Two-type branching processes with sub-exponential life-spans and SIR epidemic models

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Abstract

We consider a model of age-dependent branching stochastic process which takes into account the incubation period of the life of individuals. We demonstrate that such processes may be treated as a two-type branching process with a periodic mean matrix. In the case when the Malthusian parameter does not exist study of the process requires additional restrictions on the life and incubation time distributions which define so called sub-exponential family [2]. We obtain certain new properties of sub-exponential distributions, in particular, describe a subclass, which is closed with respect to convolution. Using these results we derive asymptotic behavior of the first and second moments and of the probability of non-extinction. We also prove a limit theorem for the process conditioned on non-extinction.

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1 Introduction

We study a modification of the branching stochastic process which takes into account the incubation period of individual's life time. Consider a population of individuals of the same type who colonize a region. We assume that at time zero we have a single individual (ancestor) of age zero. This individual lives a random time L. At the end of the time interval L the individual will die or leave the region (emigrate) after laying a random number of ν eggs (seeds). After a random incubation period τ each egg, independently of others will generate ξ individuals of age zero, with $P\{\xi = 1\} = 1 - P\{\xi = 0\} = p$. This means that each egg will generate one individual with probability p and will be "destroyed" with probability $0 \leq q < 1, p+q = 1$. These new individuals, independently each of others, will behave in the same manner as the initial ancestor, i.e. will live a random period of time and lay a random number of eggs, before they die or emigrate, and so on. Assume that pairs (L, ν) and (τ, ξ) are independent and identically distributed for distinct individuals and eggs.

It is known that "susceptible-infectious-removed" (SIR) epidemic model can successfully be approximated by branching processes, when the initial number of susceptible individuals is large Andersson and Britton [1], p. 22. More precisely, in SIR epidemic models it is assumed that individuals are at first susceptible, if they get infected, they become infectious and remain so for some time, after which they recover and become immune. An individual is said to be removed, if he (or she) has recovered and is immune or dies, and does not further participate in the epidemic. In the framework of the epidemic models L and ν may be understood as the infectious period and the number of contacts during the infectious period of a single infective individual. Naturally, the variable τ is the incubation period and q may be considered as immune rate or as the rate of vaccination.

The idea of branching process approximation of the stochastic models of the epidemic has been applied in many papers. Recent monograph by Mode and Sleeman [8] is an excellent source of the material on applications of stochastic processes in epidemiology. In particular in Chapter 2 of this monograph possible distributions of the incubation period are discussed. In Chapter 3 of Andersson and Britton [1], a systematic study of SIR models, based on the branching approximation, is provided.

The process that has been described can be given by the distributions of pairs (L, ν) and (τ, ξ). If the offspring number does not depend on the life time of the parent and the fate of the egg on the incubation period, then marginal distributions

$$G_1(t) = P\{L \leq t\},\ G_2(t) = P\{\tau \leq t\},\ t \geq 0$$

with support on $[0, \infty)$ and distributions

$$P_k = P\{\nu = k\},\ k \geq 0,\ p = P\{\xi = 1\} = 1 - P\{\xi = 0\}$$

define the process completely. We assume that $p > 0$ to exclude the trivial case, when the process will extinct in the first generation.

Realizations of the process are given by vector $\mathbf{X}(t) = (X_1(t), X_2(t))$ with $X_1(t)$ being the number of individuals (the number of infectious individuals) and $X_2(t)$ being the number of eggs (the number of individuals who had a contact with an infectious individual). The process $\mathbf{X}(t)$ can be considered as a multi-type age-dependent process with types of individuals T_1 and T_2. Individuals of type T_1 generate only individuals of type T_2 and vice versa i. e. evolution of the process has the form of transformations $T_1 \to T_2$ and $T_2 \to T_1$. The components of the vector $\mathbf{X}(t)$ are, naturally, the numbers of individuals of types T_1 and T_2 at time t.

In the case when the Malthusian parameter exists asymptotic properties of process $\mathbf{X}(t)$ can be derived using results from the theory of multi type processes. However in subcritical processes, which is the case in most epidemic models, the Malthusian parameter may do not exist. In this case the study of the process requires more delicate analysis and needs additional restrictions on the life time distributions. These restrictions define a class of so called sub-exponential distributions, which have tails that decay at a slower than exponential rate. The family of sub-exponential distributions was first introduced by Chistyakov [3], who studied asymptotic properties of the single type age-dependent process with sub-exponential life-time distributions. In [4] a class of distributions which is larger than sub-exponential is described. Some of distributions from this class may have tails which do not decay at a slower than exponential rate. We note that the sub-exponential class includes very important for applications distributions, such as, Weibull, with decreasing hazard function, Log-normal, Log-logistic, Pareto and some of other heavy tailed distributions. Possibilities of using the heavy tailed distributions in modelling of the incubation period of infectious diseases, including HIV or AIDS, discussed in Chapter 2 of [8].

