

IMMIGRATION-BRANCHING DIFFUSIONS AND THEIR EXTINCTION

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ABSTRACT

The Markov branching diffusion in which offspring move at random in a bounded space X and allowing time-dependent immigration of particles from an external source is considered. It has not been known an explicit form of the limit distribution for such processes. In the paper, using a combination of analytic methods and direct probabilistic arguments, we obtain explicit limit distributions for the vector of population sizes in disjoint subsets of the space X . Asymptotic behavior of the non-extinction probability and the expected size of the population are also studied when first moment functional of the immigration process decreases regularly.

Key words: branching diffusion, Markov process, immigration, time-dependent immigration, extinction, explicit distribution, partial process, moment functional

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1 INTRODUCTION

We consider a population evolving as a branching process where the offspring move at random on a bounded space X . Typical models that have been studied by many authors are branching random walks and branching diffusions. In branching random walks the offspring jump when they are born, and retain their position until they die or reproduce. In branching diffusions, which we are going to deal with, the offspring move throughout their lives according to diffusions, independently of each other and of the family tree, each offspring starting from the location of its birth. Each particle at the end of its life independently of others generates a population whose members may be located at any point of X . The lifetime of a particle is defined as the hitting time of either of two barriers ∂ and Δ . If the particle hits ∂ , it is instantaneously replaced by the empty population, that is disappears. If the particle hits Δ , then it is

replaced by a population of new particles distributed in X according to a probability law.

In addition a random number of new particles produced by an external source may immigrate into the population and initial positions of these new particles may also be arbitrary points of X . The state of the process at any given time is characterized by the total number n of particles at present and their positions $x_1, x_2, \dots, x_n \in X$. The development of the process in the state space \hat{X} of all finite populations of particles located in X is Markovian. In other words, the motion of a particle up to the time of reproduction is a Markov process on X .

In the literature many papers have been published on branching or immigration branching diffusions. Mathematical foundations of a general theory of stochastic population processes were given by Moyal (1962). Existence and probabilistic constructions of branching diffusions are due to Ikeda, Nagasawa and Watanabe (1965, 1966) (see also Conner(1961, 1967), Savits (1969), Hering(1973, 1978), Asmussen and Hering(1977), Kageyama and Ogura (1980)). Convergence problems for systems of critical branching Markov chains are considered by Cox (1994). It should be noted the book by Asmussen and Hering(1983) as a convenient source on branching diffusions. In that book using analytic methods limit theorems for the continuous and discrete time immigration-branching diffusions with non-stationary immigration were proved (see [2], p.231-239). However the limit distributions obtained there have no explicit form. In this paper we demonstrate that, if we use a technique developed to study Galton-Watson processes with decreasing immigration (Rahimov (1995), Ch III), it is possible to obtain explicit limit distributions for such processes when the first moment functional of the immigration decreases. The method based on a combination of analytic and probabilistic arguments. Namely the proofs of the limit theorems will be carried out in the following scheme. First using analytic methods we prove limit theorems for some "partial processes" counting only descendants of particles immigrated in the beginning of the process or immigrated "recently". Then from these theorems by direct probabilistic arguments obtain limit distributions for the basic process.

For simplicity we consider here the case of immigration at discrete time points. However it will be seen further that all statements of the paper can be obtained by the same arguments in the case of Poisson immigration process as well.

Let (X, U) be a measurable space, $X^{(n)}$ be the symmetrization of the direct product of n copies of X (see Moyal(1961)). We put $X^{(0)} = \theta$, where θ is some extra

point,

$$\hat{X} = \bigcup_{n=0}^{\infty} X^{(n)}$$

and denote \hat{U} the σ -algebra on \hat{X} induced by U . An element $x \in X$ we understand as position, energy or type of a particle diffusing in the bounded space. Then, it is clear that \hat{X} is the space of all possible populations and θ denotes an empty population.

We denote β as the Banach algebra of all bounded, complex-valued U -measurable functions ξ on X with supremum norm

$$\|\xi\| = \sup_{x \in X} |\xi(x)|$$

and put $\beta_+ = \{\xi \in \beta: \xi > 0\}$. Let ∇ be the open unit ball in β and $\bar{\nabla}$ its closure.

For $\hat{x} \in \hat{X}$ and $\xi \in \bar{\nabla}$ we define

$$\hat{x}[\xi] = \begin{cases} 0 & \text{if } \hat{x} = \theta \\ \sum_{i=1}^n \xi(x_i) & \text{if } \hat{x} = \langle x_1, \dots, x_n \rangle \end{cases} \quad (1)$$

For example, if $\xi(x) = 1_A(x)$, $A \subset X$, then $\hat{x}_t[1_A]$ is the number of particles in A at time t . Introduce the union operation for populations $\hat{x}, \hat{y} \in \hat{X}$ as following:

$$\hat{x} + \hat{y} = \begin{cases} \theta & \text{if } \hat{x} = \hat{y} = \theta \\ \langle x_1, \dots, x_n, y_1, \dots, y_n \rangle & \text{otherwise,} \end{cases} \quad (2)$$

that is, if $\hat{x} = \langle x_1, \dots, x_n \rangle$, $\hat{y} = \langle y_1, \dots, y_n \rangle$. Then the Markov branching diffusion $\{\hat{x}_t, P^{\hat{x}}\}$ is defined by the relation

$$\hat{x}_{t+s} = \sum_{i=1}^{\hat{x}_t[1]} \hat{x}_{t+s}^{t,i}, t, s > 0 \quad (3)$$

with $\hat{x}_{t+s}^{t,i}$, $i = 1, 2, \dots, \hat{x}_t[1]$ are conditionally independent, given $\mathcal{F}_t := \sigma(\hat{x}_u, u \leq t)$ and

$$P^{\hat{x}}\{\hat{x}_{t+s}^{t,j} \in \hat{A} | \mathcal{F}_t\} = P^{\langle x_j \rangle}\{\hat{x}_s \in \hat{A}\}$$

a.s. $P^{\hat{x}}$, $\hat{x}_t = \langle x_1, \dots, x_{\hat{x}_t[1]} \rangle$. Here and later on $P^{\hat{x}}$ and $E^{\hat{x}}$ denote the probability and expectation when the initial population is \hat{x} and \mathcal{F}_t is the σ -algebra generated by the branching diffusion up to time t .

Let $t \in N_0 = \{0, 1, \dots\}$ and $\hat{y}_t = \langle y_{t1}, \dots, y_{tl} \rangle$ be the population of immigrating at time t particles, where $y_{tj} \in X$ is the position (or energy) of j -th particle from \hat{y}_t . If we denote $\hat{x}_t^{\langle y_{ki} \rangle}$ the branching diffusion initiated by particle y_{ki} , then the discrete time immigration-branching diffusion can be given by the following relation:

$$\hat{Z}_t = \sum_{k=0}^t \sum_{i=1}^{\hat{y}_k[1]} \hat{x}_{t-k}^{\langle y_{ki} \rangle}. \quad (4)$$

The content of the paper is as following. In part 2 we state basic assumptions and some known results which will be used further. Part 3 is devoted to study of asymptotic behavior of the "non-extinction" probability. Results obtained there will well be used in part 5, but they are of independent interest as well. Asymptotic behavior of the expected size of population is studied in part 4. In part 5 the class of all possible explicit limit distributions is obtained.

