# Asymptotic distribution of the CLSE in a critical process with immigration

I. Rahimov *

*Department of Mathematics and Statistics, KFUPM,*
*Dhahran, KSA*

**Abstract**

It is known that in the critical case the conditional least squares estimator (CLSE) of the offspring mean of a discrete time branching process with immigration is not asymptotically normal. If the offspring variance tends to zero, it is normal with normalization factor $n^{2/3}$. We study a situation of its asymptotic normality in the case of non-degenerate offspring distribution for the process with time-dependent immigration, whose mean and variance vary regularly with nonnegative exponents $\alpha$ and $\beta$, respectively. We prove that if $\beta < 1 + 2\alpha$, the CLSE is asymptotically normal with two different normalization factors and if $\beta > 1 + 2\alpha$, its limit distribution is not normal but can be expressed in terms of the distribution of certain functionals of the time changed Wiener process. When $\beta = 1 + 2\alpha$ the limit distribution depends on the behavior of the slowly varying parts of the mean and variance.

*Key Words:* branching process, time-dependent immigration, functional, Skorokhod space, least squares estimator.

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*Postal address: Department of Mathematics and Statistics, KFUPM, Box 1339, Dhahran 31261, Saudi Arabia; E-mail address: rahimov@kfupm.edu.sa

# 1 Introduction

We consider a discrete time branching stochastic process $Z(n), n \geq 0, Z(0) = 0$. It can be defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, n, i \geq 1\}$ and $\{\xi_k, k \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1}$$

Assume that $X_{ni}$ have a common distribution for all $n$ and $i$, and families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. Variables $X_{ni}$ will be interpreted as the number of offspring of the $i$th individual in the $(n-1)$th generation and $\xi_n$ is the number of immigrating individuals to the $n$th generation. Then $Z(n)$ can be considered as the size of $n$th generation of the population.

In this interpretation $A = EX_{ni}$ is the mean number of the offspring of a single individual. Process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $A < 1, A = 1$ or $A > 1$, respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, in contrast of classical models, we do not assume that $\xi_n, n \geq 1$, are identically distributed, i.e. immigration rate may depend on the time of immigration. This makes the process $Z(n)$ inhomogeneous in time. It is known that changes in the immigration rate in time leads to essential changes of the asymptotic behavior of the process (see [15], Ch III and references therein).

It is known (see Alzaid, Al-Osh [1], Dion, at all [5] and Franke, Seligmann [6]) that in the case of Bernoulli offspring distribution the process defined in equation (1) can be considered as an integer-valued, first order autoregressive (INAR(1)) time series model with noise $\xi_k$. In this framework considered here process $Z(n)$ can be related to the INAR(1) model with a non stationary noise.

Estimation of the offspring and immigration parameters in the branching process with a stationary immigration has been an active area of the research for a long time. As a result of this activity, it has been established that a maximum likelihood approach leads to useful results, if the number of immigrating individuals $\xi_n$ and all offspring sizes $X_{ni}$ are observable. The first estimation results based on observation of the population sizes are due to Heyde and Seneta [7-10]. In the supercritical case it was shown that the Lotka-Nagaev and Harris type ratio estimators can be used to estimate the offspring mean [7, 8]. In subcritical case, using an analogy between immigration-branching process and the first order autoregressive process, the same authors derived asymptotically normal estimators for offspring and immigration means [9, 10]. However, it was shown later that in the critical or nearly critical case the conditional least squares estimator (CLSE) is not asymptotically normal (see Sriram [17], Wei and Winnicki [19, 20]). Results of [11] and [12] have shown that when the process is nearly critical and the offspring variance tends to zero, it has a normal limit distribution with normalization factor $n^{3/2}$. Assuming that the offspring variance tends to zero means that in the long run the reproduction process approaches a deterministic multiplication process. The results of [20] have been extended to the controlled branching process with a random control function (see [18]). For estimation problems in non-classical models of branching processes we also refer to [13,14] and references therein.

In this note we describe the situations of asymptotic normality of the CLSE of the offspring mean in the case when the offspring variance does not tend to zero under the assumption $A = 1$. More precisely, we prove that if the immigration mean tends to infinity depending on the time of immigration, it is possible to estimate the offspring mean by an asymptotically normal CLSE. Assuming that the immigration mean and variance vary regularly with nonnegative exponents $\alpha$ and $\beta$ respectively, we establish that if $\beta < 1 + 2\alpha$, the CLSE is asymptotically normal. If $\beta > 1 + 2\alpha$, its limit distribution is not normal but can be expressed in terms of the distribution of certain functionals of the time changed Wiener process. When $\beta = 1 + 2\alpha$, the limit distribution depends on the behavior of the slowly varying parts of $\alpha(n)$ and $\beta(n)$. The normalization factor depends on the mean number of immigrants and on relationship between mean and variance of the immigration. We also demonstrate that in important particular cases of the Poisson and geometric immigration distribution the CLSE is asymptotically normal.

A natural approach to the problem of obtaining asymptotic distributions for estimators of parameters in a branching process, when the estimators are given explicitly, is analyzing the explicit expression using results from the limit theory of the branching processes. Depending on the explicit expression of the estimator, standard or functional limit theorems for the branching process may be used. For example, proofs of results in [17] and [19,20] are based on diffusion approximation of the normalized process. In [11] and [12] the authors first prove that the process can be approximated by an Ornstein-Uhlenbeck type process, and using this approximation, obtain asymptotic normality of the estimator.

