuals of type $j, j = 1, 2, ..., n$. The process $\mathbf{X}(t) = (X_1(t), ..., X_n(t)), \mathbf{X}(0) = \delta_i = (\delta_{ij}, j = 1, ..., n)$, where $\delta_{ij} = 0, j \neq i$ and $\delta_{ii} = 1$, is called a discrete time n-type Galton-Watson process initiated by a single ancestor of type i.

Although $\mathbf{X}(t)$ the number of individuals of different types at time t is the main object of investigation in the theory of multitype branching processes, there are many other variables related to the population which are of interest as well. One example of such a variable is the time to the closest common ancestor of the whole population at given time. For a single-type Galton-Watson process this variable was considered by Zubkov (1975), who proved that, if the process is critical, the time is asymptotically uniformly distributed. Later, it turned out that the time to the closest common ancestor may be treated as a functional of so called reduced branching processes. This kind a process was introduced by Fleischmann and Siegmund-Schultze(1977) as a process that counts only individuals at a given time τ having descendants at time $t, t > \tau$. They demonstrated that in the critical single-type case the reduced process can well be approximated by a non-homogeneous pure birth process. Later on many investigations were devoted to extension of this result for more general single and multitype models of branching processes (see [17], [21] and [22], for example).

Another variable related to the genealogy of branching processes is the number of pairs (or number of r-sets) of individuals at time τ having the same number of descendants at time $t, t > \tau$. Limit distributions for this variable were obtained in [11] and [12] (see also [13], chapter IV) in critical, subcritical and supercritical cases. We note here the rice of interest in recent years to problems concerning extrema in branching stochastic processes. Most recent publications in this direction have been devoted to the asymptotic behavior of the expectation of the maxima of branching processes (Athreya (1988), Pakes (1987, 1998a,b), Borovkov and Vatutin (1996)), to the distribution of the maximum family size (Arnold and Villasenor (1996), Rahimov and Yanev (1999), Yanev and Tsokos (2000)) and to other problems. Limit distributions for the index of the first process in a sequence of branching processes exceeding some fixed or increasing levels are obtained in Rahimov and Hasan (1998). Later Rahimov (2001) showed that the stochastic process generated by the number of ancestors whose offspring in a given generation exceeds some levels can be approximated in Skorohod topology by some processes with independent increments.

In this paper we consider one more process related to the genealogy of n-type branching processes. Let $\boldsymbol{\theta}= (\theta_1(t), ..., \theta_n(t))$ be a vector of non-negative functions, τ and $t, \tau < t$ be two times of observation. Let $\nabla \subseteq \{1, 2, ..., n\}$. Then $\{i, i \in \nabla\}$ denotes a subset of "type-set". We define process $\mathbf{X}(\tau, t) = (X_1(\tau, t), ..., X_n(\tau, t))$, where $X_i(\tau, t)$ is the number of type i individuals at time τ, whose number of descendants at time t of types $j, j \in \nabla$ is greater than corresponding level, given by vector $\boldsymbol{\theta}(t-\tau)$. If, for instance, $\nabla = \{1, 2, ..., n\}$, then $X(\tau, t)$ is the number of individuals of type i at time τ whose the number of descendants at time t of all types are greater than given levels. It is clear that $\mathbf{X}(\tau, t)$ counts only "relatively productive" individuals at time τ. In the paper we obtain limit distributions for process $\mathbf{X}(\tau, t)$ as $t, \tau \to \infty$ in different cases of relationship between observation times τ and t for critical processes.

Let us give a rigorous definition of the process $\mathbf{X}(\tau, t)$. We use the following notation for individuals participating in the process. Let we have a single ancestor at time $t = 0$ of type $i, i = 1, ..., n$. We denote it by i and consider as zeroth generation. The direct offspring of the initial ancestor we denote as (i, j, m_j), where $j, j = 1, ..., n$ is the type of the direct descendant and $m_j \in N, N = \{1, 2, ...\}$ is the label (the number) of the descendant in the set of all immediate descendants of i. Thus m_{k+1}-th direct descendant of the type i_{k+1} of the individual $\boldsymbol{\alpha}= (i, i_1, m_1, ..., i_k, m_k)$ will be denoted as $\boldsymbol{\alpha}' = (\boldsymbol{\alpha}, i_{k+1}, m_{k+1})$. Here and later on for any two vectors $\boldsymbol{\alpha} = (i_1, ..., i_k)$ and $\boldsymbol{\beta}= (j_1, ..., j_m)$ the ordered pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$ we will understand as $k + m$ dimensional vector $(i_1, ..., i_k, j_1, ..., j_m)$.

If we use the above notation, the set $\Re_n \in E$, where E is the space of all finite subsets of

$$\bigcup_{k=1}^{\infty} N_1^k, N_1^k = N_1^{k-1} \times N_1, N_1 = \{i\} \times \{1, ..., n\} \times N$$

corresponds to the population of the n-th generation. It is clear that $\Re_n$ can be decomposed as $\Re_n = \cup_{i=1}^n \Re_n^{(i)}$, where $\Re_n^{(i)}$ is the population of the type i individuals of the n-th generation. Consequently components of the process $\mathbf{X}(t)$ are found as $X_i(t) = card\{\Re_n^{(i)}\}, t \in N_0$ and for any τ and t such that

$\tau < t$ we have

$$\mathbf{X}(t) = \sum_{i=1}^{n} \sum_{\boldsymbol{\alpha} \in \Re_\tau^{(i)}} \mathbf{X}^{(\boldsymbol{\alpha})}(t-\tau),$$

where $\mathbf{X}^{(\boldsymbol{\alpha})}(t) = (X_1^{(\boldsymbol{\alpha})}(t), ..., X_n^{(\boldsymbol{\alpha})}(t))$ is the n-type branching process generated by individual $\boldsymbol{\alpha}$.

Let $\Im_i([\boldsymbol{\theta}], \tau, t)$ be the set of individuals in $\Re_\tau^{(i)}$ having the number of descendants at time t of types $j, j \in \nabla$ greater than corresponding component of $\boldsymbol{\theta}$ $(t-\tau)$. It is not difficult to see that it can be described as

$$\Im_i([\boldsymbol{\theta}], \tau, t) = \{\boldsymbol{\alpha} \in \Re_\tau^{(i)} : \text{for all } j \in \nabla \; \exists \; \text{ more than } \theta_j(t-\tau)$$

$$\beta \text{ -sets such that } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Re_t^{(i)}\},$$

where $\boldsymbol{\alpha} \in N_1^\tau$, $\boldsymbol{\beta} \in N_1^{t-\tau}$. Thus the generalized reduced process is defined as $\mathbf{X}(\tau, t) = (X_i(\tau, t), i = 1, ..., n)$ with $X_i(\tau, t) = \text{card}\{\Im_i([\boldsymbol{\theta}], \tau, t)\}$.

In particular, if $\boldsymbol{\theta}(t) = \mathbf{0}$ for all t, then $\Im_i([\mathbf{0}], \tau, t)$ contains all and only individuals of type i of τ-th generation having descendants (at least of one type) in generations $\tau+1, \tau+2, ..., t-1$ and in t-th generation having descendants of all types $j, j \in \nabla$ (recall that the initial n-type process is indecomposable). For instance, if $\nabla = \{1\}$, then $\mathbf{X}(\tau, t)$ counts all individuals of τ-th generation having at least one descendant of type 1 in t-th generation. If $\nabla = \{1, 2, ..., n\}$, then it counts individuals of τ-th generation having descendants of all types in t-th generation.