This situation, in particular, explains the interest in study of the family of sub-exponential distributions by a number of authors, who have investigated various aspects of the family. However an important problem on closure of the family under convolutions was open until ninetieth. In [6] an example, demonstrating that the sub-exponential family, generally speaking, is not closed under convolution, is constructed.

In the monograph [7], p. 201, Ch. J. Mode describes an extension of the results of Chistyakov [3] to multitype branching processes, in which the life-span distributions are not the same, as an important research problem. We address this problem in a case of two-type process of a special form. For this purpose we first extend some properties of sub-exponential distributions related to the finite number of convolutions. In particular we define a subclass of the sub-exponential family, which is closed with respect to the convolution. This extension allows us to study the limiting behavior of the process $\mathbf{X}(t)$ as $t \to \infty$ and obtain results extending known limit theorems in the case when the Malthusian parameter does not exist.

2 Generating functions

Let

$$F^i(t, s_1, s_2) = E[s_1^{X_1(t)} s_2^{X_2(t)} | \mathbf{X}(0) = \varepsilon_i], \ i = 1, 2,$$

where $|s_i| \leq 1, \ \varepsilon_i = (\delta_{1i}, \delta_{2i})$ and δ_{ij} is the Kronecker delta ($\delta_{ii} = 1, \delta_{ij} = 0, i \neq j$). We also denote

$$\Phi(s) = \sum_{k=0}^{\infty} P_k s^k, \quad \varphi(s) = q + ps, \ \mathbf{s} = (s_1, s_2),$$

and $m = E\nu = \Phi'(1), \ \sigma^2 = E\nu(\nu - 1) = \Phi''(1)$.

It follows from the well known results for multi type processes [9] that the probability generating functions $F^i(t, s_1, s_2)$ for $|s_i| \leq 1, i = 1, 2$, satisfy the following non linear integral equations

$$F^1(t, s_1, s_2) = s_1(1 - G_1(t)) + \int_0^t \Phi(F^2(t - u, s_1, s_2)) dG_1(u), \tag{2.1}$$

$$F^2(t, s_1, s_2) = s_2(1 - G_2(t)) + qG_2(t) + p\int_0^t F^1(t - u, s_1, s_2) dG_2(u). \tag{2.2}$$

with initial conditions $F^i(0, s_1, s_2) = s_i, i = 1, 2$.

Let $\mathbf{X}_n = (X_n^1, X_n^2), n \geq 0$ be the process counting generations in the process $\mathbf{X}(t)$, i.e. $X_n^i, i = 1, 2$, is the number of individuals of type T_i in the nth generation. It is well known that $\mathbf{X}_n, n \geq 0$, is a two type Galton-Watson process with offspring generating functions $F^i(s_1, s_2) = E[s_1^{X_1^1} s_2^{X_1^2} | \mathbf{X}_0 = \varepsilon_i]$, $i = 1, 2$. It follows from the definition of the process that

$$F^1(s_1, s_2) = \Phi(s_2), \;\; F^2(s_1, s_2) = \varphi(s_1). \tag{2.3}$$

If $G_2(0+) = 1$, i.e. no incubation period, we obtain from (2.2) that $F^2(t, s_1, s_2) = \varphi(F^1(t, s_1, s_2))$. Consequently, the equation (2.1) will take the form

$$F^1(t, s_1, s_2) = s_1(1 - G_1(t)) + \int_0^t \Phi(\varphi(F^1(t - u, s_1, s_2)))dG_1(u).$$

In this case process $X_1(t)$ is the following modification of single type age-dependent process. The reproduction of individuals is according to usual Bellman-Harris branching process, however, after reproduction, each of new born individuals may emigrate (or may be killed) with probability q. Note that in this particular case our model is close to the branching process with disasters, considered by Kaplan *et al.* [5], where individuals, participating in the process, may disappear at renewal moments of a renewal process. If $G_2(0+) = 1$ and $p = 1$, then, naturally, we obtain the single type Bellman-Harris process.
We denote $\mathbf{M} = (EX_j^i, \;\; i, j = 1, 2)$ the mean matrix whose elements are expected numbers of type T_j offspring of a single type T_i individual in one generation. It is clear that

$$EX_j^i = \frac{\partial F^i(s_1, s_2)}{\partial s_j}|_{\mathbf{s}=\mathbf{1}}, \;\; \mathbf{s} = (s_1, s_2).$$

