Maximal number of direct offspring in simple branching processes

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Abstract

The number Y_n of offspring of the most productive particle in the nth generation of a Bienaymé-Galton-Watson process is considered. The asymptotic behaviour of Y_n as $n \to \infty$ may be viewed as an extreme value problem for i.i.d. random variables with random sample size. Limit theorems when the offspring mean is finite are proved using some convergence results for branching processes as well as transfer theorems for maxima.

Key words: Bienaymé-Galton-Watson branching process; max-stability; max-semistability; maximum with random sample size; transfer theorems.
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1 Introduction.

Let $\{Z_n\}$ be a Bienaymé-Galton-Watson process which can be defined by the reccurence

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n), \qquad n = 1, 2, \dots; \quad Z_0 \equiv 1,$$

where $\{X_i(n)\}$, $i, n = 1, 2, \dots$ are nonnegative, independent and identicaly distributed, integer-valued random variables.

Denote by $f(s) = Es^{X_i(n)}$ the offspring generating function and by $f_n(s)$ the nth iterate of $f(s)$ i.e. $f_n(s) = f(f_{n-1}(s))$, $n = 1, 2, \dots,$ $f_0(s) = s$, $0 \le s \le 1$. Additionally let $F(x) = P(X_i(n) \le x)$ be the distribution function of the 'offspring variable' which has mean $0 < m < \infty$ and variance $0 < \sigma^2 \le \infty$.

Define

$$Y_n = \max_{1 \le i \le Z_{n-1}} X_i(n), \quad n = 1, 2, \dots; \quad Y_0 \equiv 1 .$$

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It is clear that this definition is equivalent to

$$P(Y_n \le x) = \sum_{k=0}^{\infty} P(Z_{n-1} = k) F^k(x) = f_{n-1}(F(x)). \tag{1.1}$$

The study of the sequence $\{Y_n\}$ might be motivated in different ways. There have been several recent works developing results for certain kinds of extremes in branching processes, and investigating Y_n is perhaps plausible as a contribution to this program. Alternatively, a natural interpretation within the demografical framework, for example, may be given. Indeed, the random variable under question is the number of offspring in families having the largest numbers of children. Thus the asymptotic behaviour of Y_n provides some information about the influence of offspring of these families on the size of whole generation.

2 Transfer limit theorems.

Recall that a nondegenerate distribution function $H(s)$ is max-stable iff for a distribution function $F(x)$ there exist sequences of real numbers $\{a_n\}_1^\infty$, $(a_n > 0)$ and $\{b_n\}_1^\infty$ such that

$$F^n(a_n x + b_n) \to H(x) \ , \tag{2.1}$$

weakly as $n \to \infty$. If (**??**) holds then $F(x)$ is said to belong to the domain of attraction of $H(x)$; in our notation, $F \in MSD(H)$. It is well-known that, according to the classical Gnedenko's result, $H(x) = \exp\{-h(x)\}$, say, is of the type of one of the following three classes:

$$\begin{cases} (i) & h(x) = (-x)^a & \textit{for } x \in (-\infty, 0) \ , & = 1 & \textit{for } x \in [0, \infty) \ , \\ (ii) & h(x) = x^{-a} & \textit{for } x \in (0, \infty) \ , & = 0 & \textit{for } x \in (-\infty, 0] \ , \\ (iii) & h(x) = \exp\{-x\} & \textit{for } x \in (-\infty, \infty) \ , & & \end{cases} \tag{2.2}$$

where $a > 0$.