We note here that another process counting the productive individuals in a population could be defined as number of individuals at time τ of a given type whose the number of descendants of at least one type at time t is greater than the corresponding level. The particular case of this process with zero level functions would include usual reduced processes. It will be seen from further considerations that by methods, developed in this paper, limit distributions for this kind processes can also be obtained.

Main results which are given in theorems 1-3 of Section 2 describe the asymptotic behavior of the process $\mathbf{X}(\tau, t)$ for all possible relationships between observation times. The proofs of these theorems are based on theorems for random sums of independent indicator vectors obtained in Section 3. In Section 5 we demonstrate that the results of Section 3 can be used to study of the number of productive ancestors in a population. An interpretation in terms of the number of "big" trees in a Galton-Watson forest will also be discussed.

2 Limit theorems

For n -dimensional vectors $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$ we denote $\mathbf{x} \otimes \mathbf{y} = (x_1 y_1, ..., x_n y_n), \mathbf{x}^{\mathbf{y}} = (x_1^{y_1}, ..., x_n^{y_n}), \frac{\mathbf{x}}{\mathbf{y}} = \mathbf{x}/\mathbf{y} = (\frac{x_1}{y_1}, ..., \frac{x_n}{y_n}), (\mathbf{x}, \mathbf{y}) = x_1 y_1 + ... + x_n y_n, \mathbf{1} = (1, 1, ..., 1), \mathbf{0} = (0, 0, ..., 0), \mathbf{e} = (e, e, ..., e)$ and $\mathbf{x} \geq \mathbf{y}$ or $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ or $x_i > y_i$ for all i respectively.

We denote by $P^i_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in N_0^n$, the offspring distribution of the process $\mathbf{X}(t)$, i.e.

$$P^i_{\boldsymbol{\alpha}} = P\{\mathbf{X}(1) = \boldsymbol{\alpha} | \mathbf{X}(0) = \boldsymbol{\delta}_i\}$$

is the probability that an individual of type i generates the total number of $\boldsymbol{\alpha}$ of new individuals. Here $\boldsymbol{\delta}_i = (\delta_{ij}, j = 1, ..., n), \delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. We also denote

$$F^i(\mathbf{S}) = \sum_{\boldsymbol{\alpha} \in N_0^n} P^i_{\boldsymbol{\alpha}} S_1^{\alpha_1} ... S_n^{\alpha_n}, \quad \mathbf{F}(\mathbf{S}) = (F^1(\mathbf{S}), ..., F^n(\mathbf{S})),$$

$$Q^i(t) = P\{\mathbf{X}(t) \neq \mathbf{0} | \mathbf{X}(0) = \delta_i\}, \mathbf{Q}(t) = (Q^1(t), ..., Q^n(t)),$$

i.e. $Q^i(t)$ the probability of non-extinction up to time t of the process generated by a singly individual of type i.

Let for $i, j, k = 1, 2, \cdots, n$

$$a_i^j = \frac{\partial F^j(\mathbf{S})}{\partial S_i} |_{\mathbf{S}=\mathbf{1}}, b_{ik}^j = \frac{\partial^2 F^j(\mathbf{S})}{\partial S_i \partial S_k} |_{\mathbf{S}=\mathbf{1}},$$

$\mathbf{A} = \left\|a_i^j\right\|$ be the matrix of expectations, ρ be its Peron root and the right and the left eigenvectors $\mathbf{U} = (u_1, u_2, ..., u_n)$ and $\mathbf{V} = (v_1, v_2, ..., v_n)$ corresponding to the Peron root be such that

$$\mathbf{AU} = \rho \mathbf{U}, \quad \mathbf{VA} = \rho \mathbf{V}, \qquad (\mathbf{U}, \mathbf{V}) = 1, \quad (\mathbf{U}, \mathbf{1}) = 1.$$

If $\mathbf{A}$ is indecomposable, aperiodic and $\rho = 1$, the process $\mathbf{X}(t)$ is called critical indecomposable multitype branching process. We assume that the generating function $\mathbf{F}(\mathbf{S})$ satisfies the following representation

$$x - \sum_{j=1}^n v_j (\mathbf{1} - F^j(\mathbf{1} - \mathbf{U}x)) = x^{1+\alpha} L(x), \tag{1}$$

where $0 < x \leq 1, \alpha \in (0,1]$, and $L(x)$ is a slowly varying function as $x \downarrow 0$. Note that in this case $\rho = 1$, i.e. the process is critical and the second moments of the offspring distribution $b^j_{ik}, i,j,k = 1,...,n$, may not be finite. Under this assumption the following limit theorem for the process $\mathbf{X}(t)$ holds (see Vatutin (1977)).

Proposition 1. *If the offspring generating function* $\mathbf{F}(\mathbf{S})$ *satisfies representation (1) then we have*
a)

$$Q^i(t) \sim u_j t^{-1/\alpha} L_1(t)$$

as $t \to \infty$, *where* $L_1(t)$ *is a slowly varying as* $t \to \infty$ *function;*
b)

$$\lim_{t\to\infty} P\left\{\mathbf{X}(t)q(t) \leq \mathbf{x} \otimes \mathbf{V} \mid \mathbf{X}(t) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\right\} = \pi(\mathbf{x}),$$

where $q(t) = \sum_{j=1}^n v_j Q^j(t)$ *and* $\pi(\mathbf{x}) = \pi(x_1, x_2, ..., x_n)$ *a distribution having the Laplace transform*

$$\phi(\lambda) = \int_{R^n_+} e^{-(\mathbf{x},\lambda)} d\pi(\mathbf{x}) = 1 - (1 + \bar{\lambda}^{-\alpha})^{-1/\alpha}, \quad \bar{\lambda} = (\boldsymbol{\lambda}, \mathbf{1}). \tag{2}$$

where $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n), \boldsymbol{\lambda} \in R^n_+, R_+ = [0, \infty)$.

Now we introduce the operator $\Delta_{i_1...i_m} : R^n \mapsto R^n, m = 1,2,...,n$ such that for each $\mathbf{x} = (x_1, ..., x_n)$ the vector $\Delta_{i_1,...,i_m}(\mathbf{x})$ has $i_1, i_2, ...i_m - th$ coordinates as that of $\mathbf{x}$ and remaining coordinates are ∞. For instance, $\Delta_{13}(\mathbf{x}) = (x_1, \infty, x_3, \infty, ..., \infty)$. Farther we denote

$$L = \Sigma' \pi(\Delta_l(\boldsymbol{\theta})) - \Sigma'_{l\neq m} \pi(\Delta_{lm}(\boldsymbol{\theta})) + ... + (-1)^{|\nabla|-1} \pi(\boldsymbol{\theta}),$$

where $|\nabla|$ stands for cardinality of the set ∇ and Σ' is summation over values of l and m from ∇, $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$.

Now we provide the first result about $\mathbf{X}(\tau, t)$. Let $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in R^n_+, \mathbf{C} = (C_1, ..., C_n) \in R^n_+$ be some nonnegative vectors.

Theorem 1. *If condition (1) is satisfied,* $\boldsymbol{\theta}(t) = \boldsymbol{\theta} \otimes \mathbf{V}/q(t)$ *and* $t, \tau \to \infty, t - \tau \to \infty$ *such that* $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$, *then*

$$P\{\mathbf{X}(\tau, t) = \mathbf{k} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\} \to P^*_{\mathbf{k}},$$

where $\mathbf{k} = (k_1, ..., k_n) \in N_0^n$. *The probability distribution* $\{P^*_{\mathbf{k}}, \mathbf{k} \in N_0^n\}$ *has the generating function* $\phi^*(\mathbf{S}) = \phi(\mathbf{a})$, *where* $\mathbf{a} = (1-L)\mathbf{C} \otimes \mathbf{U} \otimes \mathbf{V} \otimes (\mathbf{1} - \mathbf{S}), \mathbf{S} = (S_1, ..., S_n)$.