Therefore we have due to (2.3) that

$$\mathbf{M} = \begin{pmatrix} 0 & m \\ p & 0 \end{pmatrix}$$

where $m = E\nu$. It is easy to see that $\mathbf{M}$ has eigenvalues $\pm\sqrt{pm}$. Thus $\rho = \sqrt{pm}$ is the Perron eigenvalue and corresponding positive right and left eigenvectors $\mathbf{U} = (u^1, u^2)^T, \mathbf{V} = (v_1, v_2)$ are

$$\mathbf{U} = \left(\frac{\sqrt{pm}}{p + \sqrt{pm}}, \; \frac{p}{p + \sqrt{pm}}\right)^T, \;\; \mathbf{V} = \left(\frac{p + \sqrt{pm}}{2\sqrt{pm}}, \; \frac{p + \sqrt{pm}}{2p}\right).$$

The eigenvectors are normalized such that $\mathbf{U}^T\mathbf{1} = 1, \mathbf{V}\ \mathbf{U} = 1$, where $\mathbf{1}^T = (1, 1)$.
We have $\sum_{j=1}^2 EX_j^i u^j = \rho u^i$ and $\sum_{i=1}^2 EX_j^i v_i = \rho v_j$. Concerning the second factorial moments $b_{jk}^i = E[X_j^i X_k^i],\ j \neq k$ and $b_{jj}^i = E[X_j^i(X_j^i - 1)]$, we find due to (2.3) that $b_{22}^1 = \sigma^2$ and $b_{jk}^i = 0$ for all other possible values of i, j and k. Therefore

$$b = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} v_i b_{jk}^i u^j u^k = \sigma^2 v_1 (u^2)^2 = \frac{\sigma^2 p\sqrt{\frac{p}{m}}}{2(p+\sqrt{pm})}.$$

Following the general theory, we call process $\mathbf{X}(t)$ subcritical, critical and supercritical, if $mp < 1, mp = 1, \sigma^2 > 0$ and $mp > 1$ respectively.

Since the state $\mathbf{0}$ is absorbing it can be defined as

$$q^{(i)} = P\{\mathbf{X}(t) = \mathbf{0},\ for\ some\ t > 0 | \mathbf{X}(0) = \varepsilon_i\}$$

for the process starting with one individual of type T_i, $\mathbf{q} = (q^{(1)}, q^{(2)})$. Since, when $G_i(0+) = 0, i = 1, 2$,

$$\{\mathbf{X}(t) = \mathbf{0},\ for\ some\ t > 0\} = \{\mathbf{X}_n = \mathbf{0},\ for\ some\ n > 0\},$$

it is sufficient to find the extinction probability of the Galton-Watson process $\mathbf{X}_n, n \geq 0$, constituted by generation sizes of $\mathbf{X}(t)$.

Therefore, using results for multi type processes [9], we obtain that the extinction probability $q^{(2)}$ is the smallest non-negative root of the equation

$$\Phi(x) = \frac{x - q}{p} \tag{2.4}$$

and $q^{(1)} = (q^{(2)} - q)/p$. Hence $\mathbf{q} = \mathbf{1}^T$, if and only if $mp \leq 1$.

3 Sub-exponential distributions

We consider cumulative distribution functions of positive random variables. Following Chistyakov [3] define three classes of distributions.

Definition. *Let $A(t), t \in [0, \infty)$ be cumulative distribution of a positive random variable.*

a) We say $A(t) \in \Re_0$, if for any $\varepsilon > 0$

$$\int_0^\infty e^{\varepsilon t} dA(t) = \infty. \tag{3.1}$$

b) Function $A(t) \in \Re_1$, if for any fixed $u > 0$

$$\lim_{t\to\infty} \frac{1 - A(t-u)}{1 - A(t)} = 1. \tag{3.2}$$

c) Function $A(t) \in \Re_n, n \geq 2$, if

$$\lim_{t\to\infty} \frac{1 - A^{*n}(t)}{1 - A(t)} = n. \tag{3.3}$$

We note that, as it was proved in [3], $\Re_2 \subset \Re_n$ for any $n \geq 2$ and also $\Re_2 \subset \Re_1 \subset \Re_0$. Distributions belonging to the class $\Re_2$ are called sub-exponential, describing the property of their tail which decay at a slower than exponential rate (see [2], p.147, for example). We obtain certain properties of sub-exponential distributions. Let τ_1 and τ_2 be independent nonnegative random variables and $A_i(t) = P\{\tau_i \leq t\}, i = 1, 2$.