Further we will also require the following extention of the class of max-stable distributions. A nondegenerate distribution function $G(s)$ is max-semistable (under linear transformation) iff for a distribution function $F(x)$ there exist sequences of real numbers $\{c_k\}_1^\infty$, $(c_k > 0)$ and $\{d_k\}_1^\infty$ such that

$$F^k(c_k x + d_k) \to G(x) \ , \tag{2.3}$$

weakly as $k \to \infty$, where k runs over some sequence of positive integers $k(1) < k(2) < \dots$ subject to the condition

$$\lim_{n \to \infty} \frac{k(n+1)}{k(n)} = r \ge 1 \ . \tag{2.4}$$

The case $r = 1$ corresponds to max-stable laws. If (**??**) with (**??**) holds, then $F(x)$ is said to belong to the domain of attraction of $G(x)$; in our notation, $F \in MSSD(G)$. By Theorem 2 in Grinevich (1992), $G(x) = \exp\{-g(x)\}$, say, is of the type of one of the following three classes:

$$\begin{cases} (i) & g(x) = (u - x)^{\beta} \pi(\ln(u - x)) & \textit{for } x \in (-\infty, u) \ , \\ (ii) & g(x) = (x - u)^{-\beta} \pi(\ln(x - u)) \ , & \textit{for } x \in (u, \infty) \ , \\ (iii) & g(x) = \exp\{-\beta x\} \pi(x) \ , & \textit{for } x \in (-\infty, \infty) \ , \end{cases} \tag{2.5}$$

for $u \in R$, $\beta = | c \ln r |$, where c is certain constant, and $\pi(x)$ are peroidic positive and bounded functions satisfying certain conditions (see Grinevich (1992), thm 2). Necessary and sufficient conditions for $F \in MSSD(G)$ are obtained by Grinevich (1993).

For convenience we shall give here a transfer limit theorem for maximum with random sample size (see Galambos (1987), thm 6.2.2 and Gnedenko and Gnedenko (1982)). Let us have the following three sequences:

(a) $\{\xi_i(n)\}$ - independent and identically distributed for any fixed n random variables;

(b) $\{\nu(n)\}$ - nonnegative integer-valued random variables;

(c) $\{i(n)\}$ - positive integers such that $i(n) \to \infty$ as $n \to \infty$.

Assume that $\nu(n)$ are independent of $\xi_i(n)$, $k = 1, 2, \ldots$ for any fixed n.

Theorem 2.1 *Assume that for* $x \in R$,

$$\lim_{n\to\infty} P(\max_{1\le i\le i(n)} \xi_i(n) \le x) = \Phi(x)$$

and for $x > 0$,

$$\lim_{n\to\infty} P(\frac{\nu(n)}{i(n)} \le x) = A(x) \ ,$$

where $\Phi(x)$ *and* $A(x)$ *are distribution functions. Then for* $x \in R$,

$$\lim_{n\to\infty} P(\max_{1\le i\le \nu(n)} \xi_i(n) \le x) = \int_0^\infty (\Phi(x))^y dA(y) \ .$$

We shall prove below a theorem which seems to be new and of interest itself. Let η_i, $i = 1, 2, \ldots$ be i.i.d. random variables having a common distribution function $F(x)$ and let $\nu(n)$ be a positive integer-valued random variable, independent of the η_i's. Finally, assume that $r(n)R$ is a function, tending to ∞ with n. Further on we will denote by $L(x)$ and $L_1(x)$ certain slowly varying functions (s.v.f.) as $x \to \infty$ and by $[a]$ the integral part of a.

Theorem 2.2 *Assume that there exists a sequence* $\{a_n\}_1^\infty$, $(a_n > 0)$ *such that for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le n} \eta_i}{a_n} \le x\right) = \exp\{-x^{-a}\} \qquad a > 0 \ , \tag{2.6}$$

and there exists ν, *such that*

$$\lim_{n\to\infty} \frac{\nu(n)}{r(n)} = \nu \tag{2.7}$$

where convergence is in distribution, and let $\varphi(u) = E\exp\{-u\nu\}$, $u > 0$. *Then for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i}{(r(n))^{1/a} L_1(r(n))} \le x\right) = \varphi(x^{-a}) \ , \tag{2.8}$$

where $L_1(x/L(x))/L(x) \to 1$ *as* $x \to \infty$, *for* $L(x)$ *defined by* (**??**).