Remark. It is clear that vector $\mathbf{C}$ in the condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$ necessarily has the form $\mathbf{C} = C\mathbf{1}$, where $C \geq 0$ is some constant.

Example 1. Let $\mathbf{F}(\mathbf{S})$ satisfies condition (1) with $\alpha = 1$. We shall note here that in this case the second moments of the offspring distribution still may be infinite. For this kind of a process the limit distribution $\pi(\boldsymbol{\theta})$ is exponential and the generating function $\phi^*(\mathbf{S})$ has the form $\phi^*(\mathbf{S}) = (1+d)^{-1}$, where $d = (1-L)C\sum_{j=1}^n u_j v_j (1-S_j)$, and $\pi(\Delta_{i_1...i_m}(\boldsymbol{\theta})) = 1 - \exp\{\min\{\theta_{i_1}, ..., \theta_{i_m}\}\}$. Therefore

$$L = \Sigma'(1-e^{-\theta_l}) - \Sigma'_{l\neq m}(1-e^{-\min\{\theta_l,\theta_m\}}) + ... + (-1)^{|\nabla|-1}(1-e^{-\min\{\theta_i, i\in\nabla\}}).$$

We represent the generating function as follows

$$\phi^*(\mathbf{S}) = \frac{1}{1+C(1-L)}\left(1 - \frac{C(1-L)}{1+C(1-L)}\sum_{i=1}^n u_i v_i S_i\right)^{-1}. \tag{3}$$

What is the distribution having the last probability generating function? To answer this question we consider a sequence of independent random variables $X_1, X_2, ...$ such that $P\{X_i = j\} = p_j, \quad j = 0, 1, 2, ..., n, \quad \sum_{j=0}^n p_j = 1$, where $p_0 = (1+C(1-L))^{-1}, \quad p_j = C(1-L)u_j v_j/(1+C(1-L)), \quad j = 1, 2, ..., n$. Let Δ_1 be the number of $1's$, Δ_2 be the number of $2's$ and so on Δ_n be the number of $n's$ observed in the sequence $X_1, X_2, ...$ before the first zero is obtained. Then it follows from the formula for the generating function of generalized multivariate geometric distribution in Ch. 36.9 of Johnson at. all.(1997) that the vector $(\Delta_1, ..., \Delta_n)$ has the probability generating function given by(3) i. e.

$$E\left(S_1^{\Delta_1} S_2^{\Delta_2} ... S_n^{\Delta_n}\right) = \phi^*(\mathbf{S}).$$

Hence we have the following result.

Corollary 1. If assumptions of Theorem 1 are satisfied with $\alpha = 1$, then the probability distribution $\{P^*_{\mathbf{k}}, \mathbf{k} \in N_0^n\}$ is multivariate geometric distribution defined by generating function (3) such that

$$P^*_{\mathbf{k}} = P\{\Delta_i = k_i, i = 1, ..., n\}.$$

It is clear that, if $n = 1$, the distribution is geometric, i. e. $P_k^* = pq^k, k = 0, 1, ...$ with $p = (1 + Ce^{-\theta_1})^{-1}, q = Ce^{-\theta_1}(1 + Ce^{-\theta_1})^{-1}$.

Example 2. Let assumptions of Theorem 1 be satisfied and $\tau = [\varepsilon t], 0 < \varepsilon < 1$. Using asymptotic behavior of $\mathbf{Q}(t)$ and uniform convergence theorem for the slowly varying functions (see Seneta (1985), for example) we obtain that as $t \to \infty$

$$\frac{\mathbf{Q}(t-\tau)}{\mathbf{Q}(\tau)} \to \left(\frac{\varepsilon}{1+\varepsilon}\right)^{1/\alpha} \mathbf{1}.$$

Consequently in this case the limit distribution has the generating function $\phi^*(\mathbf{S})$ with $C = (\varepsilon/(1+\varepsilon))^{1/\alpha}$. In particular we have the following result.

Corollary 2. If assumptions of Theorem 1 are satisfied and $\tau = o(t)$, then

$$\lim_{t\to\infty} P\{\mathbf{X}(\tau, t) = \mathbf{k}|\mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\} = 0$$

for all $\mathbf{k} \in N_0^n$ such that $\mathbf{k} \neq \mathbf{0}$.

It is known that in the critical case the process $\mathbf{X}(t)$ goes to extinction with probability 1. Corollary 2 shows that, if $\tau = o(t)$, even conditioned process $\mathbf{X}(\tau, t)$ given $\mathbf{X}(\tau) \neq \mathbf{0}$ vanishes with a probability approaching 1.

Theorem 1 gives a limit distribution for $\mathbf{X}(\tau, t)$ when the times of observation $\tau \to \infty$ and $t \to \infty$ such that $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau)$ has a finite limit. Now we consider the case when this limit is not finite. Let $T_i(\tau, t) = Q^i(t-\tau)/Q^i(\tau)$ and $\mathbf{T}(\tau, t) = (T_1(\tau, t), ..., T_n(\tau, t))$.

Theorem 2. *If condition (1) holds,* $\boldsymbol{\theta}(t) = \boldsymbol{\theta} \otimes \mathbf{V}/q(t)$ *and* $T_i(\tau, t) \to \infty, i = 1, 2, ..., n,$ *as* $t, \tau \to \infty, t - \tau \to \infty$ *then*

$$P\left\{\frac{\mathbf{X}(\tau, \mathbf{t})}{\mathbf{T}(\tau, \mathbf{t})} \leq \mathbf{x}|\mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\right\} \to \pi(\frac{1}{1-L}\mathbf{x}),$$

where $\pi(\mathbf{x}), \mathbf{x} \in R_+^n$, *is the distribution from Proposition 1 and L is defined just before Theorem 1.*

Remark. It follows from the asymptotic behavior of $Q^i(t)$ that, if $T_i(\tau, t) \to \infty$ for at least one i, then it holds for each $i = 1, 2, ..., n$.

Example 3. If matrix $\mathbf{A}$ is indecomposable, aperiodic, $\rho = 1$ and $b^i_{jk} < \infty, i, j, k = 1, ..., n$, then (1) is satisfied with $\alpha = 1, L(x) \to const, x \to 0$. In this case $Q^i(t) \sim 2u_i/\sigma^2 t, \quad i = 1, ..., n$ as $t \to \infty$, where $\sigma^2 = \sum_{j,m,k=1}^n v_j b^j_{mk} u_m u_k$. Consequently

$$q(t) = \sum_{j=1}^n Q^j(t) v_j \sim \frac{2}{\sigma^2 t}, t \to \infty$$

and $\boldsymbol{\theta}(t) \sim \sigma^2 t \boldsymbol{\theta} \otimes \mathbf{V}/2$. On the other hand $T_j(\tau, t) \sim \tau/(t-\tau), j = 1, ..., n$. Thus $T_j(\tau, t) \to \infty$, if, for example, $\tau \sim t$ and we obtain the following result from Theorem 2.