Since the CLSE has a form of a ratio of certain functions of the process (see equation (6)), to derive its asymptotic distribution, one needs a central limit theorem (CLT) for the process in the functional form. Therefore, application of the general limit theorems on convergence to a mixture of infinitely divisible distributions ([15], Ch 2) does not allow to obtain the limit distributions in this model. In proofs of our results we use the the functional limit theorems from [16], where it is proved that when the immigration mean approaches infinity, the normalized and centered process converges in Skorokhod metric to a deterministically time-changed Wiener process with three different covariance functions. We note that the approximation theorems in [16] are obtained using a functional CLT for martingales. Thus, in our proofs the martingale CLT is used through functional limit theorems for the branching process. This scheme allows to determine the threshold for the asymptotic normality of the CLSE and to analyze the situations, where the limit distribution is not normal. We also note that in the time homogenous models a direct use of martingale CLT is sometimes possible (see [9]).

As it was mentioned before, our results are obtained under the assumption $A = 1$. However, the scheme of proofs can be used in subcritical and supercritical cases and in an array of branching processes (nearly critical case). Of course, first one needs to establish functional limit theorems for the nonclassical processes and then apply them in obtaining of asymptotic properties of an estimator of the offspring mean. Further, as in the classical case, one may establish results for the estimator without any assumption of the criticality of the reproduction process. One more possible application of the scheme of our proofs is deriving asymptotic distributions for a weighted CLSE, which minimizes a standardized sum of squared errors. More detailed discussion on this matter we provide in concluding remarks (Section 5).

In Section 2 we provide the procedure for constructing the CLSE and formulate the theorems on possible asymptotic distributions of the estimator. Examples of the immigration process satisfying the conditions of these theorems will be discussed. In Section 3 we provide preliminary results that are needed in the proofs of main theorems. Section 4 contains the proofs of main theorems.

## 2 Main results

From now on we assume that $A = EX_{ni}$ and $B = varX_{ni}$ are finite. We also assume that $\alpha(n) = E\xi_n < \infty$, $\beta(n) = var\xi_n < \infty$ for each $n \geq 1$ and regularly varying functions as $n \to \infty$ functions, i.e. have the following form

$$\alpha(n) = n^{\alpha}L_{\alpha}(n), \quad \beta(n) = n^{\beta}L_{\beta}(n), \tag{2}$$

where $\alpha, \beta \geq 0, L_{\alpha}(n)$ and $L_{\beta}(n)$ are slowly varying as $n \to \infty$ functions. Then $A(n) = EZ(n)$ and $B^2(n) = varZ(n)$ are finite for each $n \geq 1$, and when $A = 1$,

$$A(n) = \sum_{k=1}^{n} \alpha(k), \quad B^2(n) = \Delta^2(n) + \sigma^2(n), \tag{3}$$

where

$$\Delta^2(n) = B\sum_{k=1}^{n} \alpha(k)(n-k), \quad \sigma^2(n) = \sum_{k=1}^{n} \beta(k).$$

Throughout the paper $D$, $d$ and $P$ will denote convergence of random functions in Skorokhod topology and convergence of random variables in distribution and in probability, respectively. We also denote for each $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{B^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2 \mathbf{1}_{\{|\xi_k - \alpha(k)| > \varepsilon\sigma(n)\}}]. \tag{4}$$

Let $\Im(n)$ for each $n \geq 0$ be the $\sigma$-algebra containing all the history of the process up to $n$th generation, i.e. it is generated by $\{Z(k), k = 0, 1, ..., n\}$. We obtain from (1) that

$$E[Z(n)|\Im(n-1)] = AZ(n-1) + \alpha(n), \quad n \geq 1. \tag{5}$$

If we assume that the immigration mean $\alpha(n)$ is known, then non weighted CLSE $\hat{A}_n$ of $A$ minimizes sum of squared errors

$$\sum_{k=1}^{n} (Z(k) - AZ(k-1) - \alpha(k))^2.$$

By usual arguments we obtain

$$\hat{A}_n = \frac{\sum_{k=1}^{n}(Z(k) - \alpha(k))Z(k-1)}{\sum_{k=1}^{n} Z^2(k-1)}. \tag{6}$$

We also assume that there exists $C \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha(n)} = C. \tag{7}$$

We note that if $\beta < \alpha+1$, then $C = 0$ and if $\beta > \alpha+1$, then $C = \infty$. When $\beta = \alpha + 1$ the value of $C$ depends on the relative rate of variation of the slowly varying parts of $\alpha(n)$ and $\beta(n)$.

The proofs of the limit theorems for $\hat{A}_n$ are based on the following scheme. First we write the centered $\hat{A}_n$ as a ratio of some functions of the process $Z(n)$ (see (16)). Then we express numerator and denominator of the ratio in terms of certain functionals of the process $Y_n(t)$ and some additional terms, where

$$Y_n(t) = \frac{Z([nt]) - A([nt])}{B(n)},\ t \in \mathbb{R}_+ = [0, \infty).$$

Next we study the asymptotic behavior of each term using the functional CLT for $Y_n(t)$ and properties of the regularly varying functions. If we apply the continuous mapping and Slutsky's theorems, we obtain one or another limit distribution depending on which term predominates in the ratio.

Now we provide the first result related to the case of asymptotical normality of the CLSE.