Proof. It is well-known (see e.g. Resnick (1987), prop. 1.11, p.54) that $F \in MSD(\exp\{-x^{-a}\})$, $a > 0$ if and only if (**??**) holds. In this case for $x > 0$,

$$1 - F(x) = x^{-a}L(x) \ , \quad L(x) - \text{s.v.f. as } x \to \infty \ . \tag{2.9}$$

With the convention that the infinium of an empty set is equal to $+\infty$, we define the (left continuous generalized) inverse $F^{\leftarrow} : R \to R$ of F by

$$F^{\leftarrow}(y) := \inf\{x \in R : \ F(x) \geq y\} \ .$$

Denote $U(x) = 1/(1 - F(x))$ and $d_n = U^{\leftarrow}(r(n))$. We shall prove, repeating the arguments by Resnick (1987), example after prop.0.4, p.15, that as $n \to \infty$,

$$U(d_n) = 1/(1 - F(d_n)) \sim r(n) \ . \tag{2.10}$$

Indeed, by the definition of $U^{\leftarrow}$ it follows that $z < U^{\leftarrow}(r(n))$ iff $U(z) < r(n)$. Setting $z = U^{\leftarrow}(r(n))(1 - \varepsilon)$ and then $z = U^{\leftarrow}(r(n))(1 + \varepsilon)$ we obtain

$$\frac{U(U^{\leftarrow}(r(n)))}{U(U^{\leftarrow}(r(n))(1+\varepsilon))} \leq \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \leq \frac{U(U^{\leftarrow}(r(n)))}{U(U^{\leftarrow}(r(n))(1-\varepsilon))} \ .$$

Remembering $U(x) \sim x^a/L(x)$ as $x \to \infty$. Thus

$$(1+\varepsilon)^{-a} \leq \liminf_{n\to\infty} \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \leq \limsup_{n\to\infty} \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \leq (1-\varepsilon)^{-a}$$

and (**??**) follows since $\varepsilon > 0$ is arbitrary.

Set $x_0 = \sup\{x : F(x) < 1\}$. If (**??**) holds then it is not difficult to get $x_0 = \infty$ (see also Resnick (1987), example after prop.0.4, p.15). Hence from (**??**), $d_n \to \infty$ as $n \to \infty$ and now from (**??**) we obtain for any $x > 0$,

$$\lim_{n\to\infty} \frac{1 - F(d_n x)}{(r(n))^{-1}} = \lim_{n\to\infty} \frac{1 - F(d_n x)}{1 - F(d_n)} = x^{-a} \ . \tag{2.11}$$

On the other hand, for $x > 0$,

$$P(\max_{1\leq i\leq \nu(n)} \eta_i \leq x) = v_n(F(x)) \ , \tag{2.12}$$

where $v_n(s) = Es^{\nu(n)}$ and by (**??**),

$$\lim_{n\to\infty} v_n(\exp\{-u(r(n))^{-1}\}) = \varphi(u) \ , \qquad u > 0 \ . \tag{2.13}$$

Therefore, by (**??**) - (**??**) for $x > 0$,

$$\begin{aligned} P(\max_{1\leq i\leq \nu(n)} \eta_i \leq d_n x) &= v_n(\exp\{\ln F(d_n x)\}) \\ &= v_n(\exp\{-(1 - F(d_n x))(1 + o(1))\}) \\ &= v_n\left(\exp\{-x^{-a}(r(n))^{-1}(1 + o(1))\}\right) \\ &\to \varphi(x^{-a}) \ , \end{aligned} \tag{2.14}$$

as $n \to \infty$.

Furthermore, since $U(x) \sim x^a/L(x)$ as $x \to \infty$ we get (cf. Resnick (1987), prop. 0.8(v), p.23) as $n \to \infty$,

(2.15) $$d_n = U^{\leftarrow}(r(n)) \sim (r(n))^{1/a} L_1(r(n)) \ ,$$

where (cf. Seneta (1976), lemma 1.10, p.27) $L_1(x/L(x))/L(x) \to 1$ as $x \to \infty$.