We denote $C(t) := (1 - A_1(t))/(1 - A_2(t))$. Let $A_i(t), i = 1, 2$ be such that there exists the limit

$$\lim_{t\to\infty} C(t) = C \in [0, \infty]. \tag{3.4}$$

Lemma 1. *If $A_i(t) \in \Re_2, i = 1, 2$ and (3.4) is satisfied, then*

$$\lim_{t\to\infty} \frac{1 - A_1 * A_2(t)}{1 - A_2(t)} = 1 + C. \tag{3.5}$$

Proof. First we derive from the trivial identity

$$\int_0^t \frac{1 - A(t-u)}{1 - A(t)} dA(u) = \frac{1 - A^{*2}(t)}{1 - A(t)} - 1$$

that for any $A(t) \in \Re_2$

$$\lim_{t\to\infty} \int_0^t \frac{1 - A(t-u)}{1 - A(t)} dA(u) = 1. \tag{3.6}$$

Now we consider the case of $C < \infty$. If we denote the ratio on the left side of (3.5) by $B(t)$, then

$$B(t) = 1 + \int_0^t C(t-u) \frac{1 - A_2(t-u)}{1 - A_2(t)} dA_2(u). \tag{3.7}$$

Since $C(t)$ has a finite limit, there exists a positive constant C_1 such that $C(t) \leq C_1$ for all $t > 0$. In fact, for each $\varepsilon > 0$ there is such T that $0 \leq C(t) \leq C + \varepsilon$ for $t > T$. For each T there is $K(T)$, such that $\sup_{0 \leq t \leq T} C(t) \leq K(T)$. Thus we can choose $C_1 = \max\{C + \varepsilon, K(T)\}$.

Let x be some positive number. If we partition the interval of integration in (3.7) as $[0, t] = [0, x] \cup (x, t]$, then the integral over the first part is dominated by

$$\hat{C}(t, x) \frac{1 - A_2(t-x)}{1 - A_2(t)} A_2(x),$$

where $\hat{C}(t, x) = \sup_{t-x \leq u \leq t} C(u)$. What concerns the integral over the second part, taking into account obvious inequality $P\{\tau_2 > t - u\} > P\{\tau_2 > t\}$, obtain that it is not greater than

$$C_1 \left[\int_0^t \frac{1 - A_2(t-u)}{1 - A_2(t)} dA_2(u) - A_2(x) \right].$$

Since $\hat{C}(t, x) \to C$ as $n \to \infty$ for any $x > 0$ and $\Re_2 \subset \Re_1$, taking into account (3.6) we have for each fixed $x > 0$

$$\lim_{t \to \infty} \sup B(t) \leq 1 + C A_2(x) + C_1[1 - A_2(x)].$$

Hence, sending x to infinity we conclude that

$$\lim_{t \to \infty} \sup B(t) \leq 1 + C \tag{3.8}$$

Since $\{\tau_1 > t\} \cup \{\tau_2 > t\} \subset \{\tau_1 + \tau_2 > t\}$ and random variables τ_1 and τ_2 are independent, we obtain that

$$1 - A_1 * A_2(t) \geq 1 - A_1(t) A_2(t).$$

Therefore we have

$$\lim_{t \to \infty} \inf B(t) \geq 1 + C. \tag{3.9}$$

The assertion of the lemma follows from (3.8) and (3.9).

Let now $C = \infty$. In this case we rewrite $B(t)$ as

$$B(t) = \frac{1 - A_1 * A_2(t)}{1 - A_1(t)} \cdot \frac{1 - A_1(t)}{1 - A_2(t)}.$$

Since $(1 - A_2(t))/(1 - A_1(t)) \to 0$ as $t \to \infty$, applying already proved part of the lemma we obtain that the first ratio approaches one as $t \to \infty$ and, consequently $B(t) \to \infty$, which desired to proof. Lemma is proved.

We denote $A(t) = A_1 * A_2(t)$ and by $\Re_2^*$ the subclass of $\Re_2$ such that for each pair $A_i(t) \in \Re_2^*, i = 1, 2$ the condition (3.4) is satisfied. The following result is important in the proof of main theorems.

Lemma 2.
a) If $A_i(t) \in \Re_2, i = 1, 2$ and (3.4) is satisfied, then $A(t) \in \Re_2$.
b) The subclass of sub-exponential distributions $\Re_2^*$ is closed with respect to the convolution.

Proof. Let first $C \in [0, \infty)$.We consider

$$D(t) := \frac{1 - A^{*2}(t)}{1 - A(t)} = \frac{1 - A_1^{*2} * A_2^{*2}(t)}{1 - A(t)}.$$

We rewrite $D(t)$ as

$$D(t) = \frac{1 - A_1^{*2} * A_2^{*2}(t)}{1 - A_2^{*2}(t)} \cdot \frac{1 - A_2^{*2}(t)}{1 - A_2(t)} \cdot \frac{1 - A_2(t)}{1 - A(t)}. \tag{3.10}$$

It follows from Lemma 1 that the last ratio on the right of (3.10) tends as $t \to \infty$ to $(1 + C)^{-1}$. Taking into account that $A_1(t), A_2(t) \in \Re_2$, we obtain that, when (3.4) is satisfied

$$\lim_{t\to\infty} \frac{1 - A_1^{*2}(t)}{1 - A_2^{*2}(t)} = C.$$

Therefore, if we apply Lemma 1 to first ratio in (3.10), obtain that it approaches $1 + C$ as $t \to \infty$. Thus $D(t) \to 2$ as $t \to \infty$.

