Multitype Processes with Reproduction-Dependent Immigration

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Abstract

The multitype discrete time indecomposable branching process with immigration is considered. Using a martingale approach a limit theorem proved for such processes when the totality of immigrating individuals at a given time depends on evolution of the processes generating by immigrated before individuals. Corollaries of the limit theorem obtained for the cases of finite and infinite second moments of offspring distribution in critical processes.

Key words: MULTITYPE, INDECOMPOSABLE, CRITICALITY, IMMIGRATION, DEPENDENCE, STOPPING TIME.
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1 Introduction

We consider a multitype discrete time branching stochastic process with immigration. Let $\mathbf{Y}(\mathbf{t}) = (Y_1(t), Y_2(t), ..., Y_n(t))$ be the random vector of immigrating at time $t = 0, 1, ...$ individuals of types $T_1, T_2, ..., T_n$. These immigrating individuals generate independent and identically distributed n-type branching stochastic processes. If we enumerate simultaneously immigrating individuals of the type T_j by 1, 2, ..., then the triple (k, i, j) corresponds to the ith individual of the type T_j, $\quad j = 1, 2, ..., n$, immigrating at time k. We

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shall call the multitype branching process generated by individual (k,i,j) as "(k,i,j)-process".

We denote

(1) $\mathbf{X} = \{\mathbf{X}^j_{ki}(t) = (X^{j1}_{ki}(t), X^{j2}_{ki}(t), ..., X^{jn}_{ki}(t)), k \geq 1, i \geq 1, j = 1, 2, ..., n\}$

the family of all possible (k,i,j)-processes. Here $X^{jm}_{ki}(t)$ is the number of individuals of type T_m in the (k,i,j)- process at time t. Then the branching process with n-types and immigration is defined by $\mathbf{Z}(t) = (Z_1(t), Z_2(t), ..., Z_n(t))$, $t \geq 0, \mathbf{Z}(0) = \mathbf{0}$, where

$$Z_m(t) = \sum_{j=1}^{n}\sum_{k=1}^{t}\sum_{i=1}^{Y_j(k)} X^{jm}_{ki}(t-k) \tag{2}$$

is the number of individuals of type T_m at time t.

Multitype branching processes with immigration have been studied widely in the literature (see Quine(1970), Kaplan(1974), Shurenkov(1976), Sagitov (1982)). One can find a sufficiently full bibliography of such papers in the books Mode(1971), Sevast'yanov(1971), Jagers(1975) and Badalbaev and Rahimov(1993) and in the review of Vatutin and Zubkov(1993). However, the independence of processes of reproduction and immigration was assumed in these publications. In the described above model this assumption means that the family $\mathbf{X}$ of independent and identically distributed branching processes and random vectors $\{\mathbf{Y}(t), t \geq 0\}$ are independent. Under this assumption the study of $\mathbf{Z}(t)$ can be reduced to the analyses of a relation for its generating function. If we do not assume the independence, it is not possible to get an explicit expression for the generating function of the process.

On the other hand in real branching processes often the immigration process depends on reproduction. For instance, if we consider the process of urban population growth, the number of immigrants at present depends on the lives of past immigrants and their descendants. Another example is the neutron chain reaction in a nuclear reactor with an external neutron source. If one wants to support such a process by immigration, it is apparent that the immigration process depends on reproduction.

Here the process $\mathbf{Z}(t)$ will be considered without an assumption of independence of processes of reproduction and immigration. Single-type branching processes with reproduction-dependent immigration were studied in Rahimov(1992, 1995).

Let $\Re_{k,i}^{j}(t)$ be the σ-algebra generated by the evolution of the (k,i,j)-process up to time t. We consider the $\mathbf{Y}(k,t)$ double array of vectors of immigrating individuals. In other words assume that the collection $\mathbf{Y}(k,t) = (Y_1(k,t),...,Y_n(k,t))$ of individuals immigrating at time k depends on t and for any $r = 0,1,...$ and $p = 1,2,...,n$,

$$\{Y_p(k,t) \le r\} \in \Im_{kr}(t), \tag{3}$$

where

$$\Im_{kr}(t) = \prod_{l=1}^{k-1} \prod_{q=1}^{n} \prod_{i=1}^{Y_q(l,t)} \Re_{li}^{q}(t) \times \prod_{q=1}^{n} \prod_{i=1}^{r} \Re_{ki}^{q}(t) \times \Re_0,$$