Corollary 3. If $\rho = 1, 0 < \sigma^2 < \infty$ and $t, \tau \to \infty, t - \tau \to \infty$ such that $\tau \sim t$, then

$$P\left\{\frac{t-\tau}{\tau}\mathbf{X}(\tau, t) \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\right\} \to 1 - \exp\{-\frac{x^*}{1-L}\},$$

where $\mathbf{x} \in R^n_+, x^* = \min\{x_1, ..., x_n\}$, and L is the same as in Example 1.

The above two theorems describe the asymptotic behavior of $\mathbf{X}(\tau, t)$ when $t - \tau \to \infty$. Now we consider the case $\tau = t - t_0$, where $t_0 \in (0, \infty)$ is a constant.

Theorem 3. *If condition (1) is satisfied,* $t, \tau \to \infty, \quad t - \tau = t_0 \in (0, \infty)$ *and* $\boldsymbol{\theta}(t) \equiv \boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in R^n_+$, *then*

$$P\{\mathbf{X}(\tau, t) \otimes \mathbf{Q}(\tau) \leq \mathbf{x} | \mathbf{X}(\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i\} \to \pi(\frac{\mathbf{x}}{\mathbf{R}(t_0)}),$$

where $\mathbf{x} \in R^n_+$ *and*

$$\mathbf{R}(t) = (R^1(t), ..., R^n(t)), R^i(t) = P\{\bigcap_{j=1}^n \{X_j(t) > \theta_j\} | \mathbf{X}(0) = \boldsymbol{\delta}_i\}.$$

Remark. It follows from Proposition 1 that

$$\frac{Q^i(\tau)}{Q^i(t)} \sim \left(\frac{t}{t-t_0}\right)^{1/\alpha} \frac{L_1(t-t_0)}{L_1(t)}$$

which shows that $Q^i(\tau) \sim Q^i(t)$ as $t, \tau \to \infty, t - \tau = t_0$, for each $i = 1, ..., n$. Therefore normalizing vector functions $\mathbf{Q}(\tau)$ in Theorem 3 can be replaced by $\mathbf{Q}(t)$.

3 Preliminary results

To prove results of Section 2 we use limit theorems for a random sum of random vectors which will be obtained in this section. We consider a sequence of random vectors defined as follows. Let $\{\xi_{ij}(k,m), j \geq 1\}, i = 1, 2, ..., n$, for any pair $(k,m) \in N_0^2, N\{1, 2, ...\}, N_0 = \{0\} \bigcup N$, be n independent sequences of random variables and $\{\nu_{ik}, k \in N_0\}, i = 1, 2, ..., n$, be n sequences of (not necessarily independent) random variables taking values $0, 1, ...$ and independent of family $\{\xi_{ij}(k,m)\}$. We consider the family of random vectors

$$\mathbf{W}(k,m) = (W_1(k,m), ..., W_n(k,m)), W_i(k,m) = \sum_{j=1}^{\nu_{ik}} \xi_{ij}(k,m). \qquad (4)$$

Assume that for any fixed k, m and i the variables $\xi_{ij}(k,m), j = 1, 2, ...$ are independent and identically distributed Bernoulli random variables with parameter $P_{km}^{(i)}$ (i.e. have distribution $b(1, P_{km}^{(i)})$).

We shall study the asymptotic behavior of $\mathbf{W}(k,m)$ as $k, m \to \infty$ under some assumptions on random variables ν_{ik} and $\xi_{ij}(k,m)$ in different cases of relationship between parameters k and m.

Random sums of independent random variables or random vectors have been considered by many authors. First, it is because of the interest in extending classic limit theorems of the probability theory to a more general situation and to discover new properties of the random sums caused by "randomness" of the number of summands. On the other hand many problems in different areas of probability can be connected with a sum of random number of random variables. Ruther full list of publications on random sums can be found in recent monograph by Gnedenko and Korolev (1996).

The relationship between random sums and branching stochastic processes is well known. Starting from early studies (see Harris (1966), for example) including the recent publications the fact that the number of particles in a model of branching process can be represented as a random sum have been mentioned. Some of investigations show that using this relationship in study of branching models makes possible to investigate new variables related to the genealogy of the process, to study more general modifications of branching processes and to consider different characteristics of the process from a unique point of view. This kind problems are systematically studied in the mentioned above monograph [13].

First theorem concerning the vector (4) includes the case when normalized vector $\boldsymbol{\nu}_k = (\nu_{ik}, i = 1, ..., n)$ has a limit distribution. Namely we assume that there exists a sequence of positive vectors $\mathbf{A}_k = (A_{ik}, i = 1, ..., n)$ such that $A_{ik} \to 0, k \to \infty$ and

$$\{\mathbf{A_k} \otimes \boldsymbol{\nu}_k | \boldsymbol{\nu}_k \neq \mathbf{0}\} \to \mathbf{Y} = (Y_1, ..., Y_n) \tag{5}$$

in distribution and for the vector of expectations $\mathbf{E}\boldsymbol{\xi}(k, m) = (E\xi_{ij}(k, m), i = 1, ..., n)$

$$\frac{\mathbf{E}\boldsymbol{\xi}(k, m)}{\mathbf{A}_k} \to \mathbf{a} = (a_1, ..., a_n). \tag{6}$$

Theorem 4. *If conditions (5) and (6) are satisfied, then*

$$E[\prod_{i=1}^{n} S_i^{W_i(k,m)} | \boldsymbol{\nu}_k \neq \mathbf{0}] \to \varphi(\mathbf{a} \otimes (\mathbf{1} - \mathbf{S}))$$

for any $\mathbf{0} < \mathbf{S} < \mathbf{1}$*, where* $\varphi(\boldsymbol{\lambda})$ *is the Laplace transform of the vector* $\mathbf{Y} = (Y_1, ..., Y_n)$*, i. e.* $\varphi(\boldsymbol{\lambda}) = E\mathbf{e}^{-(\boldsymbol{\lambda}, \mathbf{Y})}, \boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n) \in R_+^n$.

Proof. Since variables $\xi_{ij}(k, m), j = 1, 2, ...$ are independent and identically distributed, by total probability arguments we find for any $\mathbf{S} = (S_1, ..., S_n), \mathbf{0} < \mathbf{S} < \mathbf{1}$

$$E[\prod_{i=1}^{n} S_i^{W_i(k,m)}] = E[E[\prod_{i=1}^{n} \prod_{j=1}^{\nu_{ik}} S_i^{W_{ij}(k,m)} | \boldsymbol{\nu}_k]] = F(k, \mathbf{G}(k, m, \mathbf{S})), \tag{7}$$

where

$$\mathbf{G}(k, m, \mathbf{S}) = (G_i(k, m, S_i), i = 1, ..., n), G_i(k, m, S_i) = ES_i^{\xi_{ij}(k,m)},$$

and $F(k, \mathbf{S})$ is the probability generating function of the vector $\boldsymbol{\nu}_k$. Note here that, since $\xi_{ij}(k, m)$ are Bernoulli random variables with parameter $P_{km}^{(i)}$,

$$G_i(k, m, S_i) = 1 - P_{km}^i(1 - S_i). \tag{8}$$

It follows from condition (5) that for any $\mathbf{0} < \mathbf{S} < \mathbf{1}$

$$\frac{1 - F(k, \mathbf{e}^{-\mathbf{A}_k \otimes \boldsymbol{\lambda}_0})}{P\{\boldsymbol{\nu}_k \neq \mathbf{0}\}} \to 1 - \varphi(\boldsymbol{\lambda}_0), \tag{9}$$

where $\boldsymbol{\lambda}_0 = \mathbf{a} \otimes (\mathbf{1} - \mathbf{S})$.