Finally, from (**??**) and (**??**), using the continuouty of $\varphi(x)$, we see that

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i}{(r(n))^{1/a}L_1(r(n))} \le \frac{d_n x}{(r(n))^{1/a}L_1(r(n))} \right)$$

$$= \lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i}{(r(n))^{1/a}L_1(r(n))} \le x \right) = \varphi(x^{-a}) \ ,$$

which completes the proof of the theorem.

Corollary 2.1 *Under the assumptions of Theorem 2.2, if*

(2.16) $$r(n) = n^b (N(n))^{-1}, \qquad b > 0 \ ,$$

where $N(x)$ *is a s.v.f. as* $x \to \infty$*, then for* $x > 0$*,*

(2.17) $$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i}{a_n^b K(n)} \le x \right) = \varphi(x^{-a}) \ ,$$

where $K(n) = L_1(n^b/N(n))/\left(L_1(n)(N(n))^{1/a}\right)$ *and* $L_1(x/L(x))/L(x) \to 1$ *as* $x \to \infty$*, for* $L(x)$ *defined by (**??**).*

Proof. The statement follows immediately from Theorem 2.2, noting that (see e.g. Resnick (1987), prop.1.11, p.54)

$$a_n = (1/(1-F))^{\leftarrow}(n) \sim n^{1/a} L_1(n) \ ,$$

where $L_1(x)$ is the same s.v.f. as above.

We point out that Theorem 2.1 with $\xi_i(n) = \eta_i/a_n$ and $i(n) = n$ leads to (**??**) in the particular case when $b = 1$ and $N(n) \equiv 1$.

Finally, we shall formulate a theorem which one can find in Galambos (1987), Exercise 2 on page 312. Let, in notation of Theorem 2.2, the interrelation of basic variables $\eta_1, \eta_2, \ldots$ and $\nu(n)$ be not restricted to independence. Let $q(n)$ be an increasing function, tending to ∞ with n. The following result holds.

Theorem 2.3 *Assume that there exist sequences* $\{a_n\}_1^\infty$, $(a_n > 0)$ *and* $\{b_n\}_1^\infty$ *such that for* $x \in R$*,*

(2.18) $$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le n} \eta_i - b_n}{a_n} \le x \right) = H(x) \ ,$$

where $H(x) = \exp\{-h(x)\}$ *is defined by (??) and there exists* ν *such that*

$$\lim_{n\to\infty} \frac{\nu(n)}{q(n)} = \nu \ , \tag{2.19}$$

where the convergence is in probability, and let $\varphi(u) = E\exp\{-u\nu\}, \ \ u > 0$. *Then for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i - b_{[q(n)]}}{a_{[q(n)]}} \le x\right) = \varphi(h(x)) \ . \tag{2.20}$$

3 Subcritical process.

It is known (see e.g. Athreya and Ney, thm 1, p.16) that

$$\lim_{n\to\infty} P(Z_n = j \mid Z_n > 0) = p_j, \qquad j = 0, 1, \ldots, \tag{3.1}$$

where $\{p_j\}$ is a probability distribution with generating function $\gamma(s) = \sum_{j=0}^{\infty} p_j s^j$ which is the unique solution of the equation , for $0 < s \le 1$,

$$\gamma(f(s)) = m\gamma(s) + 1 - m, \qquad \gamma(0) = 0. \tag{3.2}$$

By (??) it is easly to prove the following theorem.

Theorem 3.1 *If* $m < 1$ *then for* $x \ge 0$,

$$\lim_{n\to\infty} P(Y_n \le x \mid Z_{n-1} > 0) = \gamma(F(x)),$$

where γ *is the unique solution of (??) among the probability generating functions.*

Proof. Using (??) and (??) we obtain for $x \ge 0$,

$$\begin{aligned} P(Y_n > x \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(x))}{1 - f_{n-1}(0)} = 1 - \frac{f_{n-1}(F(x)) - f_{n-1}(0)}{1 - f_{n-1}(0)} \\ &= 1 - E\left(F^{Z_{n-1}}(x) \mid Z_{n-1} > 0\right) \to 1 - \gamma(F(x)). \end{aligned}$$