2 BASIC ASSUMPTIONS

Let for $\hat{x} \in \hat{X}, \hat{A} \in \hat{U}, s, t \in N_0$

$$P_t(\hat{x}, \hat{A}) = P\{\hat{x}_{t+s} \in \hat{A} | \hat{x}_s = \hat{x}\}$$

be the branching transition function of \hat{x}_t . For $\eta \in \bar{\nabla}$ and $\hat{x} \in \hat{X}$ we define

$$\tilde{\eta}(\hat{x}) = \begin{cases} 1, & \hat{x} = \theta \\ \prod_{i=1}^n \eta(x_i), & \hat{x} = \langle x_1, \dots, x_n \rangle \end{cases} \quad (5)$$

Then the generating functional

$$F_t(\hat{x}, \eta) = \int_{\hat{X}} \tilde{\eta}(\hat{y}) P_t(\hat{x}, d\hat{y})$$

of $P_t(\hat{x}, \cdot)$ is well-defined on $\bar{\nabla}$.

It is known (Asmussen, Hering(1983), p.139) that the mapping $F_t : \bar{\nabla} \mapsto \bar{\nabla}$ defined by $F_t[\eta](x) := F_t(\langle x \rangle, \eta), \eta \in \bar{\nabla}, x \in X$ satisfies the semigroup relation $F_{t+s}[\eta] = F_s[F_t[\eta]], t, s \geq 0, \eta \in \bar{\nabla}$ and it is called a generating semigroup.

Assume that there exist a linear-bounded functional of ξ on β defined by the relation

$$M_t(\hat{x}, \xi) = \int_{\hat{X}} \hat{y}[\xi] P_t(\hat{x}, d\hat{y}).$$

It is also known, Asmussen, Hering (1983), that $M_{t+s} = M_t M_s$ for any $s, t \in N_0$, that is $\{M_t\}$ is a semigroup. It is called the moment semigroup of the process.

Now we introduce so called (M) and (R) assumptions which are important for our further considerations.

Assumption (M). The moment semigroup $\{M_t\}$ can be represented as

$$M_t = \rho^t P + \Delta_t, t \geq 0, \quad (6)$$

where $\rho \in (0, \infty)$, $P\xi = \Phi^*[\xi]\varphi$, $\xi \in \beta$, with $\Phi^* : \beta \mapsto \mathbf{C}$ is linear-bounded, non-negative on β_+ , $\varphi \in \beta_+$; further $\Delta_t : \beta \mapsto \beta$ such that for all $t > 0$

$$P\Delta_t\xi = \Delta_t P\xi = 0, \xi \in \beta,$$

$$-\gamma_t P\xi \leq \Delta_t\xi \leq \gamma_t P\xi, \xi \in \beta_+$$

with $\gamma_t : (0, \infty) \mapsto \mathbf{R}_+$ satisfying $\rho^{-t}\gamma_t \downarrow 0, t \uparrow \infty$.

When X is a finite set and M_t is primitive, satisfaction of (M) follows from the well-known Perron's theorem (see for details Sevastyanov (1971), Ch IV). In the general case it can also be satisfied by a wide class of branching diffusions. Examples of such processes can be seen in Hering (1978).

Suppose $\{M_t\}$ exists as a semigroup of bounded operators. Then

$$1 - F_t[\eta] = M_t[1 - \eta] - R_t(\eta)[1 - \eta] \quad (7)$$

where the mapping $R_t(\cdot)[\eta] : \bar{\nabla} \otimes \beta \mapsto \beta$ is non-increasing in the first variable and linear-bounded in the second, such that

$$0 = R_t(1)\xi \leq R_t(\eta)\xi \leq M_t\xi, (\eta, \xi) \in \bar{\nabla}_+ \otimes \beta.$$

Assumption (R). For every $t > 0$, there exist a mapping $g_t : \bar{\nabla}_+ \mapsto \beta$ such that

$$R_t(\xi)[1 - \xi] = g_t[\xi]\rho^t\Phi^*[1 - \xi]\varphi, i \in \bar{\nabla}_+,$$

$$\lim_{\|1-\xi\| \rightarrow \infty} \|g_t[\xi]\| = 0.$$

Under the assumption (M) the (R) is automatically satisfied if X is finite. In the general case it may follow from (M) under some restrictions on φ and Φ^* . Some sufficient conditions for fulfillment of (R) are given, for example, in the book by Asmussen and Hering(1983, p.160).

We also assume that the second moment functional of \hat{x}_t

$$M_2^{(2)}(\langle x \rangle, \xi, \eta) = E^{\langle x \rangle} \hat{x}_t^{(2)}[\xi, \eta]$$

is finite. Here

$$\hat{x}^{(2)}[\xi, \eta] = \begin{cases} 0, & \hat{x}[1] \leq 1, \\ \sum \sum_{i \neq j} \xi(x_i)\eta(x_j), & \hat{x} = \langle x_1, \dots, x_n \rangle, n \geq 2. \end{cases}$$

We denote

$$\mu = \frac{1}{2t} \Phi^*[M_t^{(2)}[\varphi]], M_t^{(2)}[\xi](x) = M_t^{(2)}(\langle x \rangle, \xi, \xi)$$

It is known that

$$q(x) = \lim_{t \rightarrow \infty} F_t[0](x)$$

exists for any $x \in X$, and given (M) with $\rho = 1$ the quantity μ is constant as a function of t .

3 APPROXIMATION OF NON-EXTINCTION PROBABILITY

Now we proceed to consider the immigration-branching diffusion. Let F_k^I be the generating functional of $P\{\hat{y}_k \in \hat{A}\}$, $\hat{A} \in \hat{U}$. If F_k^I has a bounded first moment functional

$$M_k^I[\xi] = \int_{\hat{X}} \hat{y}[\xi] P\{\hat{y}_k \in d\hat{y}\},$$

then it follows that (7) holds for F_k^I . Assume that $\alpha(t) = M_t^I[\varphi]$ is a regularly varying function as $t \rightarrow \infty$ and

$$r(\varepsilon) = \sup_{s>0} \frac{R_s^I((1-\varepsilon)1)[\varphi]}{\alpha(s)} \rightarrow 0, \varepsilon \rightarrow 0. \quad (8)$$

Condition (8) holds, for example, if the second factorial moment

$$M_k^{I(2)}[\xi, \eta] = \int_{\hat{X}} \hat{y}^{(2)}[\xi, \eta] P\{\hat{y}_k \in d\hat{y}\}$$

of the immigration process is finite as a symmetric bilinear-bounded functional on $\beta^{(2)} = \beta \otimes \beta$ and

$$\sup_k \frac{M_k^{I(2)}[1, 1]}{M_k^I[\varphi]} < \infty.$$

In the case of stationary immigration process, it is clear that, the state $\theta \in \hat{X}$ is a reflecting screen for the process. If the immigration process has an decreasing intensity, then the state θ may reflect or may also absorb the process. Hence it is important to know the behavior of $P\{\hat{Z}_t \neq \theta\}$ as $t \rightarrow \infty$. The first result gives us conditions under which the above probability tends to 1, to zero or to a positive limit less than 1.