If $C = \infty$, then $\lim_{t\to\infty}(C(t))^{-1} = 0$ and we just need to use exactly symmetric arguments to obtain again that $D(t)$ as $t \to \infty$ approaches 2. Part (a) is proved.

To prove Part (b), let $A(t) = A_1 * A_2(t)$ and $B(t) = B_1 * B_2(t)$ be two convolutions of sub-exponential distributions from $\Re_2^*$. Due to Part (a) $A(t), B(t) \in \Re_2$. We consider

$$\frac{1-A(t)}{1-B(t)} = \frac{1-A(t)}{1-A_1(t)} \cdot \frac{1-B_1(t)}{1-B(t)} \cdot \frac{1-A_1(t)}{1-B_1(t)}.$$

The last ratio has a limit due to $A_1(t), B_1(t) \in \Re_2^*$. Applying Lemma 1 we obtain that the first and second ratios have also finite or infinite limits. Thus $A(t), B(t) \in \Re_2^*$. The lemma is proved.

Lemma 3. *If $A_i(t) \in \Re_2, i = 1, 2$ and condition (3.4) is satisfied, then for each $i \geq 1$*

$$\lim_{t\to\infty} \frac{1 - A_2 * A^{*i}(t)}{1-A(t)} = \frac{1}{1+C} + i. \tag{3.11}$$

Proof. It follows from Lemma 1 that, if condition (3.4) holds, then

$$\frac{1-A_2(t)}{1-A(t)} \to \frac{1}{1+C}. \tag{3.12}$$

If conditions of Lemma 3 are satisfied, then $A(t) \in \Re_2$ due to Lemma 2. Taking this into account, we obtain the assertion of the lemma by consequent application of relation (3.5).

4 Asymptotic behavior of moments

If an epidemic is initiated by a single infective at time zero, what is the expected number of infective individuals and individuals who had a contact with an infective at time t? This is a standard problem in the theory of branching processes when the Malthusian parameter does exist. However, when the parameter does not exist, the problem requires more refined analysis and additional restrictions on the distributions of the incubation and infectious periods.

First we consider the following moments of the process

$$A_j^i(t) = E[X_j(t)|\mathbf{X}(0) = \varepsilon_i], B_{jk}^i(t) = E[X_j(t)(X_k(t) - \delta_{jk})|\mathbf{X}(0) = \varepsilon_i], \tag{4.1}$$

where $i, j, k = 1, 2$. We also denote $\mathbf{A}(t) = (A_j^i(t), i, j = 1, 2)$ the matrix of expected values of the process.

Assume that there exists $c \in [0, \infty]$ such that

$$\lim_{t\to\infty} \frac{1 - G_2(t)}{1 - G_1(t)} = c. \tag{4.2}$$

We also denote $G = G_1 * G_2,\ a_i = \delta_{1i} + \delta_{2i}c$ and $b_i = \delta_{1i}cm + \delta_{2i}p$. We put by definition $c(1+c)^{-1} = 1$, when $c = \infty$.

We now formulate results which are devoted to the asymptotic behavior of the first and second moments of the process. We prove a theorem in more general case when each type of individuals may generate a random number of individuals of another type. This would relate to a situation, when an egg may generate more than one new individuals.

Let $F^i(s_1, s_2) = F^i(s_j),\ i = 1, 2$ be the offspring generating functions of types T_1 and T_2 and

$$m_i = \frac{dF^i(s_j)}{ds_j}|_{s_j=1}, \quad B_i = \frac{d^2F^i(s_j)}{ds_j^2}|_{s_j=1}, \quad i, j = 1, 2$$

be the offspring mean and the second moments.

Theorem 1. *Let $m_1m_2 < 1,\ G_i(t) \in \Re_2, i = 1, 2$ and (4.2) is satisfied. Then*

a)

$$\lim_{t\to\infty} \frac{\mathbf{A}(t)}{1 - G(t)} = \frac{1}{(1+c)(1 - m_1m_2)} \begin{pmatrix} 1 & cm_1 \\ m_2 & c \end{pmatrix}$$

b) If in addition $B_i \in (0, \infty)$, then for $i, j, k = 1, 2$

$$\lim_{t\to\infty} \frac{B^i_{jk}(t)}{1 - G(t)} = 0.$$

It is easy to see that, if $G(0+) = 1$ and $m_2 = 1$, from Theorem 1 we obtain the asymptotic formula for the first moment of the single-type age dependent process (Theorem 3B(ii) in [2], p. 152). In particular, for the process with incubation we obtain the following result.