Now we consider

$$\varepsilon(k, m, \mathbf{S}) = \frac{F(k, \mathbf{G}(k, m, \mathbf{S})) - F(k, \mathbf{e}^{-\mathbf{A}_k \otimes \boldsymbol{\lambda}_0})}{P\{\boldsymbol{\nu}_k \neq \mathbf{0}\}} = E[B(k, m, \boldsymbol{\nu}_k)|\boldsymbol{\nu}_k \neq \mathbf{0}], \tag{10}$$

where

$$B(k, m, \boldsymbol{\nu}_k) = \prod_{i=1}^{n} G_i^{\nu_{ik}}(k, m, S_i) - \prod_{i=1}^{n} e^{-\nu_{ik} A_{ik} a_i (1 - S_i)}.$$

Let Δ be a positive number. Introducing the event

$$C(\Delta, \boldsymbol{\nu}_k) = \{\nu_{ik} A_{ik} < \Delta, i = 1, ..., n\}$$

we write $\varepsilon(k, m, \mathbf{S})$ as

$$\varepsilon(k, m, \mathbf{S}) = E[B(k, m, \mathbf{S})\chi|\boldsymbol{\nu}_k \neq \mathbf{0}] + E[B(k, m, \mathbf{S})(1 - \chi)|\boldsymbol{\nu}_k \neq \mathbf{0}] \tag{11}$$

where $\chi = \chi\{C(\Delta, \boldsymbol{\nu}_k)\}$. If we use inequality

$$|\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i| \leq \sum_{i=1}^{n} |a_i - b_i|, \tag{12}$$

which holds for any sets of numbers a_i, b_i such, that $|a_i| \leq 1, |b_i| \leq 1, i = 1, ..., n$, we obtain that the absolute value of the first expectation on the right side of (11) is not greater than

$$\sum_{i=1}^{n} E[|G_i^{\nu_{ik}} - e^{-\nu_{ik} A_{ik} a_i (1 - S_i)}|\chi|\boldsymbol{\nu}_k \neq \mathbf{0}] = \sum_{i=1}^{n} E[|\exp\{\nu_{ik} A_{ik} \delta_i(k, m)\} - 1|\chi|\boldsymbol{\nu}_k \neq \mathbf{0}],$$

where

$$\delta_i(k,m) = \frac{\ln G_i(k,m,S_i)}{A_{ik}} + a_i(1-S_i).$$

Taking into account the definition of event $C(\Delta, \boldsymbol{\nu}_k)$ we obtain that the last sum can be estimated by

$$\sum_{i=1}^{n} \max_{l\in D} |\exp\{lA_{ik}\delta_i(k,m)\} - 1|, \tag{13}$$

where $D = \{l \in N_0 : lA_{ik} < \Delta\}$. It follows from (8) that $1-G_i(k,m,\mathbf{S}_i) \to 0$ (since $A_{ik} \to 0$). Therefore $\ln G_i \sim -(1-G_i)$ and we conclude from condition (6) that $\delta_i(k,m) \to 0, i = 1,...,n$. Hence the first expectation on the right side of (11) tends to zero.

Now we consider the second term. Since $|B(k,m,\boldsymbol{\nu}_k)| \leq 1$ for all sample points, we obtain that the absolute value of the second expectation is not greater than

$$1 - P\{C(\Delta,\boldsymbol{\nu}_k)|\boldsymbol{\nu}_k \neq \mathbf{0}\}$$

which, according to the condition (5) and definition of $C(\Delta, \boldsymbol{\nu}_k)$, tends to

$$1 - P\{Y_1 \leq \Delta, Y_2 \leq \Delta, ..., Y_n \leq \Delta\}.$$

This estimation shows that the second expectation on the right side of (11) can be made arbitrarily small by choosing sufficiently large Δ. Thus we conclude that under conditions (5) and (6) $\varepsilon(k,m,\mathbf{S}) \to 0$. This together with (7), (9) and the fact that

$$E[\prod_{i=1}^{n} S_i^{W_i(k,m)}|\boldsymbol{\nu}_k \neq \mathbf{0}] = 1 - \frac{1 - E[\prod_{i=1}^{n} S_i^{W_i(k,m)}]}{P\{\boldsymbol{\nu}_k \neq \mathbf{0}\}} \tag{14}$$

gives the assertion of the theorem. Theorem 4 is proved.

Now we consider the case, when the limit in condition (6) is not finite. In this case we use the following vector of normalizing constants

$$\mathbf{M}(k,m) = \frac{E\boldsymbol{\xi}(k,m)}{\mathbf{A}_k} = (M_i(k,m), i = 1,...,n).$$

Theorem 5. *If condition (5) is satisfied and $k, m \to \infty$ such that $M_i(k, m) \to \infty, i = 1, 2, ..., n,$ then*

$$E[e^{-(\boldsymbol{\lambda}, \mathbf{V}(k,m))} | \boldsymbol{\nu}_k \neq \mathbf{0}] \to \varphi(\boldsymbol{\lambda}), \tag{15}$$

where $\boldsymbol{\lambda} \in R_+, \mathbf{V}(k, m) = \mathbf{W}(k, m)/\mathbf{M}(k, m)$ and $\varphi(\boldsymbol{\lambda})$ is the Laplace transform of the vector $\mathbf{Y}$.

Proof. The scheme of the proof the same as in the first case, therefore we only provide some important points.

We consider relation (14) with $\mathbf{S} = (S_1, ..., S_n)$ and $S_i = \exp\{-\lambda_i/M_i(k, m)\}, \lambda_i > 0, i = 1, ..., n$. Using (8) we obtain this time that

$$1 - G_i(k, m, S_i) \sim A_{ik}\lambda_i, k, m \to \infty, i = 1, ..., n.$$

Therefore in relation (9) we have $\boldsymbol{\lambda}$ in place of $\boldsymbol{\lambda}_0$. Since $A_{ik} \to 0$, again $\ln G_i \sim -(1 - G_i)$ and

$$\frac{\ln G_i(k, m, S_i)}{A_{ik}} \to -\lambda_i \tag{16}$$

as $k, m \to \infty$.

We consider again $\varepsilon(k, m, \mathbf{S})$ from (10) replacing λ_0 by λ and putting $S_i = \exp\{-\lambda_i/M_i(k, m)\}$. By the same arguments as in the proof of Theorem 1 we obtain that absolute value of $\varepsilon(k, m, \mathbf{S})$ can be estimated by sum (13) with

$$\delta_i(k, m) = \frac{\ln G_i(k, m, S_i)}{A_{ik}} + \lambda_i$$

and $\delta_i(k, m) \to 0, i = 1, ..., n$ due to (16). Hence $\varepsilon(k, m, \mathbf{S}) \to 0$ as $k, m \to \infty$. Again appealing to relation (14) we obtain the assertion of the theorem. Theorem 5 is proved.

The family of vectors (4) is eventually a sum of independent vectors, if vectors $\boldsymbol{\nu}_k = (\nu_{ik}, i = 1, ...n)$ have degenerate distributions. Therefore one may expect to obtain a normal limit distribution under some natural assumptions. The next theorem gives a set of conditions under which the limit of the vector $\mathbf{W}(k, m)$ is a mixture of the normal and some given distribution. Assume that

C1. There exists a sequence of positive vectors $\mathbf{r}_k = (r_{ik}, i = 1, ..., n), k \geq 1$, such that $r_{ik} \to \infty, k \to \infty, i = 1, ..., n$ and

$$P\{\frac{\boldsymbol{\nu}_k}{\mathbf{r}_k} \leq \mathbf{x} | \boldsymbol{\nu}_k \neq \mathbf{0}\} \to T(\mathbf{x}), \mathbf{x} = (x_1, ..., x_n),$$

as $k \to \infty$, where $T(\mathbf{x})$ is a proper n-dimensional distribution function.