**Theorem 1.** *If $A = 1,\ B \in (0, \infty),\ C \in [0, \infty),\ \alpha(n) \to \infty$ and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$ , then*

$$n\sqrt{\alpha(n)}(\hat{A}_n - A) \xrightarrow{d} N(0, a^2),$$

*where $N(0, a^2)$ is normal random variable with mean 0 and variance*

$$a^2 = \frac{(1+\alpha)^2(2\alpha+3)^2}{3\alpha+4}\left(\frac{B}{1+\alpha} + C\right).$$

In the case $C = 0$ we have $\sigma^2(n) = o(B^2(n))$ as $n \to \infty$, and condition $\delta_n(\varepsilon) \to 0, n \to \infty$, is automatically satisfied. In this case we obtain from Theorem 1 the assertion of Theorem 4 in [16]. In the case $C > 0$ the condition is equivalent to the Lindeberg condition for the family $\{\xi_n, n \geq 1\}$ of the number of immigrating individuals.

*Example 1.* Let $\xi_k, k \geq 1$ be Poisson with mean $\alpha(k) \to \infty, k \to \infty$, which is a regularly varying function with exponent $\alpha$. Then $\beta(n) = o(n\alpha(n)), n \to \infty$, therefore, $C = 0$ in Theorem 1 and the Lindeberg type condition is satisfied. In this case we obtain the following result.

**Corollary 1.** If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are Poisson with mean $\alpha(k) \to \infty, k \to \infty$, and $(\alpha(k))_{k=1}^{\infty}$ is regularly varying with exponent $\alpha$, then $n\sqrt{\alpha(n)}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance

$$a^2 = \frac{(1+\alpha)(2\alpha+3)^2 B}{3\alpha+4}.$$

*Example 2.* Let now the random variables $\xi_k, k \geq 1$, have positive geometric distributions with parameters $p_k = k^{-1}$ i. e. $P\{\xi_k = i\} = q_k^{i-1} p_k, i = 1, 2, ..., q_k = 1 - p_k$. In this case $\alpha(k) = k$ and $\beta(k) = q_k p_k^{-2} = k^2(1 - k^{-1})$. Consequently, we have $\Delta^2(n) \sim Bn^3/6$ and $\sigma^2(n) \sim n^3/3$. Therefore, $\sigma^2(n) \sim 2B^2(n)/(B+2)$. Now we show that the Lindeberg condition is fulfilled. Since $Es^{\xi_k} = (p_k s)(1 - q_k s)^{-1}$, we find that $(Es^{\xi_k})''' = 6p_k q_k^2 (1 - q_k s)^{-4}$. Therefore, $E\xi_k(\xi_k - 1)(\xi_k - 2) = 6q_k^2 p_k^{-3}$. From this we conclude that $E|\xi_k - \alpha(k)|^3 = O(k^3), k \to \infty$, which leads to the relation

$$C_n^3 =: \sum_{k=1}^{n} E|\xi_k - \alpha(k)|^3 = O(n^4), n \to \infty.$$

Thus, $C_n^3/\sigma^3(n) = O(n^{-1/2}), n \to \infty$, i.e. Lyapunov condition is satisfied for $\{\xi_k, k \geq 1\}$. In this case we obtain the following result from Theorem 1.

**Corollary 2.** If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are geometric with parameters $p_k = k^{-1}$, then $n^{3/2}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance $a^2 = 50(B+2)/7$.

Next theorem is related to the case $C = \infty$, but

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha^2(n)} = 0. \tag{8}$$

**Theorem 2.** *If $A = 1$, $B \in (0, \infty)$, $C = \infty$, $\alpha(n) \to \infty$, $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and (8) is satisfied, then*

$$\frac{n^{3/2}\alpha(n)}{\sqrt{\beta(n)}}(\hat{A}_n - A) \xrightarrow{d} N(0, b^2),$$

*as $n \to \infty$, where $N(0, b^2)$ is a normal random variable with mean 0 and variance*

$$b^2 = \frac{(1+\alpha)^2(2\alpha+3)^2}{2\alpha+\beta+3}.$$

It follows from condition (8) that the normalization factor in Theorem 2 tends to infinity faster than $n$.

*Example 3.* Let $\xi_k, k \geq 1$ be such that $p = P\{\xi_k = k^2\} = 1 - P\{\xi_k = 0\}, q = 1-p$. It is obvious that in this case $\alpha(n) = n^2 p,\ \beta(n) = n^4 pq$. Then simple calculations give $\Delta^2(n) \sim Bpn^4/12, \sigma^2(n) \sim pqn^5/5$ as $n \to \infty$ and, consequently, $\Delta^2(n) = o(\sigma^2(n)$. Since in this case $C_n^3 \sim pq(p^2+q^2)n^7/7$ and $\sigma^3(n) \sim (pq/5)^{3/2} n^{15/2}$, the Lyapunov condition is again satisfied. In this case the condition (8) is satisfied and we obtain from Theorem 2 that $(p/q)^{1/2} n^{3/2}(\hat{A}_n - A)$ is asymptotically normal as $n \to \infty$ with mean zero and variance $b^2 = 441/11$.

Now we consider the case

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = 0. \tag{9}$$

**Theorem 3.** *If $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and (9) is satisfied, then*

$$n(\hat{A}_n - A) \xrightarrow{d} \frac{W^2(1) - 1}{2\int_0^1 W^2(t^{1+\beta})dt},$$

*where $W(t)$ is the standard Wiener process.*

The next theorem shows that the limits of the asymptotic normality of CLSE is determined by ratio $n\alpha^2(n)/\beta(n)$. We assume that

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = d_0 \in (0,\infty). \tag{10}$$

**Theorem 4.** *If $A = 1,\ B \in (0,\infty),\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ and (10) is satisfied, then*

$$n(\hat{A}_n - A) \xrightarrow{d} \frac{2^{-1}(W^2(1) - 1) + c_0\eta}{c_0^2/(2\alpha+3) + \zeta},$$

*where*

$$c_0 = \frac{\sqrt{d_0(1+\beta)}}{1+\alpha}, \ \eta = W(1) - (1+\alpha)\int_0^1 W(t^{1+\beta})t^{\alpha}dt,$$

$$\zeta = 2c_0 \int_0^1 t^{1+\alpha}W(t^{1+\beta})dt + \int_0^1 W^2(t^{1+\beta})dt.$$

# 3 Preliminaries

We start with two results, which we need in proofs of main theorems.