4 Critical process.

Let for $0 \le s \le 1$,

$$f(s) = s + (1-s)^{1+\alpha}\mathsf{L}(1/(1-s)), \tag{4.1}$$

where $0 < \alpha \le 1$ and $\mathsf{L}(x)$ is s.v.f. as $x \to \infty$. Slack has proved (see e.g. Bingham at al. (1987), thm 8.12.3, p.395) that (??) is necassary and sufficient condition for

$$\lim_{n\to\infty} P(Q_n Z_n > y \mid Z_n > 0) = P(Z > y), \qquad y \ge 0, \tag{4.2}$$

where $Q_n = P(Z_n > 0)$, say, and Z has Laplace transform

(4.3) $$\varphi(u) = Ee^{-uZ} = 1 - (1 + u^{-\alpha})^{-1/\alpha}, \qquad u > 0.$$

In other words, (**??**) holds iff

(4.4) $$\lim_{n\to\infty} (1 - f_n(\exp\{-uQ_n\}))/Q_n = 1 - \varphi(u), \qquad u > 0.$$

In addition, if (**??**) is true then

(4.5) $$Q_n = n^{-1/\alpha} N(n),$$

where $N(n)$ is a s.v.f. as $n \to \infty$ and

(4.6) $$N^{\alpha}(x)\mathsf{L}(x^{1/\alpha}/N(x)) \to 1/\alpha, \qquad x \to \infty.$$

The special case $\alpha = 1$ is particularly important: here $\varphi(u) = 1/(1+u)$, so the limit law has exponential distribution. If $\sigma^2 < \infty$ then (**??**) holds with $\alpha = 1$ and $\mathsf{L}(x)$ asymptotically constant. If $0 < \alpha < 1$ then (**??**) is equivalent to

(4.7) $$1 - F(x) \sim x^{-(1+\alpha)} L(x), \qquad L(x) \sim \frac{\alpha}{\Gamma(1-\alpha)} \mathsf{L}(x),$$

as $x \to \infty$ (see Bingham and Doney (1974), thm A, p.716). Note that in the boundary case $\alpha = 1$, (**??**) does not imply (**??**).

We proceed to study the asymptotic behaviour of

(4.8) $$P(Y_n > a_n x + b_n \mid Z_{n-1} > 0) = \frac{1 - f_{n-1}(F(a_n x + b_n))}{Q_{n-1}}$$

as $n \to \infty$, where $\{a_n\}_1^\infty$, $(a_n > 0)$ and $\{b_n\}_1^\infty$ are certain sequences of real numbers.

Let first consider the case $\sigma^2 < \infty$. Applying Theorem 2.1 it is not hard to obtain the following result.

Theorem 4.1 *Assume that $m = 1$ and $\sigma^2 < \infty$. If there exist sequences of real numbers $\{a_n\}_1^\infty$, $(a_n > 0)$ and $\{b_n\}_1^\infty$ such that for any real x,*

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le n} X_i(n) - b_n}{a_n} \le x\right) = H(x) \ ,$$

*where $H(x) = \exp\{-h(x)\}$ is defined by (**??**, then for any real x,*

(4.9) $$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x \mid Z_{n-1} > 0\right) = \left(1 + \frac{\sigma^2}{2} h(x)\right)^{-1} .$$

Proof. Since $m = 1$ and $\sigma^2 < \infty$ we have (see e.g. Athreya and Ney (1972), thm 2, p.20) for $x \ge 0$,

$$\lim_{n\to\infty} P\left(\frac{Z_{n-1}}{n} \le x \mid Z_{n-1} > 0\right) = P(Z \le x)$$

weakly, where $\varphi(u) = E\exp\{-uZ\} = 1/(1+\sigma^2 u/2)$.

Now, appealing to Theorem 2.1 with $i(n) = n$, $\nu(n) \equiv Z_{n-1}$ and $\xi_i(n) \equiv (X_k(n) - b_n)/a_n$, we get, for any real x,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x \mid Z_{n-1} > 0\right) = \int_0^\infty \exp\{-yh(x)\} dP(Z \le y) = \varphi(h(x))$$

and the proof is completed.