$\Re_0$ is some σ-algebra and the direct products of the random number of σ-algebras we shall understand as

$$\prod_{i=1}^{Y} \Re_i = \{A : A \bigcap \{Y = j\} \in \prod_{i=1}^{j} \Re_i\}, \quad \prod_{i=1}^{0} \Re_i = \{\emptyset, \Omega\}.$$

Under the condition (**??**) the totality of immigrating individuals at time k may depend on evolution of the processes generated by the individuals which immigrated up to time k.

2 The basic theorem and corollaries

For n-dimesional vectors $\mathbf{x} = (x_1,x_2,...,x_n), \mathbf{y} = (y_1,y_2,...,y_n)$ we denote $\mathbf{x} \oplus \mathbf{y} = (x_1y_1,...,x_ny_n), \mathbf{x}^{\mathbf{y}} = (x_1^{y_1},...,x_n^{y_n}), (\mathbf{x},\mathbf{y}) = x_1y_1 + ... + x_ny_n, \mathbf{1} = (1,1,...,1), \mathbf{0} = (0,0,...,0), \mathbf{e} = (e,e,...,e)$ and $\mathbf{x} \ge \mathbf{y}$ or $\mathbf{x} > \mathbf{y}$ if $x_i \ge y_i$ or $x_i > y_i, i = 1,2,...,n$, respectively.

We also denote for $j = 1,2,...,n$,

$$F_j(t,\mathbf{S}) = ES_1^{X_{ki}^{j1}(t)} S_2^{X_{ki}^{j2}(t)} \dots S_n^{X_{ki}^{jn}(t)}, \quad \mathbf{S} = (S_1,S_2,...,S_n)$$

the generating functions of the (k,i,j)-processes.

Assume that

$$\sup_t EX_{ki}^{jl}(t) \le C_0 < \infty,$$

and for any fixed $\lambda = (\lambda_1, ..., \lambda_n) > \mathbf{0}$ and for some non-increasing functions $\mathbf{Q}(t) = (Q_1(t), ..., Q_n(t))$

$$\lim_{t \to \infty} \frac{1 - F_j(t, \mathbf{e}^{-\lambda \oplus \mathbf{Q}(t)})}{Q_j(t)} = 1 - \varphi(\lambda), \tag{4}$$

where $\varphi(\lambda) = \varphi(\lambda_1, ..., \lambda_n)$ is the Laplace transform of a random vector having finite expectation, $\mathbf{Q}(t) \to \mathbf{0}$ and for any $x \in [0, 1]$,

$$\lim_{t \to \infty} \frac{Q_j(t)}{Q_j(tx)} = \pi(x) \tag{5}$$

and this convergence is uniform in each interval of the form $[\varepsilon, 1)$ for any $\varepsilon > 0$ and $\pi(x)$ is a continuous function for $x \in (0, 1]$.

Conditions (**??**) and (**??**) can be satisfied for critical or close to critical (in the case of transition phenomena) multitype branching processes. The limit function $\pi(x)$ in (**??**) necessarily has the form x^α for some $\alpha \in [0, \infty)$ (see S.I Resnick(1987), p.14).

For the immigration process we assume that

$$\sum_{j=1}^{n} Q_j(t) \sum_{m=0}^{[tx]} Y_j(m, t) \xrightarrow{P} T(x), \tag{6}$$

as $t \to \infty$ for $0 \leq x \leq 1$, where $T(x)$ is some $\Re_0$- measurable, stochastically continuous for $x = 1$ stochastic process with non-decreasing trajectories, $T(0) = 0$ and $T(1) < \infty$ almost everywhere.

Theorem. If conditions (3) - (6) are satisfied, then

$$\mathbf{W}(t) = \mathbf{Q}(t) \oplus \mathbf{Z}(t) \xrightarrow{\mathcal{D}} \mathbf{W} = (W_1, W_2, ..., W_n),$$

where

$$Ee^{-(\lambda, \mathbf{W})} = E \exp\left\{ - \int_0^1 \frac{1 - \varphi(\lambda_1 \pi(1-x), ..., \lambda_n \pi(1-x))}{\pi(1-x)} dT(x) \right\}$$

(the value of the integrand at $x = 1$ is defined by continuity).