**Lemma 1.** *If $\alpha(n)$ and $\beta(n)$ are regularly varying functions of exponents $\alpha \geq 0$ and $\beta \geq 0$, respectively, then as $n \to \infty$*
*a)*

$$\Delta^2(n) \sim \frac{B\alpha(n)n^2}{(\alpha+1)(\alpha+2)}, \quad \sigma^2(n) \sim \frac{n\beta(n)}{\beta+1}. \tag{11}$$

*b) For each $\gamma \geq 0$*

$$\sum_{k=1}^{n} A^{\gamma}(k) \sim \frac{n}{\gamma\alpha+\gamma+1}A^{\gamma}(n), \ \ A(n) \sim \frac{n\alpha(n)}{\alpha+1}. \tag{12}$$

The proof of Lemma 1 is based on Karamata's theorem on regularly varying functions (see [4], Theorem 1.5.11) and can be seen in [16]. The next lemma is also proved in [16] and we provide it for a ready reference.

**Lemma 2.** *If $\alpha(n) \to \infty$ as $n \to \infty$, then for each $t \in \mathbb{R}_+ = [0, \infty)$*
*a)*

$$B^{-4}(n)var\left(\sum_{k=1}^{[nt]} Z(k)\right) \to 0; \tag{13}$$

*b)*

$$B^{-4}(n)\sum_{k=1}^{[nt]} EZ^2(k) \to 0. \tag{14}$$

Now we state a theorem from [16] on convergence to a deterministically time-changed Wiener process in a appropriate form.

**Theorem A.** *If* $A = 1,\ B \in (0, \infty),\ \alpha(n) \to \infty,\ \delta_n(\varepsilon) \to 0$ *as* $n \to \infty$ *for any* $\varepsilon > 0$ *and* $C$ *be the limit in (7), then*

$$Y_n(t) \stackrel{D}{\to} Y(t)$$

*as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$*, where*

$$Y(t) = \begin{cases} W(t^{2+\alpha}), & \text{if } C = 0, \\ W(t^{2+\alpha}) = W(t^{1+\beta}), & \text{if } C \in (0, \infty), \\ W(t^{1+\beta}), & \text{if } C = \infty. \end{cases}$$

As it was mentioned before, when $C = 0$, the Lindeberg type condition for $\{\xi_n, n \geq 1\}$ holds automatically.

We also need the following result, which gives a sufficient condition for convergence to a functional of the continuous process. The proof of this theorem is based on continuous mapping theorem and can be found in [2]. If measurable mappings $\Psi$ and $\Psi_n, n = 1, 2, ...$ defined as $D(\mathbb{R}_+, \mathbb{R}^k) \mapsto D(\mathbb{R}_+, \mathbb{R}^l)$ such that $\|\Psi_n(x_n) - \Psi(x)\|_\infty \to 0$ as $n \to \infty$ for all $x, x_n \in D(\mathbb{R}_+, \mathbb{R}^k)$ with $\|x_n - x\|_\infty \to 0$, we shall write $\Psi_n \Rightarrow \Psi$. Here $\|.\|_\infty$ stands for the supremum norm.

**Theorem B.** *Let* $\Psi$ *and* $\Psi_n,\ n = 1, 2, ...$ *be measurable mappings such that* $\Psi_n \Rightarrow \Psi$ *as* $n \to \infty$ *and* $Y(t),\ Y_n(t), n = 1, 2, ...$ *be stochastic processes with values in* $D(\mathbb{R}_+, \mathbb{R}^k)$*. If* $Y_n(t) \stackrel{D}{\to} Y(t)$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R}^k)$ *and almost all trajectories of* $Y(t)$ *are continuous, then* $\Psi_n(Y_n) \stackrel{D}{\to} \Psi(Y)$ *as* $n \to \infty$ *in* $D(\mathbb{R}_+, \mathbb{R}^l)$*.*

# 4 Proofs of theorems

**Proof of Theorem 1.** We denote $M(k) = Z(k) - E[Z(k)|\Im(k-1)]$. It follows from (1) and (5) that

$$Z(k) - E[Z(k)] = Z(k-1) - E[Z(k-1)] + M(k).$$

By consequent application of this equality, we obtain

$$Y_n(t) = \sum_{k=1}^{[nt]} \frac{M(k)}{B(n)}. \tag{15}$$

It follows from (6) that

$$\hat{A}_n - 1 = \frac{\sum_{k=1}^{n} Z(k-1)M(k)}{\sum_{k=1}^{n} Z^2(k-1)} =: \frac{D(n)}{Q(n)}. \tag{16}$$

Taking into account (15), we rewrite the numerator as $D(n) = D_1(n) + D_2(n)$, where

$$D_1(n) = \sum_{k=2}^{n} \sum_{i=1}^{k-1} M(i)M(k), \quad D_2(n) = \sum_{k=2}^{n} A(k-1)M(k).$$