Let consider the case $\sigma^2 = \infty$. The proof of the next theorem will follow as an application of Theorem 2.2 (via Corollary 2.1).

Theorem 4.2 *Assume that* $m = 1$ *and for* $0 < \alpha \le 1$ *as* $x \to \infty$,

$$1 - F(x) \sim x^{-(1+\alpha)} L(x), \qquad L(x) - s.v.f. \text{ as } x \to \infty \ . \tag{4.10}$$

Then for $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{c_n} \le x \mid Z_{n-1} > 0\right) = 1 - (1 + x^{\alpha(1+\alpha)})^{-1/\alpha} \tag{4.11}$$

and

$$c_n = n^{1/(\alpha(1+\alpha))}(N(n))^{-1/(1+\alpha)} L_1(n^{1/\alpha}(N(n))^{-1}) \ , \tag{4.12}$$

where $N(n)$ *is defined by (**??**) and* $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$.

Proof. Assumption (**??**) is equivalent to $F \in MSD(\exp\{-x^{-(1+\alpha)}\}, \ x > 0$ and similarly as in the proof of Corollary 2.1, one can get the norming constants

$$a_n = (1/(1-F))^{\leftarrow}(n) \sim n^{1/(1+\alpha)} L_1(n)$$

as $n \to \infty$, where $L_1(x/L(x))/L(x) \to 1$ as $x \to \infty$.

To make use the Theorem 2.2 let put, in its notation, $\eta_i \equiv X_i(n)$, $n = 1, 2, \ldots$, and $\nu_n = Z_{n-1} I\{Z_{n-1} > 0\}$, where $I\{A\}$ denote the indicator variable of the event A. Appealing to (**??**) and (**??**) we see that the condition (**??**) of Corollary 2.1 is fulfiled with $b = 1/\alpha$ and by (**??**), $\varphi(u) = 1 - (1 + u^{-\alpha})^{-1/\alpha}$. Now, it is easily verified that Corollary 2.1 leads to (**??**) and (**??**). The proof is completed.

Le us remark that in the assumptions of Theorem 4.2 there is not restriction on σ^2 to be infinite. It is also worth noting that if σ^2 is finite then the assertion of Theorem 4.2 coincides with those of Theorem 4.1 provided (**??**). Indeed, suppose that (**??**) holds and $\sigma^2 < \infty$. Then $\alpha = 1$ and $F(x) \in MSD(\exp\{-x^{-2}\})$, $x > 0$. Now, in the notation of Theorem 4.1, (see e.g. Resnick (1987), prop.1.11, p.54) $b_n = 0$ and similarly to (**??**) we get, as $n \to \infty$,

$$a_n = (1/(1-F)) \leftarrow (n) \sim n^{1/2} L_1(n) \ ,$$

where $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$ and (**??**) becomes for $y > 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{n^{1/2} L_1(n)} \le y \mid Z_{n-1} > 0\right) = \left(1 + \frac{\sigma^2}{2} y^{-2}\right)^{-1} \ . \tag{4.13}$$

On the other hand, since $\sigma^2 < \infty$ we have $1/Q_n \sim \sigma^2 n/2$ as $n \to \infty$ and, in the notation of Theorem 4.2, we get as $n \to \infty$,

$$c_n \sim (\frac{\sigma^2}{2})^{1/2} n^{1/2} L_1(\frac{\sigma^2}{2} n) \ ,$$

where $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$. Hence (**??**) becomes for $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{(\frac{\sigma^2}{2})^{1/2} n^{1/2} L_1(\frac{\sigma^2}{2} n)} \le x \mid Z_{n-1} > 0 \right) = (1 + x^{-2})^{-1} \ . \tag{4.14}$$

Finally, putting $x = \sigma y/2^{1/2}$, it is clear that (**??**) is equivalent to (**??**).

In general case when either $\sigma < \infty$ or $\sigma^2 = \infty$ the following result holds.