Theorem 1 Let (M) , (R) and (8) be satisfied, $\rho = 1, 0 < \mu < \infty, q = 1, \alpha(t) \rightarrow 0, t \rightarrow \infty$.

1°. If $\alpha(t) \ln t \rightarrow 0$, then

$$\lim_{t \rightarrow \infty} P\{\hat{Z}_t \neq \theta\} = 0.$$

2°. If $\alpha(t) \ln t \rightarrow C \in (0, \infty)$, then

$$\lim_{t \rightarrow \infty} P\{\hat{Z}_t \neq \theta\} = 1 - \exp\left\{-\frac{C}{\mu}\right\}.$$

3°. If $\alpha(t) \ln t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} P\{\hat{Z}_t \neq \theta\} = 1.$$

Proof. First we prove part 2°. If we denote $H_t[\xi]$ the generating functional of the process \hat{Z}_t , then it follows from (4) that

$$H_t[\xi] = \prod_{k=0}^t F_k^I[F_{t-k}[\xi]]. \quad (9)$$

We consider the sum

$$A = \sum_{k=0}^t M_k^I[1 - F_{t-k}[0]] = A_1 + A_2 + A_3, \quad (10)$$

where $A_i, i = 1, 2, 3$, are sums with respect to $0 \leq k < t/\ln t, t/\ln t \leq k \leq t - \ln t$ and $t - \ln t < k \leq t$.

Further we use the following results from Asmussen, Hering (1983, pp. 184, 202).

Lemma 1 If (M) and (R) are satisfied and $q = 1$, then for every $t > 0$ there exist a mapping $h_t : \bar{\nabla}_+ \rightarrow \mathcal{B}$ such that

$$1 - F_t[\xi] = (1 + h_t[\xi])\Phi^*[1 - F_t[\xi]]\varphi, \xi \in \bar{\nabla}_+, \quad (11)$$

where $\lim_{t \rightarrow \infty} \|h_t[\xi]\| = 0$ uniformly in $\xi \in \bar{\nabla}_+$.

The next result is a generalization of so called basic lemma in the theory of Galton-Watson processes (see Jagers(1983), p.25).

Lemma 2 If (M) and (R) are satisfied, $0 < \mu < \infty, \rho = 1, q = 1$ then for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n\delta} \left\{ \Phi^*[1 - F_{n\delta}[\xi]]^{-1} - \Phi^*[1 - \xi]^{-1} \right\} = \mu \quad (12)$$

uniformly in $\xi \in \bar{\nabla}_+ \cap \{\Phi^*[1 - \xi] > 0\}$.

It follows from (11) and (12) that

$$(n+1)(1 - F_n[0]) = (1 + \varepsilon_n) \frac{\varphi}{\mu}, \quad (13)$$

where $\lim_{n \rightarrow \infty} \|\varepsilon_n\| = 0$.

Since M_t^I is a linear-bounded functional, using (13), we have

$$A_1 = \sum_{k \in E_1} \frac{1}{t - k + 1} M_k^I \left[(1 + \varepsilon_{t-k}) \frac{\varphi}{\mu} \right],$$

where $E_1 = \{k : 0 \leq k < t/\ln t\}$. Using here the simple estimates

$$1 - \|\varepsilon\| \leq 1 + \varepsilon \leq 1 + \|\varepsilon\| \quad (14)$$

we obtain that A_1 is non-greater than

$$\mu^{-1} (1 + \sup_{k \in E_1} \|\varepsilon_{t-k}\|) \sum_{k \in E_1} \alpha(k) (t - k + 1)^{-1}$$

which tends to zero as $t \rightarrow \infty$.

Now we consider A_2 . Again using (13) and (14), we obtain

$$A_2 \leq \mu^{-1} (1 + \sup_{k \in E_2} \|\varepsilon_{t-k}\|) \sum_{k \in E_2} \alpha(k) (t - k + 1)^{-1},$$

where $E_2 = \{k : t/\ln t < k \leq t - \ln t\}$. Since under the our conditions

$$\lim_{t \rightarrow \infty} \sum_{k \in E_2} \alpha(k) (t - k + 1)^{-1} = C,$$

we have $\limsup_{t \rightarrow \infty} A_2 \leq C/\mu$. If we use the left side of (14), we obtain that $\liminf A_2$ is non-less than C/μ . Thus A_2 tends to C/μ as $t \rightarrow \infty$.

By similar arguments it can be verified that $A_3 \rightarrow 0$ as $t \rightarrow \infty$.

If we use (7) written for F_t^I , we have that under the condition (8)

$$\lim_{t \rightarrow \infty} \sum_{k=0}^t \{1 - F_k^I[F_{t-k}[0]]\} = \lim_{t \rightarrow \infty} A. \quad (15)$$

Thus assertion of the part 2^o follows from (15) and the simple relation

$$\ln H_t[0] = \sum_{k=0}^t \ln F_k^I[F_{t-k}[0]].$$

Now we prove part 1^o. For any $\varepsilon > 0$ and fixed μ we can find such a $C > 0$ that $1 - \exp\{-C/\mu\} < \varepsilon$. Let \hat{Z}_t^* be the immigration-branching diffusion with the same

branching transition function and with immigration process such that $\alpha^*(t) \ln t \rightarrow 0$. Then it is clear that for any $\varepsilon > 0$

$$\limsup_{t \rightarrow \infty} P\{\hat{Z}_t^* \neq \theta\} \leq \lim_{t \rightarrow \infty} P\{\hat{Z}_t \neq \theta\} \leq \varepsilon.$$

From the last relation we have the assertion of the part 1^o.

Part 3^o can also be derived from part 2^o. To do this one has to choose $C > O$ such that $\exp\{-C/\mu\} < \varepsilon$ and have to compare the "non-extinction" probability of the process \hat{Z}_t and a new process whose immigration process satisfies conditions of the part 3^o. The theorem is proved.