First we consider $D_1(n)$. Since

$$D_1(n) = \sum_{k=1}^{n} \sum_{i=1}^{k} M(i)M(k) - \sum_{k=1}^{n} M^2(k),$$

using simple identity

$$(\sum_{k=1}^{n} M(k))^2 = D_1(n) + \sum_{k=1}^{n} \sum_{i=1}^{k} M(i)M(k),$$

we obtain

$$2D_1(n) = (\sum_{k=1}^{n} M(k))^2 - \sum_{k=1}^{n} M^2(k).$$

Taking into account (15), we have

$$D_1(n) = \frac{B^2(n)}{2} Y_n^2(1) - \frac{1}{2} \sum_{k=1}^{n} M^2(k). \tag{17}$$

Since $E[Z(k)|\Im(k-1)] = Z(k-1) + \alpha(k)$, we obtain from (1) that

$$M(k) = \sum_{i=1}^{Z(k-1)} (X_{ki} - 1) + \xi_k - \alpha(k). \tag{18}$$

Using (18) and independence of $X_{ki},\ i = 1, 2, ...,$ we derive

$$\frac{1}{B^2(n)}\sum_{k=1}^{n} E[M^2(k)|\Im(k-1)] = \frac{B}{B^2(n)}\sum_{k=1}^{n} Z(k-1) + \frac{\sigma^2(n)}{B^2(n)}. \tag{19}$$

Now we consider the mean of the first term in (19). Let $C \in (0, \infty)$. In this case we obtain from Lemma 1 that as $n \to \infty$

$$\sum_{k=1}^{n} A(k) \sim \frac{n^2\alpha(n)}{(\alpha+1)(\alpha+2)},\ B^2(n) \sim \frac{Bn^2\alpha(n)}{(\alpha+1)(\alpha+2)} + \frac{Cn^2\alpha(n)}{1+\beta}. \tag{20}$$

Therefore, we have

$$\lim_{n\to\infty} \frac{B}{B^2(n)}\sum_{k=1}^{n} EZ(k-1) = \frac{B(\beta+1)}{B(\beta+1)+d}\ , \tag{21}$$

where $d = C(1+\alpha)(2+\alpha)$. It follows from Lemma 2 that the variance of the first term in (19) tends to zero as $n \to \infty$, which yields its convergence as $n \to \infty$ to the right side of (21) in probability. Using again (20) and Lemma 1, we obtain

$$\lim_{n\to\infty} \frac{\sigma^2(n)}{B^2(n)} = \theta =: \frac{d}{B(\beta+1)+d}. \tag{22}$$

Thus, we conclude that as $n \to \infty$

$$\frac{1}{B^2(n)}\sum_{k=1}^{n} M^2(k) \xrightarrow{P} 1. \tag{23}$$

In the case $C = 0$, the right side of (21) equals 1, $\theta = 0$ and we come again to convergence (23).

It follows from Theorem A and the continuous mapping theorem that $Y_n^2(1) \xrightarrow{d} W^2(1)$ as $n \to \infty$. Taking this into account in (17) and appealing to Slutsky's theorem, we obtain that

$$\frac{1}{B^2(n)} D_1(n) \xrightarrow{d} \frac{1}{2}(Y^2(1) - 1). \tag{24}$$

Now we consider $D_2(n)$. It is not difficult to see that

$$D_2(n) = \sum_{k=2}^{n}\sum_{i=1}^{k-1} \alpha(i)M(k) = \sum_{i=1}^{n-1} \alpha(i) \sum_{k=i+1}^{n} M(k).$$

Therefore, taking into account (15), it can be written as

$$\frac{1}{K(n)} D_2(n) = \int_0^1 [Y_n(1) - Y_n(t)] dA_n(t), \tag{25}$$

where $K(n) = A(n)B(n)$ and $A_n(t) = A([nt])/A(n), n \geq 1$, are non-decreasing functions of $t$.

Now we consider a sequence of functionals $\Psi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}, n \geq 1$, defined by

$$\Psi_n(x) = \int_0^1 [x(1) - x(t)] dA_n(t).$$

Since $A(n)$ is regularly varying function with exponent $1 + \alpha$, sequence $A_n(t) \to t^{1+\alpha}$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$. Therefore, for all $x, x_n \in D(\mathbb{R}_+, \mathbb{R})$ such that $\|x_n - x\|_\infty \to 0, n \to \infty$, we have $\|\Psi_n(x_n) - \Psi(x)\|_\infty \to 0$ as $n \to \infty$, where

$$\Psi(x) = (1 + \alpha) \int_0^1 [x(1) - x(t)] t^\alpha dt.$$

It follows from theorems A and B that $\Psi_n(Y_n) \xrightarrow{d} \Psi(Y)$ as $n \to \infty$, where $Y(t) = W(t^{1+\beta})$, when $C \in (0, \infty)$ and $Y(t) = W(t^{2+\alpha})$, when $C = 0$. Hence, we conclude that

$$\frac{1}{K(n)} D_2(n) \xrightarrow{d} \eta \tag{26}$$

as $n \to \infty$, where

$$\eta = (1 + \alpha) \int_0^1 [Y(1) - Y(t)] t^\alpha dt.$$

Now we consider $Q(n)$, which is defined in (16). It can be written as $Q(n) = Q_1(n) + 2Q_2(n) + Q_3(n)$, where

$$Q_1(n) = \sum_{k=1}^n A^2(k-1), \quad Q_2(n) = \sum_{k=1}^n A(k-1)(Z(k-1) - A(k-1)),$$

$$Q_3(n) = \sum_{k=1}^n (Z(k-1) - A(k-1))^2.$$

It follows from Lemma 1 that

$$\lim_{n\to\infty} \frac{Q_1(n)}{nA^2(n)} = (2\alpha+3)^{-1}. \tag{27}$$

To estimate $Q_2(n)$, we consider

$$\frac{Q_2(n)}{nK(n)} = \sum_{k=1}^{n-1} A_n(\frac{k}{n}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} Y_n(t)dt.$$