Theorem 4.3 *Assume that* $m = 1$ *and for* $0 < \alpha \le 1$,

$$f(s) = s + (1-s)^{1+\alpha} \mathsf{L}(1/(1-s)), \qquad \mathsf{L}(x) - s.v.f. \ as \ x \to \infty \ .$$

If

$$\lim_{n\to\infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n+1)} = 1 \tag{4.15}$$

then for $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{U(Y_n)}{a_n} \le x \mid Z_{n-1} > 0 \right) = 1 - (1 + x^{\alpha})^{-1/\alpha} \tag{4.16}$$

and

$$a_n = n^{1/\alpha} (N(n))^{-1} \ ,$$

where $N(n)$ *is defined by (**??**) and* $U(x) = 1/(1 - F(x))$.

Proof. Since $\lim_{n\to\infty} Q_n = 0$ we have (cf. Leadbetter at al. (1983), thm 1.7.13, p.24) that (**??**) is necessary and sufficient condition for the existence of a sequence $\{u_n\}$ such that for $x > 0$,

$$\lim_{n\to\infty} \frac{1 - F(u_n)}{Q_n} = x \ . \tag{4.17}$$

Further, by (**??**) and (**??**), since $1 - F(u_n) \to 0$ as $n \to \infty$ we get

$$\begin{aligned} P(Y_n > u_n \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(u_n))}{Q_{n-1}} = \frac{1 - f_{n-1}(\exp\{\ln F(u_n)\})}{Q_{n-1}} \\ &= \frac{1 - f_{n-1}(\exp\{-(1 - F(u_n))(1 + o(1))\})}{Q_{n-1}} \\ &= \frac{1 - f_{n-1}(\exp\{-x Q_{n-1}(1 + o(1))\})}{Q_{n-1}} \\ &\to 1 - \varphi(x) \ , \end{aligned}$$

as $n \to \infty$, where $\varphi(x)$ is defined by (??). Now, from (??) (u_n are chosen not integers) and using Lemma 2.2.1 by Galambos (1987) one can obtain for $x > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\frac{1-F(Y_n)}{Q_{n-1}} \le x \mid Z_{n-1} > 0\right) \\
&= \lim_{n\to\infty} P\left(\frac{1-F(Y_n)}{Q_{n-1}} \le \frac{1-F(u_n)}{Q_{n-1}} + x - \frac{1-F(u_n)}{Q_{n-1}} \mid Z_{n-1} > 0\right) \\
&= \lim_{n\to\infty} P(F(Y_n) \ge F(u_n) \mid Z_{n-1} > 0) \\
&= \lim_{n\to\infty} P(Y_n > u_n \mid Z_{n-1} > 0) = 1 - \varphi(x) \ .
\end{aligned}$$

From here, taking into account (??) and (??), it is not difficult to get (??) and complete the proof of the theorem.

5 Supercritical process.

It is well-known (see e.g. Athreya and Ney (1972), thm 3, p.30) that if $1 < m < \infty$ then always exists a sequence of constants $\{C_n\}$ with $\lim_{n\to\infty} C_n = \infty$ such that $\{Z_n/C_n\}$ converges almost surely to a non-degenerate limit W. The Laplace transform $\psi(u) = E\exp\{-uW\}$, $u > 0$, of the limiting random variable, is the unique, up to a scale factor, solution of the equation

$$\psi(u) = f\left(\psi\left(\frac{u}{m}\right)\right) \ . \tag{5.1}$$

The constants C_n take the form (see Cohn (1982), thm 4)

$$C_n = m^n / L\left(m^n\right), \tag{5.2}$$

where $L(x) = \int_0^x P(W > y)dy$ $L(x)$ is s.v.f. as $x \to \infty$.

We shall prove the following result.