We now consider the asymptotic behavior of the non-extinction probability in the case 1^o i.e. when $\alpha(t) \ln t \rightarrow 0, t \rightarrow \infty$. Introduce the following notation:

$$\alpha(t) = M_t^I[\varphi] = \frac{l(t)}{t^\alpha + 1}, \alpha \geq 0, a(t) = \sum_{k=0}^t \alpha(k)$$

$$\beta(t) = P^{<x>}\{\hat{x}_t \neq \theta\}, b(t) = \sum_{k=0}^t \beta(k).$$

Here $l(t)$ is a slowly varying as $t \rightarrow \infty$ function. It follows from (13) that under the our assumptions

$$t\mu\beta(t) \sim \varphi, t \rightarrow \infty. \quad (16)$$

Note that $a(t)$, when $\alpha \geq 1$, and $b(t)$ are slowly varying functions. Therefore, it follows from the Karamata representation of slowly varying functions that there exist positive, integer valued functions $L_i(t), i = 1, 2$, such that $L_i(t) \rightarrow \infty, L_i(t) = o(t), t \rightarrow \infty$ and (see also Lemma 3 below)

$$a(L_1(t)) \sim a(t), b(L_2(t)) \sim b(t), t \rightarrow \infty. \quad (17)$$

Introduce the following "partial" processes:

$$\hat{Z}_t^{(j)} = \sum_{k \in I_j^t} \sum_{i=1}^{\hat{y}_k^{(1)}} \hat{x}_{t-k}^{<y_{ki}>},$$

where

$$I_1^t = \{k \in N : 0 \leq k \leq L_1(t)\}, I_3^t = \{k \in N : t - L_2(t) \leq k \leq t\},$$

$$I_2^t = \{k \in N : L_1(t) + 1 \leq k \leq t - 1 - L_2(t)\}.$$

It is clear that

$$\hat{Z}_t = \hat{Z}_t^{(1)} + \hat{Z}_t^{(2)} + \hat{Z}_t^{(3)} \quad (18)$$

Theorem 2 Let assumptions of Theorem 1 are satisfied. Then

1°. If $\alpha \geq 1$, then

$$P\{\hat{Z}_t^{(1)} \neq \theta\} \sim (\mu t)^{-1} a(t),$$

if $\alpha < 1$, then

$$P\{\hat{Z}_t^{(1)} \neq \theta\} = o\left(\frac{a(t)}{t}\right);$$

2°.

$$P\{\hat{Z}_t^{(3)} \neq \theta\} \sim \mu^{-1} \alpha(t) \ln t;$$

3°.

$$P\{\hat{Z}_t \neq \theta\} \sim \frac{1}{\mu t} a(t) + \frac{1}{\mu} \alpha(t) \ln t.$$

Remark. It should be noted that part 3° of the theorem shows that event $\{\hat{Z}_t \neq \theta\}$ may occur because of either $\{\hat{Z}_t^{(1)} \neq \theta\}$ or $\{\hat{Z}_t^{(2)} \neq \theta\}$ and, when $\alpha(t) \sim t^{-1} \text{const}$ these two possibilities are asymptotically equiprobable.

Proof of Theorem 2. First we prove part 1°. It follows from (7) and (13) that

$$0 \leq 1 - F_k^I[F_{t-k}[0]] \leq \frac{1}{t-k+1} M_k^I[(1 + \varepsilon_{t-k}) \frac{\varphi}{\mu}] \rightarrow 0 \quad (19)$$

as $t \rightarrow \infty$ uniformly in $k \in I_1$. Therefore

$$- \sum_{k \in I_1} \ln F_k^I[F_{t-k}[0]] \sim \sum_{k \in I_1} (1 - F_k^I[F_{t-k}[0]]).$$

Using (7), (13) and that M_k^I and R_k^I are linear-bounded functionals we obtain that the last sum as $t \rightarrow \infty$ is equivalent to

$$\delta_t = \frac{1 + \|\varepsilon_t\|}{\mu} \sum_{k \in I_1} \frac{M_k^I[\varphi]}{t-k+1} [1 - r(k, t)],$$

where

$$r(k, t) = \frac{R_k^I[F_{t-k}[0]][\varphi]}{M_k^I[\varphi]}.$$

Taking into account the choice of $L_1(t)$ and condition (8) we have that δ_t is equivalent to $(\mu t)^{-1} a(t)$ when $\alpha \geq 1$. If $\alpha < 1$, then $a(L_1(t)) = o(a(t))$, and thus in this case $\delta_t = o(t^{-1} a(t))$. The assertion of part 1° follows from this and the simple relation $1 - e^{-x} \sim x, x \rightarrow 0$.

Let us prove part 2°. It is not difficult to see that (19) holds as $t \rightarrow \infty$ uniformly in $k \in I_3$ also. Thus we have to consider

$$A = \sum_{k \in I_3} (1 - F_k^I[F_{t-k}[0]]) = A_1 + A_2, \quad (20)$$

where

$$A_1 = \sum_{k \in I_3} M_k^I [1 - F_{t-k}[0]], A_2 = \sum_{k \in I_3} R_k^I (F_{t-k}[0]) [1 - F_{t-k}[0]].$$

Using again (13), (14) and that M_k^I is linear-bounded functional we obtain that $A_1 \sim \mu^{-1} \alpha(t) \ln t, t \rightarrow \infty$. Now we consider A_2 . Since R_k^I is linear-bounded functional, using (13) and (14) we have that it is non-greater than

$$\text{const} \cdot \sum_{k \in I_3} (t - k + 1)^{-1} R_k^I (F_{t-k}[0]) [\varphi].$$

It follows from condition (8) that for any $\varepsilon > 0$ there exist $\delta > 0$ such that $r(\Delta) < \varepsilon$ for all $\Delta < \delta$. Since $F_k[0] \rightarrow 1$, there exist $m \in N$ such that $1 - F_{t-k}[0] < \delta$ for $t - k > m$. Now we partition I_3 as $I_3 = I_3^{(1)} \cup I_3^{(2)}$, where

$$I_3^{(1)} = \{k : t - L_2(t) \leq k \leq t - m\}, I_3^{(2)} = \{k : t - m < k \leq t\}.$$

Then we have

$$\sum_{k \in I_3^{(1)}} (t - k + 1)^{-1} R_k^I (F_{t-k}[0]) [\varphi] \leq \varepsilon \alpha(t) \ln t.$$

Since $R_t^I(\eta)\xi \leq M_t^I \xi$, the second part of the sum is non-greater than

$$\sum_{k \in I_3^{(2)}} (t - k + 1)^{-1} \alpha(k) = o(\alpha(t) \ln t)$$

as $t \rightarrow \infty$ for any fixed m . Therefore we have $A_2 = o(\alpha(t) \ln t), t \rightarrow \infty$. Thus we obtain the assertion of part 2^o from (20) and estimates for A_1 and A_2 . Now we prove part 3^o. First we consider

$$P_2 = P\{\hat{Z}_t^{(2)} \neq \theta\} = 1 - \prod_{k \in I_2} F_k^I [F_{t-k}[0]].$$