Now we define the functionals $\Phi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}, n \geq 1$ by

$$\Phi_n(x) = \sum_{k=1}^{n-1} A_n(\frac{k}{n}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} x(t)dt.$$

It is easy to see that for any $x, x_n \in D(\mathbb{R}_+, \mathbb{R})$ such that $\|x_n - x\| \to 0, n \to \infty$, we have $\|\Phi_n(x_n) - \Phi(x)\| \to 0$ as $n \to \infty$, where

$$\Phi(x) = \int_0^1 t^{1+\alpha} x(t)dt.$$

It is obvious that

$$\frac{Q_2(n)}{nK(n)} = \Phi_n(Y_n).$$

Therefore, using again theorems A and B, we obtain

$$\frac{Q_2(n)}{nK(n)} \xrightarrow{d} \int_0^1 t^{1+\alpha} Y(t)dt. \tag{28}$$

Now we consider $Q_3(n)$. Taking into account the definition of $Y_n(t)$, we have

$$\frac{Q_3(n)}{nB^2(n)} = \int_0^1 Y_n^2(t)dt.$$

It follows from Theorem A and continuous mapping theorem that

$$\frac{Q_3(n)}{nB^2(n)} \xrightarrow{d} \int_0^1 Y^2(t)dt \tag{29}$$

as $n \to \infty$. Recall that $K(n) = A(n)B(n)$. When $C \in [0, \infty)$, it follows from Lemma 1 that $B(n) = o(A(n))$ as $n \to \infty$. Consequently, $B^2(n) = o(K(n))$

and from (24) and (26) we obtain that $D(n)/K(n)$ as $n \to \infty$ converges to $\eta$ in distribution. On the other hand, since $K(n) = o(A^2(n))$, we obtain from (27)-(29) that $Q(n)/nA^2(n)$ converges to $(2\alpha+3)^{-1}$ in probability. Using Lemma 1, we obtain that

$$\lim_{n\to\infty} \frac{B(n)}{A(n)}\sqrt{\alpha(n)} = (1+\alpha)\left(\frac{B}{(1+\alpha)(2+\alpha)} + \frac{C}{1+\beta}\right)^{1/2}. \qquad (30)$$

We note that when $C \in (0,\infty)$, we have $\beta = 1+\alpha$. Therefore, the quantity $1+\beta$ in (30) can be replaced by $2+\alpha$. Hence, appealing to Slutsky's theorem, we conclude that

$$n\sqrt{\alpha(n)}(\hat{A}_n - 1) \xrightarrow{d} \eta\frac{(2\alpha+3)(1+\alpha)}{\sqrt{2+\alpha}}\left(\frac{B}{1+\alpha} + C\right)^{1/2}. \qquad (31)$$

Applying Ito formula to

$$\eta = (1+\alpha)\int_0^1 [W(1) - W(t^{2+\alpha})]t^\alpha dt,$$

we have

$$\eta = \int_0^1 t^{1+\alpha} dW(t^{2+\alpha}).$$

Therefore, $\eta$ has normal distribution with mean zero. To find its variance, we use identity

$$E\eta^2 = \int_0^1\int_0^1 s^\alpha t^\alpha R(t,s)dsdt,$$

where

$$R(t,s) = (1+\alpha)^2 E[(W(1) - W(s^{2+\alpha}))(W(1) - W(t^{2+\alpha}))].$$

We consider

$$E\eta^2 = \int_0^1\int_0^t s^\alpha t^\alpha R(t,s)dsdt + \int_0^1\int_t^1 s^\alpha t^\alpha R(t,s)dsdt. \qquad (32)$$

By a standard technique, we obtain that the first term on the right side of (32) is equal to

$$\int_0^1\int_0^t s^\alpha t^\alpha E[(W(1) - W(t^{2+\alpha}))^2]dsdt = \frac{\alpha+2}{2(3\alpha+4)(\alpha+1)^2}.$$

In a similar way we derive that the second term on the right of (32) is also equal to $(\alpha+2)/(2(3\alpha+4))$. Hence, we have

$$E\eta^2 = \frac{\alpha+2}{3\alpha+4}\ ,$$

which implies the desired result for the variance of the limiting random variable. Theorem 1 is proved.

**Proof of Theorem 2.** The proof follows the same scheme as the proof of the first theorem. We again consider (16). When $C = \infty$, we have $\Delta^2(n) = o(B^2(n))$ as $n \to \infty$, On the other hand, there exist a constant $C_1 \in (0,\infty)$ such that

$$\sum_{k=1}^{n} A(k) \sim C_1 \Delta^2(n),\ n \to \infty.$$

Therefore, the first term on the right side of (19) as $n \to \infty$ converges to zero in probability. The second term on the right side of (19) tends to 1 and we again have (23). It follows from Theorem A and (23) that (24) remains true when $C = \infty$.

As in the proof of the Theorem 1, we obtain that (26) also holds with

$$\eta = (1+\alpha)\int_0^1 [W(1) - W(t^{1+\beta})]t^{\alpha}dt. \tag{33}$$

Obviously, relations (27)-(29) remain true when $C = \infty$ with $Y(t) = W(t^{1+\beta})$. Since $B^2(n) \sim \sigma^2(n)$ as $n \to \infty$, appealing to Lemma 1, we obtain

$$\frac{B(n)}{A(n)} \sim \frac{1+\alpha}{\alpha(n)}\sqrt{\frac{\beta(n)}{n(1+\beta)}},$$

which, due to condition (8), shows that $B(n) = o(A(n))$ as $n \to \infty$. Consequently, $D(n)/K(n)$ converges to $\eta$ as $n \to \infty$ in distribution.