Theorem 5.1 *Assume that* $1 < m < \infty$. *If there exist sequences of real numbers* $\{a_n\}_1^\infty$, $(a_n > 0)$ *and* $\{b_n\}_1^\infty$ *such that for any real* x,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le [C_n]} X_i(n) - b_n}{a_n} \le x\right) = G(x) \ , \tag{5.3}$$

where $G(x) = \exp\{g(x)\}$ *is defined by (??) with* $r = m$, *then for any real* x,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x\right) = \psi(g(x)) \ ,$$

where the Laplace transform $\psi(u)$ *satysfies (??).*

Proof. Since

$$\frac{C_{n+1}-1}{C_n} \le \frac{[C_{n+1}]}{[C_n]} \le \frac{C_{n+1}}{C_n - 1}$$

we get

$$\lim_{n\to\infty} \frac{[C_{n+1}]}{[C_n]} = \lim_{n\to\infty} \frac{C_{n+1}}{C_n} = m > 1 \ .$$

Hence, from (**??**) and (**??**), $F \in MSSD(G)$ with, in notation of the definition (**??**), $c_{k(n)} = a_n$ and $d_{k(n)} = b_n$, where $k(n) = [C_n]$.

Further, we have for $x > 0$,

$$P\left(\frac{Z_n}{C_n} \le x\right) \ge P\left(\frac{Z_n}{[C_n]} \le x\right) \ge P\left(\frac{Z_n}{C_n} \le x(1 - \frac{1}{C_n})\right) \ .$$

Since $\{Z_n/C_n\}$ tends almost surely, and hence weakly, to W which has an absolutely continuously distribution on $(0, \infty)$ (see e.g. Atreya and Ney, cor. 12.1, p.52) we get for $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{Z_n}{[C_n]} \le x\right) = P(W \le x) \ .$$

Now, it is easily verified that by Theorem 2.1, putting $i(n) = [C_n]$, $\nu(n) \equiv Z_{n-1}$ and $\xi_i(n) \equiv (X_i(n) - b_n)/a_n$, it follows that for any real x,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x\right) = \int_0^\infty \exp\{-yg(x)\} dP(W \le y) = \psi(g(x)) \ .$$

The theorem is proved.

The following result is a corollary of Theorem 2.3.

Theorem 5.2 *Assume that* $1 < m < \infty$. *If there exist sequences* $\{a_n\}_1^\infty$, $(a_n > 0)$ *and* $\{b_n\}_1^\infty$ *such that for* $x \in R$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le n} X_i(n) - b_n}{a_n} \le x\right) = H(x) \ ,$$

where $H(x) = \exp\{-h(x)\}$ *is defined by (**??**), then for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_{[C_n]}}{a_{[C_n]}} \le x\right) = \varphi(h(x)) \ . \tag{5.4}$$

Proof. Since $\{Z_n/C_n\}$ tends almost surely, and hence in probability, to W and $\lim_{n\to\infty} C_{n+1}/C_n = m > 1$ we get (**??**) from Theorem 2.3 with $\eta_i \equiv X_i(n)$, $n = 1, 2, \ldots$, $\nu(n) = Z_n$ and $q(n) = C_n$.

Comment. It is known (cf. Galambos (1987), cor. 2.4.1) that if (**??**) does not hold then do not exist any constants $a_n > 0$ and b_n such that $(Y_n - b_n)/a_n$ to tend weakly to a nondegenerate limit. So, if $F \in MSD(H)$ then (**??**) is fulfiled. Hence the assumptions of Theorems 4.1 and 4.2 as well as of Theorems 5.1 and 5.2 also imply (**??**). It is not difficult to verify that (??) is not true for geometric and Poisson distributions (see e.g. Galambos (1987)). On the other hand, by Theorems 3.48 and 3.50 in Nielson (1995) we have that if the distribution function F_X of a random variable X belongs to the domain of attraction of $H(x) = \exp\{-x^{-a}\},\ x > 0,\ a > 0$ then so does $F_{[X]}$. Furthermore, $F_X \in MSD(\exp\{-\exp\{-x\}\}),\ x \in R$ iff $F_{[X]} \in MSD(\exp\{-\exp\{-x\}\}),\ x \in R$ provided $x_0 = \sup\{x : F_X(x) < 1\} = \infty$.

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