Using simple inequality $\prod_i (1 - \varepsilon_i) \geq 1 - \sum_i \varepsilon_i, \varepsilon_i \geq 0$ and relation (7) we obtain that P_2 is non-greater than

$$\sum_{k \in I_2} M_k^I [1 - F_{t-k}[0]].$$

Since M_k^I is a linear functional we have from (13) and (14) that

$$P_2 \leq \frac{1}{\mu} (1 + \sup_{k \in I_2} \|\varepsilon_{t-k}\|) \sum_{k \in I_2} (t - k + 1)^{-1} M_k^I [\varphi].$$

It follows from properties $M_k^I [\varphi]$ that there exist $C(\alpha) > 0$ such that

$$\min_{0 \leq k \leq t} \left\{ \frac{t - k + 1}{t + 1} + \frac{M_t^I [\varphi]}{M_k^I [\varphi]} \right\} \geq C(\alpha) > 0.$$

Therefore

$$\sum_{k \in I_2} \frac{1}{t-k+1} \alpha(k) \leq \frac{1}{C(\alpha)} \left\{ \frac{1}{t} \sum_{k \in I_2} \alpha(k) + \alpha(t) \sum_{k \in I_2} \frac{1}{t-k} \right\},$$

where

$$\frac{1}{t} \sum_{k \in I_2} \alpha(k) \leq \frac{\text{const}}{t} [a(t) - a(L_1(t))] = o\left(\frac{1}{t} a(t)\right),$$

$$\alpha(t) \sum_{k \in I_2} \frac{1}{t-k} \leq \text{const} \cdot \alpha(t) [b(t) - b(L_2(t))] = o(\alpha(t) \ln t).$$

Thus we can conclude that

$$P_2 = o\left(\frac{1}{t} a(t) + \alpha(t) \ln t\right). \quad (21)$$

Further we use the following equality

$$\{\hat{Z}_t \neq \theta\} = \bigcup_{i=1}^3 \{\hat{Z}_t^{(i)} \neq \theta\} \quad (22)$$

where events on the right hand side are independent. If we denote $P_i, i = 1, 2, 3$ the probability of the i -th event, then

$$P\{\hat{Z}_t \neq \theta\} = \sum_{i=1}^3 P_i - \sum_{i \neq j} P_i P_j + P_1 P_2 P_3. \quad (23)$$

The assertion of the part 3^o follows from (23), if we use results of parts 1^o and 2^o and relation (21). Theorem 2 is proved.

4 THE LIMIT THEOREMS

Now we turn our attention to limit theorems for the vector of population sizes in disjoint subsets of the space X . Let $\{A_i, i = 1, \dots, j\}$ be a measurable decomposition of X , that is

$$X = \cup_{i=1}^j A_i, A_l \cap A_m = \phi.$$

If $\chi(A)$ is the indicator function of the set $A \subseteq X$, then it is clear that $\hat{Z}_t[\chi(A)]$ is the number of particles in A at time t .

Theorem 3 Let (M), (R) and (8) be satisfied, $\rho = 1, 0 < \mu < \infty$ and $q = 1$. If $\alpha(t) \rightarrow 0$ such that $\alpha(t) \ln t \rightarrow \infty$, then for every finite measurable decomposition $\{A_i, i = 1, \dots, j\}$ of X

$$\lim_{t \rightarrow \infty} P \left\{ \left(\frac{\hat{Z}_t[\chi(A_i)]}{t} \right)^{\alpha(t)/\mu} \leq x_i, i = 1, \dots, j \right\} = P\{\nu_1 \leq x_1, \dots, \nu_j \leq x_j\},$$

where $\nu_1 = \dots = \nu_j$ with probability 1 and ν_k has the uniform distribution on $[0, 1]$.

Further we need the following result, which can be proved using the well known representation for slowly varying functions.

Lemma 3 If $L(x)$ is a slowly varying as $x \rightarrow \infty$ function, then there is another slowly varying function $l(x) \sim L(x), x \rightarrow \infty$ for which it is possible to find function $\lambda_l(x) \rightarrow \infty, \lambda_l(x) = o(x)$ such that as $x \rightarrow \infty$

$$\sup_{u \in \Delta(x)} \left| \frac{l(u)}{l(x)} - 1 \right| \rightarrow 0, \Delta(x) = \left[\frac{x}{\lambda_l(x)}, x \right].$$

In some sense the Lemma 3 is an extension of so called uniform convergence theorem for slowly varying functions.

Proof of Theorem 4. It follows from the condition $\alpha(t) \ln t \rightarrow \infty$ that $\alpha(t)$ is a slowly varying function as $t \rightarrow \infty$. Thus there is a function $\lambda_\alpha(t)$ for which the extended uniform convergence theorem holds (see Lemma 3). If $0 < x_i < 1, i = 1, \dots, j$ and $x = \min\{x_1, \dots, x_j\}$, then the function $L(t) = tx^{\mu/\alpha(t)} \rightarrow \infty, L(t) = o(t)$ as $t \rightarrow \infty$. Now we consider the sum

$$\sum_{k=0}^t M_k^I [1 - F_{t-k}[\xi_t]], \xi_t = \exp \left\{ - \frac{\sum_{i=1}^j \lambda_i \chi(A_i)}{tx_i^{\mu/\alpha(t)}} \right\},$$

where $\lambda_i \geq 0, i = 1, \dots, j$. We decompose the sum as following

$$A = A_1 + A_2 + A_3 \tag{24}$$

where $A_i, i = 1, 2, 3$ are sums of the terms when k belongs to $K_1 = \{k : 0 \leq k < t/\lambda_\alpha(t)\}, K_2 = \{k : t/\lambda_\alpha(t) \leq k < t - L(t)\}$ and $K_3 = \{k : t - L(t) \leq k \leq t\}$ respectively. In this case we obtain from (11) and (12) the following relation

$$1 - F_n[\xi] = \frac{\varphi}{\mu n + \Phi^*[1 - \xi]^{-1}} (1 + \varepsilon_n^*[\xi]), \tag{25}$$

where $\|\varepsilon_n^*\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $\xi \in \bar{\nabla}_+ \cap \{\Phi^*[1 - \xi] > 0\}$. The following estimates are also true for any ξ from the above set:

$$1 - \|\varepsilon_n^*[\xi]\| \leq 1 + \varepsilon_n^*[\xi] \leq 1 + \|\varepsilon_n^*[\xi]\| \quad (26)$$

First we consider A_2 . Using (25), right side of (26) and Lemma 3 we get

$$\limsup_{t \rightarrow \infty} A_2 \leq \limsup_{t \rightarrow \infty} \alpha(t) \sum_{k \in K_2} \left\{ \mu k + \Phi^*[1 - \xi_t]^{-1} \right\}^{-1}.$$

It follows from the choice of $L(t)$ and linearity of Φ^* that

$$\lim_{t \rightarrow \infty} L(t)\Phi^*[1 - \xi_t] = \delta(E_0) \equiv \sum_{i \in E_0} \lambda_i \Phi^*[\chi(A_i)], \quad (27)$$

where $E_0 = \{i : x_i = x\}$. Thus we have from the above estimate that

$$\limsup_{t \rightarrow \infty} A_2 \leq \limsup_{t \rightarrow \infty} \frac{\alpha(t)}{\mu} \ln(1 + \mu \delta(E_0) x^{-\mu/\alpha(t)}) = -\ln x.$$

If we use the left side of (26), by the same arguments obtain that the \liminf of A_2 is non-less than $-\ln x$. Consequently $A_2 \rightarrow -\ln x$ as $t \rightarrow \infty$.