Taking into account relation $B(n) = o(A(n))$ again, we see that $Q(n)/nA^2(n)$ converges to $(2\alpha+3)^{-1}$ in probability. Since $K(n)/A^2(n) = B(n)/A(n)$, we have

$$\frac{K(n)}{nA^2(n)} \sim \frac{1+\alpha}{n^{3/2}\alpha(n)}\sqrt{\frac{\beta(n)}{1+\beta}}$$

as $n \to \infty$. By the similar arguments, as in the proof of Theorem 1, we derive equality

$$E\eta^2 = \frac{1+\beta}{2\alpha+\beta+3}\ .$$

This gives the desired result for the variance of the limiting normal distribution. Theorem 2 is proved.

**Proof of Theorem 3.** As in the proof of previous theorem, it is easy to see that when condition (9) is fulfilled, $\sigma^2(n) \sim B^2(n)$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{B^2(n)} \sum_{k=1}^{n} EZ(k-1) = 0.$$

It follows from (19) that relation (23) remains true. Since $A(n) = o(B(n))$ as $n \to \infty$ , we obtain from (26) that

$$\frac{D_2(n)}{B^2(n)} = \frac{1}{B^2(n)} \sum_{k=2}^{n} A(k-1)M(k) \xrightarrow{P} 0 \tag{34}$$

as $n \to \infty$ . Similarly, relations (27) and (28) give

$$\lim_{n\to\infty} \frac{Q_1(n)}{nB^2(n)} = 0, \tag{35}$$

$$\frac{Q_2(n)}{nB^2(n)} \xrightarrow{P} 0,\ n \to \infty. \tag{36}$$

We denote

$$\omega(n) = Q_3(n)/nB^2(n) = \int_0^1 Y_n^2(t)dt$$

and consider equality

$$n(\hat{A}_n - 1) = \frac{S_1(n) + S_2(n) + S_3(n)}{1 + S_4(n) + S_5(n)}, \tag{37}$$

where

$$S_1(n) = \frac{Y_n^2(1) - 1}{2\omega(n)}\ ,\ S_2(n) = \frac{1}{2\omega(n)}(1 - \frac{1}{B^2(n)} \sum_{k=1}^{n} M^2(k)),$$

$$S_3(n) = \frac{\sum_{k=2}^{n} A(k-1)M(k)}{B^2(n)\omega(n)}\ ,\ S_4(n) = \frac{2Q_2(n)}{nB^2(n)\omega(n)}\ ,$$

$$S_5(n) = \frac{Q_1(n)}{nB^2(n)\omega(n)}.$$

Since due to Theorem A

$$\omega(n) \xrightarrow{d} \int_0^1 W^2(t^{1+\beta})dt$$

as $n \to \infty$ and $\int_0^1 W^2(t^{1+\beta})dt \neq 0$ almost surely, we obtain from (23) and (34)-(36) that

$$S_i(n) \xrightarrow{P} 0 \tag{38}$$

as $n \to \infty$ for $i = 2, 3, 4, 5$. Now we consider the functional $\Psi : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$, which is defined by

$$\Psi(x) = \frac{x^2(1) - 1}{2\int_0^1 x^2(t)dt}$$

for each $x \in D(\mathbb{R}_+, \mathbb{R})$. It follows from Theorem A and the continuous mapping theorem ([3], p.30, Theorem 5.1) that $\Psi(Y_n) \xrightarrow{D} \Psi(Y)$ as $n \to \infty$ in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ with $Y(t) = W(t^{1+\beta})$. Therefore,

$$S_1(n) \xrightarrow{d} \frac{W^2(1) - 1}{2\int_0^1 W^2(t^{1+\beta})dt}. \tag{39}$$

We obtain the assertion of Theorem 3 from (37)-(39).

**Proof of Theorem 4.** When condition (10) is satisfied, it follows from Lemma 1 that

$$A(n) \sim c_0 B(n) \tag{40}$$

as $n \to \infty$, where $c_0 = \sqrt{d_0(1+\beta)}/(1+\alpha)$. Using notation in the proof of Theorem 1, we have:

$$n(\hat{A}_n - 1) = \frac{T_1(n) + T_2(n)}{T_4(n) + T_5(n) + T_6(n)} + \frac{T_3(n)}{T_4(n) + T_5(n) + T_6(n)}, \tag{41}$$

where

$$T_1(n) = 2^{-1}(Y_n^2(1) - 1),\ T_2(n) = \frac{A(n)}{B(n)} \int_0^1 [Y_n(1) - Y_n(t)]dA_n(t),$$

$$T_3(n) = 2^{-1}(1 - \frac{1}{B^2(n)} \sum_{k=1}^{n} M^2(k)),\ T_4(n) = \frac{1}{nB^2(n)} \sum_{k=1}^{n} A^2(k-1),$$

$$T_5(n) = \frac{2A(n)}{B(n)} \sum_{k=1}^{n-1} A_n(\frac{k}{n}) \int_{k/n}^{(k+1)/n} Y_n(t)dt,\ T_6(n) = \int_0^1 Y_n^2(t)dt.$$

Now we consider a sequence of functionals $\Phi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}, n \geq 1$, defined by

$$\Phi_n(x) = \frac{2^{-1}(x^2(1) - 1) + (A(n)/B(n))\Omega_n^{(1)}(x)}{Q_1(n)/(nB^2(n)) + (2A(n)/B(n))\Omega_n^{(2)}(x) + \Omega^{(3)}(x)}$$

for any $x \in D(\mathbb{R}_+, \mathbb{R})$, where

$$\Omega_n^{(1)}(x) = \int_0^1 [x(1) - x(t)]dA_n(t),$$

$$\Omega_n^{(2)}(x) = \sum_{k=1}^{n-1} A_n(\frac{k}{n}) \int_{k/n}^{(k+1)/n} x(t)dt,\ \Omega^{(3)}(x) = \int_0^1 x^2(t)dt\ .$$