Corollary. *Let $mp < 1$, $G_i(t) \in \Re_2, i = 1, 2$ and (4.2) is satisfied. Then for $i, j, k = 1, 2$ and expected values of the process defined in (4.1)*

a)

$$\lim_{t\to\infty} \frac{A^i_j(t)}{1 - G(t)} = \frac{\delta_{ij}a_i + (1 - \delta_{ij})b_i}{(1 + c)(1 - mp)}.$$

b) If in addition $\sigma^2 \in (0, \infty)$, then for $i, j, k = 1, 2$

$$\lim_{t\to\infty} \frac{B^i_{jk}(t)}{1 - G(t)} = 0.$$

Proof of Theorem 1. First we prove Part (a). In this case equations (2.1) and (2.2) will be exposed to a little change:

$$F^i(t, s_1, s_2) = s_i(1 - G_i(t)) + \int_0^t F^i(F^j(t - u, s_1, s_2))dG_i(u), i = 1, 2. \quad (4.3)$$

We obtain from (4.3) the following equations

$$A^i_i(t) = 1 - G_i(t) + m_i \int_0^t A^j_i(t - u)dG_i(u),$$

$$A^j_i(t) = m_j \int_0^t A^i_i(t - u)dG_j(u).$$

From these equations, using the associativity of convolutions, we obtain the following linear equations for the expected values.

$$A^i_j(t) = m_i(G_i(t) - G(t)) + m_1m_2 \int_0^t A^i_j(t - u)dG(u), \quad i \neq j, \qquad (4.4)$$

$$A^i_i(t) = 1 - G_i(t) + m_1m_2 \int_0^t A^i_i(t - u)dG(u), \qquad (4.5)$$

where $G(t) = G_1 * G_2(t)$.

For simplicity we consider just $A^1_2(t)$. It follows from the renewal theory that a solution of equation (4.4) can be represented as

$$A^1_2(t) = m_1(G_1 - G) * U_a(t),$$

where $a = m_1 m_2$ and $U_a(t) = \sum_{i=0}^{\infty} a^i G^{*i}(t)$. From this, by simple manipulations, we get

$$\frac{A_2^1(t)}{1-G(t)} = m_1 \sum_{i=0}^{\infty} a^i \left[\frac{1-G^{*(i+1)}(t)}{1-G(t)} - \frac{1-G_1 * G^{*i}(t)}{1-G(t)} \right]. \tag{4.6}$$

Since $G(t) \in \Re_2 \subset \Re_n$ for any $n \geq 2$, the first ratio on the right side of (4.6) approaches $i+1$ as $t \to \infty$. When condition (4.2) is satisfied

$$\frac{1-G_1(t)}{1-G(t)} \to \frac{1}{1+c}$$

as $t \to \infty$, due to Lemma 1. Therefore, appealing to Lemma 3, we find that the second ratio on the right side of (4.6) approaches $(1+c)^{-1} + i$ as $t \to \infty$. Thus

$$\lim_{t \to \infty} \frac{A_2^1(t)}{1-G(t)} = \frac{m_1 c}{(1+c)(1-a)}. \tag{4.7}$$

To justify taking the limit in (4.6) consider

$$\frac{A_2^1(t)}{1-G(t)} < \sum_{i=0}^{\infty} a^i \frac{1-G^{*(i+1)}(t)}{1-G(t)}.$$

According to Lemma IV.4.7 in [2] for any given $\varepsilon > 0$ there is a positive constant D such that

$$1 - G^{*(i+1)}(t) \leq D(1+\varepsilon)^{i+1}(1-G(t))$$

for all $i \geq 0$ and $t > 0$. Hence

$$\frac{A_2^1(t)}{1-G(t)} < D(1+\varepsilon) \sum_{i=0}^{\infty} (a(1+\varepsilon))^i.$$

Since $a < 1$ we can choose ε such that $a(1+\varepsilon) < 1$, then the series on the right side is convergent, which justifies taking the limit in (4.6).