We denote $P_\alpha^j, \quad j=1,2,...,n$, the offspring distribution of the (k,i,j)-process, that is

$$P_\alpha^j = P\{\mathbf{X}_{ki}^j(1)=\alpha\}, \quad \alpha=(\alpha_1,\alpha_2,...,\alpha_n), \quad \alpha_i \in \mathcal{N}_0=\{0,1,...\},$$

is the probability that an individual of the type T_j generates the totality α of new individuals. We also denote

$$F^j(\mathbf{S}) = \sum_{\alpha\in\mathcal{N}_0^n} P_\alpha^j S_1^{\alpha_1}...S_n^{\alpha_n}, \quad \mathbf{F}(\mathbf{S})=(F^1(\mathbf{S}),...,F^n(\mathbf{S}))$$

Let for $i,j,k=1,2,\cdots,n$

$$a_i^j = \frac{\partial F^j(\mathbf{S})}{\partial S_i}\,|_{\mathbf{S}=\mathbf{1}}\,, b_{ik}^j = \frac{\partial^2 F^j(\mathbf{S})}{\partial S_i \partial S_k}\,|_{\mathbf{S}=\mathbf{1}}$$

be finite, $\mathbf{A}=\left\|a_i^j\right\|$ be the matrix of expectations, ρ be its Peron root and the right and the left eigenvectors $\mathbf{U}=(u_1,u_2,...,u_n)$ and $\mathbf{V}=(v_1,v_2,...,v_n)$ corresponding to the Peron root be such that

$$\mathbf{AU}=\rho\mathbf{U}, \quad \mathbf{VA}=\rho\mathbf{V}, \qquad \sum_{i=1}^n u_i v_i = 1, \quad \sum_{i=1}^n u_i = 1.$$

If $\mathbf{A}$ is indecomposable, aperiodic and $\rho=1$, then the limit theorem for the critical multitype branching processes holds (see Sevast'yanov(1971), for example). In this case the conditions (**??**) and (**??**) are satisfied with

$$\begin{aligned} \varphi(\lambda) &= (1+\sum_{j=1}^n \lambda_j)^{-1}, \\ Q_j(t) &= P\{\mathbf{X}_{ki}^j(t)\neq\mathbf{0}\}(u_j v_j)^{-1} \sim \frac{2}{btv_j}, \qquad t\to\infty, \\ \pi(x) &= x, \end{aligned}$$

where $b=\sum_{j,m,k=1}^n v_j b_{mk}^j u_m u_k$ and we have the following result.

Corollary 1. If conditions (3) and (6) are satisfied, then

$$\lim_{t\to\infty} P\left\{\frac{2Z_j(t)}{btv_j} \le y_j, j=1,2,...,n\right\} = A(\mathbf{y}),$$

where the distribution $A(\mathbf{y})$, $\quad \mathbf{y} = (y_1, y_2, \ldots, y_n)$, *has the Laplace transform*

$$\int_{R_+^n} e^{-(\mathbf{y},\lambda)} dA(\mathbf{y}) = E \exp\left\{ -\sum_{j=1}^{n} \lambda_j \int_0^1 (1 + (1-x)\sum_{j=1}^{n} \lambda_j)^{-1} dT(x) \right\}.$$

Let now the generating functions $F^j(\mathbf{S})$ such that

$$x - \sum_{j=1}^{n} v_j(\mathbf{1} - F^j(\mathbf{1} - \mathbf{U}x)) = x^{1+\alpha} L(x), \tag{7}$$

where $0 < x \leq 1, \alpha \in (0, 1]$, and $L(x)$ is a slowly varying function as $x \downarrow 0$. In this case the following limit theorem for the critical multitype branching process holds (see Vatutin (1977)).