Using (25) and right side of (26) we can see that $\limsup \lambda_\alpha(t)A_1 < \infty$, that is A_1 tends to zero as $t \rightarrow \infty$.

Now we consider A_3 . Again using (25), (26) and the fact that $\alpha(t)$ slowly varies, we obtain, for some positive constant C , the following estimate

$$A_3 \leq C\alpha(t) \sum_{k \in K_3} \left\{ \mu(t - k) + \Phi^*[1 - \xi_t]^{-1} \right\}^{-1} < C\delta(E_0)\alpha(t),$$

which shows that A_3 also tends to zero as $t \rightarrow \infty$. Thus it follows from (24) that $A \rightarrow -\ln x$ as $t \rightarrow \infty$.

Using relation (7) written for $F_k^I[\xi]$ we have

$$\sum_{k=0}^t \{1 - F_k^I[F_{t-k}[\xi_t]]\} = A - \sum_{k=0}^t R_k^I[F_{t-k}[\xi_t]][1 - F_{t-k}[\xi_t]].$$

The relations (9), (24) and the simple approximation $\ln(1 - x) \sim -x$, $x \rightarrow 0$, gives us that

$$\lim_{t \rightarrow \infty} \ln H_t[\xi_t] = -\lim_{t \rightarrow \infty} A = \ln x.$$

Now we use the standard arguments allowing to obtain the limit theorem with large deviations from the behavior of the Laplace transform (see [15], for example). The number $0 < x < 1$ is the Laplace transform of the random vector $\tau = (\tau_1, \dots, \tau_j)$

such that $P\{\tau = \mathbf{0}\} = \mathbf{x}$, $\mathbf{P}\{\cup_{i=1}^j \{\tau_i = \infty\}\} = \mathbf{1} - \mathbf{x}$ and $P\{\cap_{i=1}^j \{\tau_i < \infty\}, \tau \neq \mathbf{0}\} = 0$. Hence by the continuity theorem for Laplace transforms

$$\lim_{t \rightarrow \infty} P \left\{ \hat{Z}_t[\chi(A_i)] t^{-1} x_i^{-\mu/\alpha(t)} \leq y_i, i = 1, \dots, j \right\} = x$$

for any finite positive numbers y_1, \dots, y_j . The assertion of the theorem follows from the last relation if we put $y_1 = \dots = y_j = 1$. The theorem is proved.

The following theorem gives a limit distribution when $\alpha(t) \sim C(\ln t)^{-1}$.

Theorem 4 Let (M), (R) and (8) be satisfied, $\rho = 1$, $0 < \mu < \infty$ and $q = 1$. If $\alpha(t) \rightarrow 0$ such that $\alpha(t) \ln t \rightarrow C \in (0, \infty)$ as $t \rightarrow \infty$, then for every finite measurable decomposition $\{A_i, i = 1, \dots, j\}$ of X

$$\lim_{t \rightarrow \infty} P \left\{ \frac{\hat{Z}_t[\chi(A_i)]^{\alpha(t)/\mu} - 1}{e^{C/\mu} - 1} \leq x_i, i = 1, \dots, j \right\} = P\{\nu_i \leq x_i, i = 1, \dots, j\},$$

where $\nu_1 = \dots = \nu_j$ with probability 1 and $P\{\nu_i \leq x\} = e^{-C/\mu} + x(1 - e^{-C/\mu}), 0 \leq x \leq 1$.

Proof. We consider the sum from (24) with

$$\xi_t = \exp\left\{-\sum_{i=1}^j \lambda_i \chi(A_i/C_t(x_i))\right\},$$

where $C_t(x_i) = (1 + x_i(e^{C/\mu} - 1))^{\mu/\alpha(t)}$, $0 < x_i < 1$. It is not difficult to see that under the conditions of Theorem 5 $C_t(x_i) \rightarrow \infty$, $C_t(x_i) = o(t)$ as $t \rightarrow \infty$. If we decompose the sum as in (24) with $\lambda_\alpha(t) = \ln t$ and $L(t) = \ln \ln t$, using again (25) and (26) obtain that as $t \rightarrow \infty$

$$A_2 \sim \frac{\alpha(t)}{\mu} \ln \frac{\mu t \Phi^*[1 - \xi_t] + 1}{\mu \Phi^*[1 - \xi_t] \ln \ln t + 1}. \quad (28)$$

Taking into account the fact that $\lim_{t \rightarrow \infty} C_t(x) \Phi^*[1 - \xi_t] = \delta(E_0)$, where $x = \min\{x_1, \dots, x_j\}$ and $\delta(E_0)$ the same as in (27), we have from (28) that A_2 tends to $C\mu^{-1} - \ln(1 + x(e^{C/\mu} - 1))$. The same arguments as in the proof of Theorem 4 shows that A_1 and A_3 tend to zero as $t \rightarrow \infty$. Thus

$$\lim_{t \rightarrow \infty} H_t[\xi_t] = e^{-C/\mu} + x(1 - e^{-C/\mu}).$$

The assertion of the theorem follows from the last relation by continuity theorem for the Laplace transforms. The theorem is proved.

We proceed a further study in the following scheme. We consider the "partial processes" introduced in Section 3. It is clear from the definition that $Z_t^{(1)}$ and $Z_t^{(3)}$

are branching diffusions generated by the particles immigrated in the beginning of the process and by the recent immigrants, respectively. First, we prove limit theorems for partial processes using analytic technique as in the proof of theorems 4 and 5. After that we deduce the explicit limit theorem for the basic process from those results by the direct probability arguments.