To derive the asymptotic behavior of $A_1^2(t)$, we realize that, when (4.2) is satisfied,

$$\frac{1-G_2(t)}{1-G(t)} \to \frac{1}{1+c^{-1}}$$

as $t \to \infty$. Therefore by similar arguments we obtain

$$\lim_{t\to\infty} \frac{A_1^2(t)}{1-G(t)} = \frac{m_2}{(1+c)(1-a)}. \tag{4.8}$$

Now we consider $A_i^i(t)$. The renewal theory allows to write a solution of equation (4.5) as $A_i^i(t) = (1 - G_i) * U_a(t)$, which leads to equality

$$\frac{A_i^i(t)}{1-G(t)} = \sum_{j=0}^{\infty} a^j \left[\frac{1 - G_i * G^{*j}(t)}{1-G(t)} - \frac{1-G^{*j}(t)}{1-G(t)}\right] \tag{4.9}$$

The second ratio on the right side tends to j as $t \to \infty$. What concerns the first ratio, when condition (4.2) is satisfied it approaches $(1+c)^{-1} + j$ for $i = 1$ and $c(1+c)^{-1} + j$ for $i = 2$. Hence in this case we have

$$\lim_{t\to\infty} \frac{A_i^i(t)}{1-G(t)} = \frac{\delta_{1i} + \delta_{2i}c}{(1+c)(1+a)}, \quad i = 1, 2. \tag{4.10}$$

The assertion of Part (a) of the theorem follows from relations (4.7), (4.8) and (4.10) with $m_1 = m$ and $m_2 = p$.

Proof of Part (b). We obtain from equations (4.8) by differentiation the following equations:

$$B_{jk}^1(t) = B_1 \int_0^t A_{jk}^2(t-u)dG_1(u) + m_1 \int_0^t B_{jk}^2(t-u)dG_1(u),$$

$$B_{jk}^2(t) = B_2 \int_0^t A_{jk}^1(t-u)dG_2(u) + m_2 \int_0^t B_{jk}^1(t-u)dG_2(u),$$

where

$$B_i = \frac{d^2F^i(s_j)}{ds_j^2}|_{s_j=1}, \quad A_{jk}^i(t) = A_j^i(t)A_k^i(t).$$

From these equations due to associativity of the convolution we conclude that $B_{jk}^1(t)$ is a solution of equation

$$B_{jk}^1(t) = B_1 \int_0^t A_{jk}^2(t-u)dG_1(u) + m_1 m_2 \int_0^t \left[\frac{B_2}{m_2} A_{jk}^1(t-u) + B_{jk}^1(t-u)\right] dG(u).$$

By substitution

$$C(t) = \frac{B_2}{m_2} A_{jk}^1(t) + B_{jk}^1(t), \quad D(t) = B_1 \int_0^t A_{jk}^2(t-u)dG_1(u) + \frac{B_2}{m_2} A_{jk}^1(t)$$

we get renewal equation

$$C(t) = D(t) + m_1 m_2 \int_0^t C(t-u)dG(u),$$

the solution of which can be represented as

$$C(t) = D * \sum_{n=0}^{\infty} a^n G^{*n}(t) = \sum_{n=0}^{\infty} a^n \int_0^t D(t-u)dG^{*n}(u). \tag{4.11}$$

Now we consider

$$\int_0^t D(t-u)dG^{*n}(u) = I_1 + I_2, \tag{4.12}$$

where

$$I_1 = B_1 \int_0^t A_{jk}^2(t-u)d(G_1 * G^{*n}(u)), \quad I_2 = \frac{B_2}{m_2} \int_0^t A_{jk}^1(t-u)dG^{*n}(u).$$

Since $A_j^i(t) \sim constant \cdot (1 - G(t)),\ i, j = 1$, as $t \to \infty$, there is a positive constant C_0 such that

$$A_{jk}^i(t) \le C_0(1 - G(t))^2 \tag{4.13}$$

for all $t \ge 0$ and $i, j, k = 1, 2$.

Let T be a positive number. Using (4.13) we have

$$\frac{I_1}{B_1 C_0 (1 - G(t))} \le \int_0^T R_2(u,t)d\Psi_{1n}(u) + \int_T^t R_2(u,t)d\Psi_{1n}(u), \tag{4.14}$$

where $\Psi_{1n}(t) = G_1 * G^{*n}(t)$ and $R_i(u,t) = (1 - G(t-u))^i/(1 - G(t))$. Since the first integral on the right side of (4.14) is dominated by $R_2(T,t)\Psi_{1n}(T)$ and $G(t) \in \Re_2 \subset \Re_1$, it tends to zero as $t \to \infty$ for each $T > 0$.

What concerns the second integral, it is not greater than

$$\int_0^t R_1(u,t)d\Psi_{1n}(u) - \int_0^T R_1(u,t)d\Psi_{1n}(u).$$

Since $R_1(u,t) \ge R_1(0,t) = 1$, we easily obtain that it is dominated by

$$\frac{1 - \Psi_{1n+1}(t)}{1 - G(t)} - \frac{1 - \Psi_{1n}(t)}{1 - G(t)} - \Psi_{1n}(T).$$