C2. For given sequence of positive vectors $\mathbf{r}_k$ there exists a sequence $\mathbf{l}_k = (l_{ik}, i = 1, ..., n), k \geq 1$ such that $r_{ik}/l_{ik} \to \infty, k \to \infty, i = 1, ..., n$ and

$$l_{ik} E\xi_{i1}(k, m)(1 - E\xi_{i1}(k, m)) \to C_i, i = 1, ..., n$$

as $k, m \to \infty$, where $\mathbf{C} = (C_i, i = 1, ..., n)$ is a positive vector of constants.

Theorem 6. *If conditions C1 and C2 are satisfied, then*

$$P\{\frac{W_i(k, m) - \nu_{ik} P_{km}^{(i)}}{\sqrt{r_{ik} C_i / l_{ik}}} \leq x_i, i = 1, ..., n | \nu_k \neq \mathbf{0}\} \to K(x_1, ..., x_n)$$

as $k, m \to \infty$ *for any* $(x_1, ..., x_n) \in R^n$, *where*

$$K(x_1, ..., x_n) = \int_0^\infty ... \int_0^\infty \prod_{i=1}^n \Phi(\frac{x_i}{\sqrt{y_i}}) dT(y_1, ..., y_n)$$

and $\Phi(x)$ *is the standard normal distribution.*

Proof. Let $\mathbf{W}^*(k, m) = (W_i^*(k, m), i = 1, ..., n)$ with

$$W_i^*(k, m) = \sum_{j=1}^{r_{ik}} \xi_{ij}, i = 1, ..., n.$$

In the proof we use the following proposition.

Proposition 2. Assume that there exist sequences $\{l_{ik}, k \geq 1\}, i = 1, ..., n$ for which condition C2 is satisfied. Then for each $i = 1, ..., n$ variable $(W_i^*(k, m) - r_{ik} P_{km}^i)/\sqrt{r_{ik} C_i / l_{ik}}$ is asymptotically normal as $k, m \to \infty$.

Proof. The assertion follows directly from central limit theorem and from trivial identities:

$$EW_i^*(k,m) = r_{ik}P_{km}^{(i)}, varW_i^*(k,m) = r_{ik}P_{km}^{(i)}(1 - P_{km}^{(i)}).$$

Now we continue the proof of Theorem 6. Let $L(k, m, \mathbf{x})$ be the conditional distribution in Theorem 6 and for $i = 1, ..., n$

$$F_i(k,m,t_i,x_i) = P\left\{\frac{V_i(k,m) - t_iP_{km}^{(i)}}{\sqrt{r_{ik}C_i/l_{ik}}} \le x_i\right\},$$

where $t_i \in N_0$ and $V_i(k,m) = \xi_{i1}(k,m) + ... + \xi_{it_i}(k,m)$. Using independence of sequences $\{\xi_{ij}(k,m), j \ge 1\}$ by total probability arguments we obtain for any $q > 0$

$$L(k,m,\mathbf{x}) = \sum_{\mathbf{h}\in N_0^n} \prod_{i=1}^{n} \sum_{t_i\in\Delta} F_i(k,m,t_i,x_i)P_k(\mathbf{t}), \tag{17}$$

where $\mathbf{h} = (h_1, ..., h_n), \mathbf{t} = (t_1, ..., t_n), P_k(\mathbf{t}) = P\{\nu_k = \mathbf{t}|\nu_k \ne \mathbf{0}\}$ and

$$\Delta_i = \{t_i \in N_0 : \frac{h_i}{q} \le \frac{t_i}{r_{ik}} < \frac{h_i + 1}{q}\}.$$

Let now $p > 0$ be such that pq is an integer. We partition the sum on the right side of (17) as

$$L(k,m,\mathbf{x}) = \Sigma' + \Sigma'' = I_1 + I_2, \tag{18}$$

where Σ' is the sum over all vectors $\mathbf{h} \in N_0^n$ such that $h_i \le pq, i = 1, ..., n$ and Σ'' is the sum over all such vectors that at least one of coordinates is greater that pq.

First we consider I_1. Using the monotonicity of the distribution function we obtain the following estimate

$$I_1 \le \Sigma' \prod_{i=1}^{n} \sum_{t_i\in\Delta_i} P\{\frac{V_i(k,m) - t_iP_{km}^{(i)}}{\sqrt{t_iC_i/l_{ik}}} \le x_i\alpha(i,q)\}P_k(\mathbf{t}),$$

where $\alpha(i,q) = \sqrt{q/h_i}$, if $x_i > 0$ and it is equal to $\sqrt{q/(h_i + 1)}$, if $x_i < 0$. We denote $\mathbf{Y} = (Y_1, ..., Y_n)$ a random vector having distribution $T(x_1, ..., x_n)$.

Since $h_i r_{ik}/q \le t_i < (h_i+1)r_{ik}/q$ for $t_i \in \Delta_i$, if $r_{ik} \to \infty$, then so does t_i and $t_i/l_{ik} \to \infty$. Consequently, if we use Proposition 2 and condition C1, we get

$$\limsup_{k,m\to\infty} I_1 \le \Sigma' \prod_{i=1}^{n} \Phi(x_i\alpha(i,q))P\{\frac{h_i}{q} \le Y_i < \frac{h_i+1}{q}, i=1,...,n\}. \tag{19}$$

Repeating similar arguments we obtain that

$$\liminf_{k,m\to\infty} I_1 \ge \Sigma' \prod_{i=1}^{n} \Phi(x_i\beta(i,q))P\{\frac{h_i}{q} \le Y_i < \frac{h_i+1}{q}, i=1,...,n\}, \tag{20}$$

where $\beta(i,q) = \sqrt{q/(h_i+1)}$,if $x_i > 0$ and it is equal to $\sqrt{q/h_i}$ otherwise. Since for each fixed p and $q \to \infty$ right sides of (19) and (20) have the same limit, we conclude that

$$\lim_{k,m\to\infty} I_1 = \int_0^p \cdots \int_0^p \prod_{i=1}^{n} \Phi(\frac{x_i}{\sqrt{y_i}})dT(y_1,...,y_n). \tag{21}$$

Now we consider I_2. Recall that Σ'' is the sum over all vectors $\mathbf{h} \in N_0^n$ such that at least one of coordinates is greater than pq. Let h_j be the coordinate of $\mathbf{h}$ which is greater than pq. Then it is not difficult to see that

$$I_2 \le \sum_{h_j=pq+1}^{\infty} P\{\frac{h_j}{q} \le \frac{\nu_{jk}}{r_{jk}} < \frac{h_j+1}{q} | \boldsymbol{\nu}_k \ne \mathbf{0}\} \le P\{\frac{\nu_{jk}}{r_{jk}} > p | \boldsymbol{\nu}_k \ne \mathbf{0}\}.$$

From here due to condition C1 we obtain that

$$\limsup_{k,m\to\infty} I_2 \le 1 - T(p), \tag{22}$$

where $T(p) = P\{Y_i < \infty, i \ne j, Y_j \le p\}$. It is clear that the difference on the right side of (22) can be made arbitrarily small by choosing p sufficiently large. Therefore $I_2 \to 0$ as $k,m \to \infty$. Theorem 6 is proved.