We denote by $H_t^{(i)}[\xi]$ generating functionals of processes $\hat{Z}_t^{(i)}, i = 1, 2, 3$, that is

$$H_t^{(i)}[\xi] = \prod_{k \in I_i} F_k^I[F_{t-k}[\xi]].$$

Theorem 5 Let (M), (R) and (8) be satisfied, $\rho = 1, 0 < \mu < \infty, q = 1$. If $\alpha(t) \rightarrow 0$ such that $\alpha(t) \ln t \rightarrow 0, \alpha \geq 1$, then for any finite measurable decomposition $\{A_i, i = 1, \dots, j\}$ of X

$$\begin{aligned} \lim_{t \rightarrow \infty} P \left\{ \frac{\hat{Z}_t^{(1)}[\chi[A_i]]}{t} \leq x_i, i = 1, \dots, j | \hat{Z}_t^{(1)} \neq \theta \right\} = \\ = P \{ \nu_i \Phi^*[\chi(A_i)] \leq x_i, i = 1, \dots, j \}, \end{aligned}$$

where $\nu_1 = \dots = \nu_j$ with probability 1 and $P\{\nu_i \leq x\} = 1 - e^{-x/\mu}, x \geq 0$

Theorem 6 Let (M), (R) and (8) be satisfied, $\rho = 1, 0 < \infty, q = 1$. If $\alpha(t) \rightarrow 0$ such that $\alpha(t) \ln t \rightarrow 0$, then for any finite measurable decomposition $\{A_i, i = 1, \dots, j\}$ of X

$$\lim_{t \rightarrow \infty} P \left\{ \frac{\ln \hat{Z}_t^{(3)}[\chi(A_i)]}{\ln t} \leq x_i, i = 1, \dots, j | \hat{Z}_t^{(3)} \neq \theta \right\} = P \{ \nu_i \leq x_i, i = 1, \dots, j \}$$

where (ν_1, \dots, ν_j) are the same as in Theorem 4.

Proof of Theorem 6. We consider the sum

$$A = \sum_{k \in I_1} \left(1 - F_k^I[F_{t-k}[\xi_t]] \right), \xi_t = \exp \left\{ -t^{-1} \sum_{i=1}^j \lambda_i \chi(A_i) \right\},$$

where $\lambda_i \geq 0, i = 1, \dots, j$ and I_1 is the set defined in Section 3. Using (7) we can write

$$A = A_1 + A_2 \tag{29}$$

where

$$A_1 = \sum_{k \in I_1} M_k^I[1 - F_{t-k}[\xi_t]], A_2 = \sum_{k \in I_1} R_k^I(F_{t-k}[\xi_t])[1 - F_{t-k}[\xi_t]].$$

Let us consider A_1 . Using relations (25), (26) and the definition of the set I_1 , we get that A_1 is equivalent as $t \rightarrow \infty$ to $a(L_1(t)) \{\mu t + \Phi^*[1 - \xi_t]^{-1}\}^{-1}$ where $a(t)$ is defined in the Section 3. Since $\Phi^*[\cdot]$ is linear-bounded and nonnegative on β_+ (see Assumption (M)), $t\Phi^*[1 - \xi_t] \rightarrow \delta(E)$ as $t \rightarrow \infty$, where $E = \{1, \dots, j\}$ and δ is the same as in (27). Thus from here and from (17) we conclude that

$$A_1 \sim \frac{\delta(E)}{1 + \mu\delta(E)} a(t)t^{-1}, \quad t \rightarrow \infty. \quad (30)$$

Now, we consider A_2 . Since $R_k^I(\cdot)[\cdot]$ is a non-increasing function of the first argument and a linear bounded functional with respect to the second argument, using relations (25), (26) we find that for some positive constant C and sufficiently large t

$$A_2 \leq C \sup_k \frac{R_k^I((1 - \varepsilon_t)1)[\varphi]}{\alpha(k)} t^{-1} a(t),$$

where $\varepsilon_t > 0$ such that $\varepsilon_t \rightarrow 0$ as $t \rightarrow \infty$. The last inequality shows that under condition (8), $A_2 = o(t^{-1}a(t))$ as $t \rightarrow \infty$.

It is not difficult to see that the Laplace transform of the conditional distribution in Theorem 6 equals

$$1 - \frac{1 - H_t^{(1)}[\xi_t]}{P\{\hat{Z}_t^{(1)} \neq \theta\}}.$$

Using the approximation $\ln(1-x) \sim -x$, $x \rightarrow 0$ we have that $1 - H_t^{(1)}[\xi_t] \sim A$, $t \rightarrow \infty$ and, consequently, taking into account (29), (30) and part 1^o of Theorem 2, we obtain that the limit of the above above Laplace transform is $(1 + \mu\delta(E))^{-1}$. The theorem is proved.

Proof of Theorem 7. First we consider the sum

$$C_t(m) = \sum_{k=0}^m \left\{ \mu k + \Phi^*[1 - \xi_t]^{-1} \right\}^{-1}, \quad \xi_t = \exp \left\{ - \sum_{i=1}^j \lambda_i \chi(A_i) t^{-x_i} \right\}.$$

It is not difficult to see that $C_t(t) - C_t(L_2(t)) \leq b(t) - b(L_2(t))$ and the last difference has the order $o(\ln t)$ according to relation (17). If we denote $q(x, t) = \{\mu x + \Phi^*[1 - \xi_t]^{-1}\}^{-1}$, then $C_t(t)$ can be written as follows:

$$C_t(t) = \int_0^{t-1} q(x, t) dx + C^{(1)}(t), \quad 0 \leq C^{(1)}(t) \leq \sum_{k=0}^{t-1} [q(k, t) - q(k+1, t)] \quad (31)$$

and the sum on the right side tends to zero as $t \rightarrow \infty$. Since $t^x \Phi^*[1 - \xi_t] \rightarrow \delta(E_0)$, where $x = \min\{x_1, \dots, x_j\}$ and $\delta(E_0)$ the same as in (27), we find that the integral

in (31) is equivalent to $\mu^{-1} \ln(1 + \mu\delta(E_0)t^{1-x})$ as $t \rightarrow \infty$. Hence we have

$$C_t(L_2(t)) \sim \frac{1-x}{\mu} \ln t, t \rightarrow \infty. \quad (32)$$

Now we consider

$$C^{(2)}(t) = \sum_{k \in I_3} \|\varepsilon_{t-k}^*[\xi_t]\| q(t-k, t),$$

where $\varepsilon_t^*[\cdot]$ is the functional from (25). Since $\|\varepsilon_k^*[\xi]\| \rightarrow 0$ as $k \rightarrow \infty$ uniformly in $\xi \in \overline{\nabla}_+ \cap \{\Phi^*[1-\xi] > 0\}$, there exists a positive integer m such that $\|\varepsilon_{t-k}^*[\xi_t]\| < \varepsilon$ for $t-k \geq m$.