Applying Lemma 3 to the above ratios we conclude that

$$\lim_{t\to\infty} \sup \frac{I_1}{B_1 C_0 (1 - G(t))} \leq 1 - \Psi_{1n}(T),$$

which means that as $t \to \infty$

$$I_1 = o(1 - G(t)). \tag{4.15}$$

Since due to inequality (4.13)

$$I_2 \leq \frac{B_2 C_0}{B_1} \int_0^t (1 - G(t-u))^2 dG^{*n}(u),$$

one can prove by similar arguments that as $t \to \infty$

$$I_2 = o(1 - G(t)). \tag{4.16}$$

Relations (4.12), (4.15) and (4.16) allow us to write

$$\lim_{t\to\infty} \int_0^t \frac{D(t-u)}{1 - G(t)} dG^{*n}(u) = 0,$$

which implies due to (4.11) that

$$\lim_{t\to\infty} \frac{C(t)}{1 - G(t)} = 0. \tag{4.17}$$

The possibility of taking the limit in (4.11) can be justified as in the proof of Part (a).

Thus the assertion of Part (b) for $B^i_{jk}(t)$ when $i = 1$ follows from (4.17). Considering the case of $i = 2$ makes no change in the above proof. Theorem is proved.

5 The limit theorem

Now we prove the limit theorem which gives asymptotic behavior of the non-extinction probability $Q^i(t) = P\{\mathbf{X}(t) \neq \mathbf{0} | \mathbf{X}(0) = \varepsilon_i\}$ and the limiting distribution of the process conditioned on non-extinction. We denote for vectors $\mathbf{s} = (s_1, s_2)$ and $\mathbf{x} = (x_1, x_2)$ the quantity $\mathbf{s}^{\mathbf{x}} = s_1^{x_1} s_2^{x_2}$.

Theorem 2. *If* $mp < 1, \sigma^2 \in (0,\infty),\ G_i(t) \in \Re_2,\ i = 1,2$ *and (4.2) is satisfied, then*

a)

$$\lim_{t\to\infty} \frac{Q^i(t)}{1-G(t)} = \frac{a_i+b_i}{(1+c)(1-mp)};$$

b)

$$\lim_{t\to\infty} E[\mathbf{s}^{\mathbf{X}(t)}|\mathbf{X}(t)\neq 0, \mathbf{X}(0)=\varepsilon_i] = \frac{a_i s_i + b_i s_j}{a_i+b_i}.$$

Remarks. 1. It follows from Theorem 2 that, if the process does not become extinct, in the long run with probability one a single individual is alive. It is the individual of the initial type with probability $a_i(a_i+b_i)^{-1}$ and of the opposite type with probability $b_i(a_i+b_i)^{-1}$. Recall that $a_i = \delta_{1i} + \delta_{2i}c,\ b_i = \delta_{1i}cm + \delta_{2i}p$. In terminology of epidemics the results illustrate the following situation. By preventive measures, such as isolating of infective individuals or increasing of the immunization rate one can ensure that $mp < 1$, which leads to extinction of the epidemic with probability one. However, if an epidemic initiated by a single infective does not cease, then in the long run one infective may exist with probability $(1+cm)^{-1}$. With probability $cm(1+cm)^{-1}$ an individual who had a contact with an infective may exist in the long run.

2. If in particular $G_2(0+) = 1$ and $p = 1$ from Part (b) of Theorem 2 we obtain the result for subcritical single-type process (Theorem 2 in [2], p. 171).

Proof of Theorem 2. We use the following expansion:

$$F^i(t,s_1,s_2) = 1 + \sum_{j=1}^{2} A^i_j(t)(s_j-1) + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \tilde{B}^i_{jk}(t,\mathbf{s})(s_j-1)(s_k-1),$$

where $\tilde{B}^i_{jk}(t,\mathbf{s}) \le B^i_{jk}(t)$ for $\mathbf{0}\le\mathbf{s}\le\mathbf{1}$.

Taking into account Part (b) of Theorem 1 we have for any fixed $\mathbf{0}\le\mathbf{s}\le\mathbf{1}$ as $t\to\infty$

$$1 - F^i(t,s_1,s_2) \sim \sum_{j=1}^{2} A^i_j(t)(1-s_j). \tag{5.1}$$

From this, putting $\mathbf{s} = \mathbf{0}$ and using Part (a) of Theorem 1 we obtain the assertion of Part (a).

We derive from (5.1) that

$$\lim_{t\to\infty} \frac{1 - F^i(t, s_1, s_2)}{Q^i(t)} = \frac{a_i(1 - s_i) + b_i(1 - s_j)}{a_i + b_i}. \tag{5.2}$$

Since

$$E\left[\mathbf{s}^{\mathbf{X}(t)} | \mathbf{X}(t) \neq \mathbf{0}, \mathbf{X}(0) = \varepsilon_i\right] = 1 - \frac{1 - F^i(t, s_1, s_2)}{Q^i(t)},$$

we have the assertion of Part (b). The theorem is proved.

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