4 Proofs of theorems of Part 2

Proof of Theorem 1. It follows from the definition of $\mathbf{X}(\tau,t)$ in introductory part that its i-th component can be written as

$$X_i(\tau,t) = \sum_{\boldsymbol{\alpha}\in R_\tau^{(i)}} \chi(\bigcap_{j\in\nabla} \{X_j^{(\boldsymbol{\alpha})}(t-\tau) > \theta_j(t-\tau)\}). \tag{23}$$

Since $card\{R_\tau^{(i)}\} = X_i(\tau)$, from here we can see that it can be presented in the form (4) with $\nu_{i\tau} = X_i(\tau)$ and

$$\xi_{ij}(\tau,t) = \chi(\bigcap_{l\in\nabla}\{X_{il}^j(t-\tau) > \theta_l(t-\tau)\}), \tag{24}$$

where $X_{il}^j(t)$ is the number of individuals of type l at time t in the process initiated by j-th individual of type i. Hence Theorem 4 can be applied. It follows from Proposition 1 that condition (5) satisfied with $\mathbf{A}_\tau = q(\tau)\mathbf{1}/\mathbf{V}$ and with

$$\varphi(\boldsymbol{\lambda}) = Ee^{-(\boldsymbol{\lambda},\mathbf{Y})} = \phi(\boldsymbol{\lambda}),$$

where $\phi(\boldsymbol{\lambda})$ is defined in (2). Further it is not difficult to see that

$$E\xi_{ij}(\tau,t) = P(\bigcap_{l\in\nabla}\{X_{il}^j(t-\tau) > \theta_l(t-\tau)\})$$

$$= P(\bigcap_{l\in\nabla}\{X_{il}^j(t-\tau)q(t-\tau) > \theta_l v_l\}, \mathbf{X}(t-\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i).$$

Now we put

$$P^*(A) = P(A|\mathbf{X}(t-\tau) \neq \mathbf{0}, \mathbf{X}(0) = \boldsymbol{\delta}_i), A_l = \{X_{il}^j(t-\tau) \leq \theta_l(t-\tau)\}.$$

It follows from simple identity about the probability of union of several events that

$$P^*(\bigcup_{l\in\nabla} A_l) = \Sigma' P^*(A_l) - \Sigma'_{l\neq m} P^*(A_l \bigcap A_m) + ... + (-1)^{|\nabla|-1} P^*(\bigcap_{l\in\nabla} A_l).$$

We remaind here that $|\nabla|$ is the number of elements of the set ∇ and $\sum'$ is summation over the values of l and m in ∇. It follows from Proposition 1 that for any set $(i_1, ..., i_m), m = 1, 2, ..., n$

$$P^*(\bigcap_{l=1}^m A_{i_l}) \to \pi(\Delta_{i_1...i_m}(\boldsymbol{\theta})), \boldsymbol{\theta} = (\theta_1, ...\theta_n). \tag{25}$$

If we take into account this fact, obtain from the above equality that

$$P^*(\bigcup_{l\in\nabla} A_l) \to L$$

as $t-\tau \to \infty$ where L is defined just before Theorem 1. Thus we obtain

$$E\xi_{ij}(\tau,t) \sim (1-L)Q^i(t-\tau), t-\tau \to \infty.$$

Since $\mathbf{A}_\tau = q(\tau)\mathbf{1}/\mathbf{V}$ using asymptotic behavior of $\mathbf{Q}(t)$ again, we obtain that when $\tau, t \to \infty, t-\tau \to \infty$

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \sim (1-L)v_i\frac{Q^i(t-\tau)}{q(\tau)}.$$

From here, taking into account condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$, we conclude that

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \to (1-L)C_i u_i v_i,$$

which shows that condition (6) of Theorem 4 is also satisfied with $\mathbf{a} = (1-L)\mathbf{C}\otimes\mathbf{U}\otimes\mathbf{V}$. Consequently the assertion of Theorem 1 follows from Theorem 4. Theorem 1 is proved.

Proof of Theorem 2. We now use Theorem 5. As it was shown in the proof of Theorem 1, condition (5) of Theorem 5 is satisfied with $\mathbf{A}_\tau = q(\tau)\mathbf{1}/\mathbf{V}$. Now we consider

$$M_i(\tau,t) = \frac{E\xi_{ij}(\tau,t)}{q(\tau)}v_i,$$

where $\xi_{ij}(\tau,t)$ is the same as in (24). Appealing again to asymptotic behavior of $\mathbf{Q}(t)$, we obtain that

$$M_i(\tau,t) \sim (1-L)v_i T_i(\tau,t)\frac{Q^i(\tau)}{q(\tau)} \tag{26}$$

as $t,\tau \to \infty, t-\tau \to \infty$. It follows from (26) that condition $M_i(\tau,t) \to \infty$ of Theorem 5 is also satisfied when $T_i(\tau,t) = Q^i(t-\tau)/Q^i(\tau) \to \infty$. The assertion of Theorem 2 follows now from Theorem 5. Theorem 2 is proved.

Proof of Theorem 3. We use again Theorem 5. As in the proofs of the preceding theorems condition (5) follows from Proposition 1. If $t-\tau \in (0,\infty)$, we obtain from (24) that

$$M_i(\tau,t) = \frac{R_i(\Delta)}{q(\tau)}v_i.$$

Thus we have that $M_i(\tau,t) \to \infty$ as $t, \tau \to \infty, t-\tau \in (0,\infty)$. We obtain from Theorem 5 that

$$E[\prod_{i=1}^{n} e^{-\lambda_i X_i(\tau,t)Q^i(\tau)} | \mathbf{X}(\tau) \neq \mathbf{0}] \to \varphi(\boldsymbol{\lambda} \otimes \mathbf{R}(\Delta)).$$

This yields the assertion of Theorem 3. Theorem 3 is proved.

5 The number of productive ancestors

Now we consider a population containing at time $t=0$ a random number $\nu_i(t), i=1,...,n, t \in N_0$ individuals (ancestors) of n different types $1,2,...,n$. Each of these individuals generates a discrete time indecomposable n type branching stochastic process. Let $\boldsymbol{\theta}(t) = (\theta_1(t),...,\theta_n(t))$ be a vector of non-negative functions, ∇ be a subset of $\{1,2,...,n\}$ as it was defined in the introductory part. In how many processes generated by these ancestors the number of descendants at time t of types $j, j \in \nabla$ will exceed the corresponding level given by $\boldsymbol{\theta}(t)$? To answer the question we investigate process $\mathbf{Y}(t) = \mathbf{Y}([\boldsymbol{\theta}],t) = (Y_1(t),...,Y_n(t))$, where $Y_i(t)$ is the number of initial individuals of type i whose number of descendants at time t of types $j, j \in \nabla$ is greater than corresponding component of the vector $\boldsymbol{\theta}(t)$. It is clear that $\mathbf{Y}(t)$ takes into account only "relatively productive" ancestors regulated by family of levels $\boldsymbol{\theta}(t), t \in N_0$ and set of types ∇.

Process $\mathbf{Y}(t)$ may be associated with the following scheme describing growth of n-type trees in a forest. Suppose at time zero we have $\nu_i(t), i = 1,...,n$, one branch trees of types i. Each of these trees will grow and give new branches of types $1,2,...,n$ according to independent, indecomposable n-type branching processes. Then process $\mathbf{Y}(t) = (Y_1(t),...,Y_n(t))$ will count the number of "big trees": $Y_i(t)$ is the number of big trees of type i having more than $\theta_j(t)$ new branches at time t for $j \in \nabla$. When $\nabla = \{1,2,...,n\}$ process $\mathbf{Y}(t)$ counts "biggest" trees having large number of branches of all types.