Taking into account (40) and that

$$\lim_{n\to\infty} \frac{Q_1(n)}{nB^2(n)} = \frac{c_0^2}{2\alpha + 3}\ ,\ \lim_{n\to\infty} A_n(t) = t^{1+\alpha}\ ,$$

we see that $\|\Phi_n(x_n) - \Phi(x)\|_\infty \to 0$ as $n \to \infty$ for any sequence $x_n \in D(\mathbb{R}_+, \mathbb{R})$ such that $\|x_n - x\|_\infty \to 0, n \to \infty$. Here the functional $\Phi : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$ is defined by

$$\Phi(x) = \frac{2^{-1}(x^2(1) - 1) + c_0(1 + \alpha) \int_0^1 [x(1) - x(t)]t^\alpha dt}{c_0^2(2\alpha + 3)^{-1} + 2c_0 \int_0^1 t^{1+\alpha}x(t)dt + \int_0^1 x^2(t)dt}.$$

Since the trajectories of the limit process $Y(t)$ in Theorem A are almost surely continuous, we conclude due to theorems A and B that $\Phi_n(Y_n) \xrightarrow{D} \Phi(Y)$ as $n \to \infty$ in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$. Thus, the first ratio in (41) as $n \to \infty$ converges to

$$\frac{2^{-1}(W^2(1) - 1) + c_0\eta}{c_0^2/(2\alpha + 3) + \zeta}$$

in distribution.

As in the proof of Theorem 3, it follows from (19), (23) and (40) that $T_3(n)$ as $n \to \infty$ converges to zero in probability. Appealing again to theorems A and B and using Slutsky's theorem, we obtain the assertion of Theorem 4.

# 5 Concluding remarks

The estimator (6) minimizes the sum of squared errors $\sum_{k=1}^{n} \varepsilon^2(k)$, where $\varepsilon(k)$ are defined by stochastic regression equation

$$Z(k) = AZ(k-1) + \alpha(k) + \varepsilon(k). \tag{42}$$

It is easy to find that $E[\varepsilon(k)|\Im(k-1)] = 0$. Using definition of the process given in (1), we find

$$Var[\varepsilon(k)|\Im(k-1)] = BZ(k-1) + \beta(k),$$

which shows that the conditional variance depends not only on $Z(k)$, but also influenced by the immigration variance. It is known that so-called weighted CLSE is a better estimator from the point of view of the variance of the asymptotic distribution and of the convergence rate. Construction of the weighted CLSE is based on standardization of $\sum_{k=1}^{n} \varepsilon^2(k)$. In order to do this, we divide equation (42) by root of an estimator of the conditional variance $Var[Z(k)|\Im(k-1)] = Var[\varepsilon(k)|\Im(k-1)]$. More or less accurate estimator used in the classical processes (see [20]) is $Z(k-1)+1$, which converts equation (42) into

$$\frac{Z(k)}{(Z(k-1)+1)^{1/2}} = \frac{AZ(k-1)+\alpha(k)}{(Z(k-1)+1)^{1/2}} + \varepsilon^*(k), \tag{43}$$

where $\varepsilon^*(k) = \varepsilon(k)(Z(k-1)+1)^{-1/2}$. It is obvious that $E[\varepsilon^*(k)|\Im(k-1)] = 0$ and

$$Var[\varepsilon^*(k)|\Im(k-1)] = \frac{BZ(k-1)}{Z(k-1)+1} + \frac{\beta(k)}{Z(k-1)+1}.$$

Therefore, if the immigration variance is uniformly bounded, then $Var[\varepsilon^*(n)|\Im(n-1)] \to B$ as $n \to \infty$ on the set $\{Z(n) \to \infty\}$. By the standard technique we derive the following weighted CLSE

$$\hat{A}_n^{(1)} = \sum_{k=1}^{n} \frac{(Z(k)-\alpha(k))Z(k-1)}{Z(k-1)+1} \left[\sum_{k=1}^{n} \frac{Z^2(k-1)}{Z(k-1)+1}\right]^{-1}. \tag{44}$$

It is clear that on the set $\{Z(n) > 0, n \geq 1\}$ one may choose $Z(k-1)$ as the weight and this would lead to a simpler than (44) weighted CLSE. However, this estimator is defined on the set of trajectories of the process, which do not return to the state zero. We obtain a weighted CLSE, which is defined on the set of all the trajectories, if we use $\tau(k) = Z(k) + \mathbf{1}_{\{Z(k)=0\}}, k \geq 0$ as the weight. Taking into account trivial identities $Z(k)/\tau(k) = \mathbf{1}_{\{Z(k)>0\}}$ and $Z^2(k)/\tau(k) = Z(k)$, we obtain another weighted CLSE

$$\hat{A}_n^{(2)} = \frac{\sum_{k=1}^n (Z(k) - \alpha(k))\mathbf{1}_{\{Z(k-1)>0\}}}{\sum_{k=1}^n Z(k-1)}. \tag{45}$$

Following the scheme of the proofs of Theorems 1-4, one may derive asymptotic distributions for (44) and (45) in the critical case. To get full spectrum of the limit distributions in a noncritical case, in addition to Theorem A, one needs to establish appropriate functional limit theorems for the process.

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