Proposition. Under the condition (7) we have a)

$$P\{\mathbf{X}_{ki}^{j}(t) \neq \mathbf{0}\} \sim u^j t^{-1/\alpha} L_1(t)$$

as $t \to \infty$, *where* $L_1(t)$ *is a slowly varying as* $t \to \infty$ *function; b)*

$$\lim_{t\to\infty} P\left\{ X_{ki}^{jl}(t) q(t) \leq x_l v_l, i = 1, 2, ..., n \mid \mathbf{X}_{ki}^{j}(t) \neq \mathbf{0} \right\} = G(\mathbf{x}),$$

where

$$q(t) = \sum_{j=1}^{n} v_j (1 - F^j(t, \mathbf{0}),$$

and $G(\mathbf{x}) = G(x_1, x_2, ..., x_n)$ *a distribution having the Laplace transform*

$$\int_{R_+^n} e^{-(\mathbf{x},\lambda)} dG(\mathbf{x}) = 1 - (1 + \bar{\lambda}^{-\alpha})^{-1/\alpha}, \bar{\lambda} = \sum_{j=1}^{n} \lambda_j.$$

It follws from the Proposition 1 that under the assumption (7) conditions (4) and (5) are satisfied with

$$\begin{aligned}
\varphi(\lambda) &= 1 - (1 + (\sum_{j=1}^{n} \lambda_j)^{-\alpha})^{-1/\alpha}, \\
Q_j(t) &= P\{\mathbf{X}_{ki}^{j}(t) \neq \mathbf{0}\}(u_j v_j)^{-1} \sim \frac{L_1(t)}{v_j t^{1/\ \alpha}}, \qquad t \to \infty, \\
\pi(x) &= x^{1/\alpha},
\end{aligned}$$

and in this case we have the folowing corollary.

Corollary 2. If conditions (3), (6) and (7) are satisfied, then

$$\lim_{t\to\infty} P\left\{Q_j(t)Z_j(t) \leq y_j, j = 1,2,...,n\right\} = B(y_1, y_2, ..., y_n),$$

where the distribution $B(\mathbf{y}), \mathbf{y} = (y_1, y_2, ..., y_n)$ *has the Laplace transform*

$$\int_{R_+^n} e^{-(\mathbf{y},\lambda)} dB(\mathbf{y}) = E\exp\left\{ -\int_0^1 [1 - x + (\sum_{j=1}^n \lambda_j)^{-\alpha}]^{-1/\alpha} dT(x) \right\}$$

Corollaries 1 and 2 give examples of branching processes for which conditions (4) and (5) of the theorem are fulfilled. Now we consider some examples of immigration processes satisfying condition (6).

Example 1. Let $\mathbf{Y}(k,t) \equiv \mathbf{Y}(k)$ and for $r = 0,1,...,p = 1,2,...,n$

$$\{Y_p(k) \leq r\} \in \mathcal{B}_r(k,t) = \prod_{i=1}^r \prod_{q=1}^n \Re_{ki}^q(t).$$

Since $(k,i,j)-$processes are independent, the vector $\mathbf{Y}(k)$ and processes $\{\mathbf{X}_{li}^j(t), l, i \geq 1, l \neq k, j = 1,...,p\}$ are also independent. If $\mathbf{Y}(k), k = 1,2,...$ have the same distribution, $E\mathbf{Y}(k) = \mathbf{a} = (a_1, a_2, ..., a_n)$ is finite and $\rho = 1, b \in (0,\infty)$, then it follows from the law of large numbers that condition (6) holds with $T(x) = 2xb^{-1}\sum_{j=1}^n a_j(v_j)^{-1}$. It is not difficult to see that in this case the Laplace transform in Corollary 1 is

$$(1+\bar{\lambda})^{-\alpha}, \quad \alpha = 2b^{-1}\sum_{j=1}^n a_j(v_j)^{-1}.$$

Therefore we have the following result which is a generalization of well-known theorem on convergence to the gamma distribution.

Corollary 3. If $\rho = 1, b \in (0,\infty)$*, the coordinates of the vector* $\mathbf{Y}(k)$ *are stopping times with respect to the family* $\mathcal{B}_r(k,t), r = 0,1,...,$ *it has the same distribution for different* k *and* $\mathbf{a} < \infty$*, then*

$$\mathbf{W}(t) = (\frac{2Z_j(t)}{btv_j}, j = 1,2,...,n) \xrightarrow{\mathcal{D}} \mathbf{W} = (W_1, ..., W_n)$$

where $W_i = W_j$ with probability 1 and W_i has the gamma distridution of the parameter α.