It is not difficult to see that the components of the process $Y_i(t)$ can be presented as

$$Y_i(t) = \sum_{j=1}^{\nu_i(t)} \xi_{ij}(t), \tag{27}$$

where $\xi_{ij}(t) = \chi(\bigcap_{l\in\nabla}\{X_{il}^j(t) > \theta_l(t)\})$ and $X_{il}^j(t)$ is, as before, the number of individuals of type l at time t in the process initiated by j-th ancestor of type i. Consequently theorems proved for random sum (4) may be applied to this process.

Let all assumptions from Part 2 on n-type branching process $\mathbf{X}(t), t \in N_0$ be satisfied and the generating function corresponding to probability distribution P_α^i, $\boldsymbol{\alpha} \in N_0^n$ satisfies equation (1).

Theorem 7. *Let condition (1) be satisfied and* $\boldsymbol{\theta}(t) = \boldsymbol{\theta}\otimes\mathbf{V}/q(t), \theta \in R_+^n$. *If condition C1 is satisfied and for the normalizing coefficients in C1*

$$r_{it}Q^i(t) \to \infty \tag{28}$$

as $t \to \infty$ *for* $i = 1, ..., n$, *then*

$$P\{\frac{Y_i(t) - \nu_{it}a_i(t)}{\sqrt{\nu_{it}a_i(t)}} \le x_i, i = 1, ..., n|\boldsymbol{\nu}_t \ne \mathbf{0}\} \to K(\mathbf{x}),$$

where $\mathbf{x} \in R^n, \boldsymbol{\nu}_t = (\nu_i(t), i = 1, ..., n), a_i(t) = (1-L)Q^i(t)$, *and* $K(\mathbf{x})$ *defined in Theorem 6.*

Proof. We demonstrate that conditions of Theorem 6 are satisfied. It is clear that we just need to show that condition C2 holds for the variables defined in (27). As in the proof of Theorem 1 we easily obtain that

$$E\xi_{ij}(t) = P(\bigcap_{l\in\nabla}\{X_{il}^j(t) > \theta_l(t)\}) =$$

$$P(\bigcap_{l\in\nabla}\{X_{il}^j(t)q(t) > \theta_l v_l\}, \mathbf{X}(t) \ne \mathbf{0}) \sim (1-L)Q^i(t).$$

Consequently, if we take $l_{it} = 1/Q^i(t)$, then for $i = 1, ..., n$ as $t \to \infty$

$$l_{it}E\xi_{ij}(t)(1 - E\xi_{ij}(t)) \to 1 - L.$$

On the other hand $r_{it}/l_{it} \to \infty, i = 1, ..., n$ as $t \to \infty$ due to condition (28). Hence condition C2 of Theorem 6 is satisfied and assertion of the theorem follows from Theorem 6.

In conclusion of this section we note that according to condition (28) the assertion of Theorem 7 holds when the initial population is large enough.

One may obtain limit distributions for $\mathbf{Y}(t)$ when this condition is not satisfied. To do it one needs to apply theorems 4 and 5 instead of Theorem 6.

Concluding remarks. In this paper we considered the subpopulation of productive individuals in n-type Galton-Watson processes. It is not difficult to see that the methods developed here could be used in study of more general models of branching processes, such as continuous time Markov branching processes, Bellman-Harris processes, the Sevastyanov model and Crump-Mode-Jagers processes. For this purpose one needs slightly modify theorems from Section 3 and use known limit theorems for one or another model of branching processes. Considering a subpopulation of "unproductive" individuals in the population seems to be another direction of investigation of the genealogy of branching processes.

Acknowledgment. This research project has been funded by King Fahd University of Petroleum and Minerals under Project No MS/STOCHASTIC/254. The author thanks a referee for careful reading and for suggestions which improved presentation of the paper.

REFERENCES

[1]. Arnold, B. C. and Villasenor, J. A. (1996). The tallest man in the world. In "Statistical Theory and Applications": Papers in Honor of Herbert A. David, ed. H. N. Nagaraja, P. Sen, D. Morrison. Springer, Berlin, 81–88.
[2]. Athreya, K.B.(1988) On the maximum sequence in a critical branching process, Ann. Prob., 16, 502–507.
[3]. Borovkov, K. A. and Vatutin, V. A. (1996). On distribution tails and expectations of maxima in critical branching processes, J. Appl. Prob., 33, 614–622.
[4]. Fleischmann, K. and Siegmund-Schultze, R. (1977). The structure of reduced critical Galton-Watson processes, Math. Nachr. 79, 233–241.
[5].Gnedenko, B.V., Korolev, V. Yu. (1996). Random Summation. CRC Press, 267 p.
[6].T.E. Harris (1966). The theory of branching processes, Springer Verlag, New York.
[7]. Johnson, N. L., Kotz,S., Balakrishnan, N. (1997). Discrete Multivariate Distributions, Wiley Series in Probability and Statistics, New York, 299 pp.

[8]. Pakes, A. G. (1987). Remarks on the maxima of a martingale sequence with applications to the simple critical branching processes, J. Appl. Prob. 24, 768-772.
[9].Pakes, A. G. (1988a). A limit theorem for the maxima of the para-critical simple branching process. Adv. Appl. Prob. 30, 740–756.
[10]. Pakes, A. G. (1988b). Extreme order statistics on Galton-Watson trees. Metrika, 47, 95–117.
[11].Rahimov, I.(1987). Limit theorems for random sums of dependent indicators and their applications, Theory Probab. Appl., 32, 317–326.
[12]. Rahimov, I. (1989). Asymptotic behavior of families of particles in branching random processes, Soviet Math. Dokl., 39, 322–325.
[13]. Rahimov, I. (1995). Random Sums and Branching Stochastic Processes, Springer, LNS 96, New York.
[14]. Rahimov, I. and Yanev, G. P. (1999). On maximum family size in branching processes. J. Appl. Prob. 36, 632–643.
[15]. Rahimov, I. and Hasan, H. (1998). Limit theorems for exceedances of a sequence of branching processes. Bull. Malaysian. Math. Soc. (Second ser.) 21, 37–46.
[16]. Rahimov, I. (2001). Approximation of exceedance processes in large populations. Stochastic Models, 17, 147–156.
[17]. Sagitov, S. M. (1985). Reduced critical Bellman-Harris branching processes with several types of particles. Theory Probab. Appl. 30, 737–749.
[18]. Seneta, E. (1985). Regularly Varying Functions, Nauka, Moscow.
[19].Topchij V. A. and Vatutin,V. A. (1997). Left-continuous random walk and the maximum of the critical Galton-Watson processes. Theory Probab. Appl., 42,17-27.
[20]. Vatutin V. A. (1977). Limit theorems for critical Markov branching processes with several types of particles and infinite second moments. Math. USSR- Sbornik, 32, 215-225.
[21].Yakymov, A. L. (1980). Reduced branching processes. Theory Probab. Appl., 25, 593–596.
[22]. Yakymov, A. L. (1985). Asymptotic properties of subcritical and supercritical reduced branching processes. Theory Probab. Appl., 30, 183–188.
[23]. Yanev G. P. and Tsokos, Ch. (2000). Family size order statistics in branching processes with immigration, Stochastic Anal. Appl., 18, 655–670.
[24]. Zubkov, A. M. (1975). Limit distribution of the distance up to closest common ancestor, Theory Probab. Appl., 20, 614–623.