We partition I_3 as $I_3 = I_3^{(1)} \cup I_3^{(2)}$, where $I_3^{(1)} = \{k : t - L_2 \leq k \leq t - m\}$ and $I_3^{(2)} = \{t - m < k < t\}$. It follows from the choice of m that the sum of terms with $k \in I_3^{(1)}$ is less than $\varepsilon C_t(L_2(t))$ and the sum of terms with $k \in I_3^{(2)}$ tends to zero as $t \rightarrow \infty$ for any fixed m . Thus using (25) and (26), we obtain that as $t \rightarrow \infty$

$$A_1 \equiv \sum_{k \in I_3} M_k^I [1 - F_{t-k}[\xi_t]] \sim \alpha(t) C_t(L_2(t)). \quad (33)$$

The same arguments as in the proof of Theorem 6, if we take into account (32) and (33), give us that under condition (8), $1 - H_t^{(3)}[\xi_t] \sim \mu^{-1}(1-x)\alpha(t) \ln t$ as $t \rightarrow \infty$. Consequently, using part 2⁰ of Theorem 2, we conclude that the Laplace transform of the conditional distribution in Theorem 7 tends to $x = \min\{x_1, \dots, x_j\}$ as $t \rightarrow \infty$. How from this fact the assertion of the theorem follows can be shown as in the proof of Theorem 5. The theorem is proved.

We now formulate the limit theorem for the basic branching diffusion. It turns out that the form of limiting distributions and the normalizing functions depends on the behavior at infinity of the function $\Theta(t) = t\alpha(t) \ln t/a(t)$.

Theorem 7 Let $(M), (R)$ and (8) be satisfied, $\rho = 1, 0 < \mu < \infty, q = 1, \alpha(t) \ln t \rightarrow 0, t \rightarrow \infty$ and $\{A_i, i = 1, \dots, j\}$ be a finite measurable decomposition of X .

I⁰. If $\liminf_{t \rightarrow \infty} \Theta(t) > 0$, then for any $x_i \in [0, 1], i = 1, \dots, j, x = \min\{x_1, \dots, x_j\}$ as $t \rightarrow \infty$

$$\begin{aligned} P \left\{ \frac{\ln \hat{Z}_t[\chi[A_i]]}{\ln t} = \leq x_i, i = 1, \dots, j | \hat{Z}_t \neq \theta \right\} \\ = \frac{x\Theta(t)}{1 + \Theta(t)} (1 + o(1)). \end{aligned}$$

2⁰. If $\limsup_{t \rightarrow \infty} \Theta(t) < \infty$, then for any $x_i \geq 0, i = 1, \dots, j$ as $t \rightarrow \infty$

$$P \left\{ \frac{\hat{Z}_t[\chi[A_i]]}{t} \leq x_i, i = 1, \dots, j | \hat{Z}_t \neq \theta \right\} = \frac{\Theta(t) + G(x_1, \dots, x_j)}{1 + \Theta(t)} (1 + o(t)),$$

where $G(x_1, \dots, x_j)$ is the limiting distribution in Theorem 6.

Examples. 1⁰. If $\alpha < 1$, then $\Theta(t) \rightarrow \infty$ and we obtain from part 1⁰ of the theorem that the limit distribution is the same as in Theorem 7. If $\alpha > 1$, then $\Theta(t) \rightarrow 0$ and it follows from part 2⁰ of the theorem that the limit distribution is the same as in Theorem 6.

2⁰. Let now $\alpha = 1, l(t) \equiv C_0 \in (0, \infty)$. Then $\Theta(t) \rightarrow 1$ as $t \rightarrow \infty$. In this case it follows from parts 1⁰ and 2⁰ of Theorem 8 that the limit distributions there have atoms of the same mass 2^{-1} at points 1 and zero respectively. The limit Theorems 6 and 7 obtained for the partial processes explain the cause of the appearance of these atoms.

Proof of Theorem 8. Let $B_i = B_i(t)$ be the event that i -th partial process is not equal to θ at time t . We consider the following events

$$C_i = B_i \cap \bar{B}_j \cap \bar{B}_k, C_{ij} = B_i \cap B_j \cap \bar{B}_k, C_{ijk} = B_i \cap B_j \cap B_k,$$

where $i, j, k = 1, 2, 3, i \neq j, i \neq k, j \neq k$.

If we denote $D = \{\hat{Z}_t \neq \theta\}$, then it is not difficult to see that

$$D = \bigcup_{i=1}^3 C_i \cup \bigcup_{\substack{i,j=1 \\ i \neq j}}^2 C_{ij} \cup C_{123}. \quad (34)$$

Since the events on the right side of (39) are disjoint, for any event A we have

$$P\{A|D\} = \sum_{i=1}^3 P\{A|C_i\} \frac{P\{C_i\}}{P(D)} + \sum_{\substack{i,j=1 \\ i \neq j}}^2 \frac{P\{A \cap C_{ij}\}}{P\{D\}} + \frac{P\{A \cap C_{123}\}}{P\{D\}}. \quad (35)$$

It follows from the definition of the process that the partial processes $\hat{Z}_t^{(i)}, i = 1, 2, 3$, are independent. Therefore, using the assertion of Theorem 1, we get, under the conditions of Theorem 8, that

$$P\{C_i\} \sim P\{B_i\}, P\{C_{ij}\} \sim P\{B_i\}P\{B_j\}, P\{C_{123}\} \sim P\{B_1\}P\{B_2\}P\{B_3\}.$$

Let us prove part 1⁰. In order to do it we put in (40) that

$$A = \left\{ \frac{\ln \hat{Z}_t[\chi(A_i)]}{\ln t} \leq x_i, i = 1, \dots, j \right\}, 0 < x_i < 1.$$

Using Theorems 6 and 7 we have that $P\{A|C_3\} \rightarrow x = \min\{x_1, \dots, x_j\}$ and $P\{A|C_1\} \rightarrow 0$ as $t \rightarrow \infty$. On the other hand, as it was shown in the proof of Theorem 2, $P\{C_2\} \sim P\{B_2\} = o(P\{D\})$, $t \rightarrow \infty$. Thus the first sum in (35) is equivalent to $xP\{B_3\}/P\{D\}$ as $t \rightarrow \infty$.

Taking into account results of Theorem 2 again, we see that $P\{C_{ij}\} = o(P\{D\})$ and $P\{C_{123}\} = o(P\{D\})$ as $t \rightarrow \infty$. Hence, it follows from these relations and from (35) that as $t \rightarrow \infty$

$$P\{A|D\} = \frac{x\Theta(t)}{1 + \Theta(t)} + o(1)$$

which gives the assertion of part 1⁰ when $\liminf \Theta(t) > 0$. Part 1⁰ of the theorem is proved.

Now we prove part 2⁰. We now put in (35) $A = \{t^{-1}\hat{Z}_t[\chi(A_i)] \leq x_i, i = 1, \dots, j\}$. In this case we obtain from Theorems 6 and 7 that $P\{A|C_1\}$ tends to $G(x_1, \dots, x_j)$ and $P\{A|C_3\} \rightarrow 1$ as $t \rightarrow \infty$.

It follows from the condition $\limsup \Theta(t) < \infty$ that $\alpha \geq 1$. Consequently we can use part 1⁰ of Theorem 2 and thus, the first sum in (35) as $t \rightarrow \infty$ is equivalent to $(G(x_1, \dots, x_j) + \Theta(t))(1 + \Theta(t))^{-1}$. The theorem is proved.

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