Example 2. Let again coordinates of the vector $\mathbf{Y}(k)$ are stopping times with respect to $\mathrm{B}_r(k,t)$ and

$$\frac{1}{t}\sum_{k=0}^{t}\mathbf{Y}(k)\xrightarrow{P}\mathbf{Y}=(Y_1,...,Y_n).$$

In this case (6) holds with $T(x) = 2xb^{-1}\sum_{j=1}^{n}Y_j(v_j)^{-1}$ and the Laplace transform in Corollary 2 is

$$E[(1+\bar{\lambda})^{-2T/b}],\qquad T=(\mathbf{V^{-1}},\mathbf{Y}).$$

Since $Y_j, j=1,...,n$, have the positive infinitely divisible distributions T also has such a distribution. Therefore (see Feller (1967), Sec.7, Chap. XII) its Laplace transform is of the form

$$Ee^{-uT}=\exp\left\{-\int_0^\infty\frac{1-e^{-ux}}{x}P(dx)\right\},$$

where $P(x)$ is a measure such that $\int_0^\infty x^{-1}P(dx)<\infty$. Since

$$E[(1+\bar{\lambda})^{-2T/b}]=E\exp\{-\frac{2T}{b}\log(1+\bar{\lambda})\},$$

using this fact we obtain the following result.

Corollary 4. If coordinates of $\mathbf{Y}(k)$ are stopping times with respect to the family $\mathcal{B}_r(k,t), r=0,1,...$, then, under the assumptions mentioned above $\mathbf{W}(t)\xrightarrow{\mathcal{D}}\mathbf{W}, t\to\infty$, where

$$Ee^{-(\lambda,\mathbf{W})}=\exp\left\{-\int_0^\infty\frac{(1+\bar{\lambda})^{2x/b}-1}{x(1+\bar{\lambda})^{2x/b}}P(dx)\right\}.$$

3 The proof of the basic result

First we consider the function

$$H(t,\lambda)=\prod_{k=1}^{t}\prod_{j=1}^{n}[F_j(t-k,\mathbf{e}^{-\lambda\oplus\mathbf{Q}(t)})]^{Y_j(k,t)}$$

and we shall prove the following lemma.

Lemma 1. If conditions (4), (5) and (6) are satisfied, then

$$H(t,\lambda) \xrightarrow{P} H(\lambda) = \exp\left\{ -\int_0^1 \frac{1-\varphi(\pi(1-x)\lambda)}{\pi(1-x)} dT(x) \right\}. \tag{8}$$

Proof. We choose $\varepsilon \in (0,1)$, put $a = 1-\varepsilon$ and consider

$$A_1 = \sum_{j=1}^{n} \sum_{k=1}^{[ta]} Y_j(k,t) \log F_j(t-k, \mathbf{e}^{-\lambda \oplus \mathbf{Q}(t)}) \tag{9}$$

If we denote

$$\pi_j(t,x) = \frac{Q_j(t)}{Q_j(tx)}, \quad \pi(t,x) = (\pi_1(t,x), ..., \pi_n(t,x)), \theta = \theta_k(t) = 1 - \frac{k}{t},$$

then for sufficiently large t and $1 \le k \le [ta]$

$$0 < \pi(\varepsilon) - \varepsilon_1 \le \pi_j(t,\theta) \le 1 \tag{10}$$

for some $\varepsilon_1 > 0$ and $j = 1,2,...,n$. Therefore it follows from (10) and the condition (4) that

$$\sup_{1 \le k \le [at]} \left| \frac{1 - F_j(t-k, \mathbf{e}^{-\lambda \oplus \mathbf{Q}(t)})}{Q_j(t-k)} - 1 + \varphi((\lambda, \pi(t,\theta)) \right| \to 0 \tag{11}$$

as $t \to \infty$. On the other hand, since $\varepsilon \le \theta_k(t) < 1$ for $1 \le k \le [ta]$, it follows from (5) that

$$\sup_{1 \le k \le [ta]} |\pi_j(t,\theta) - \pi(\theta)| \to 0 \tag{12}$$

as $t \to \infty$ for $j = 1,2,...,n$. We obtain from (11) and (12) the following representation

$$-\frac{\log F_j(t-k, \mathbf{e}^{-\lambda \oplus \mathbf{Q}(t)})}{Q_j(t)} = \frac{1-\varphi(\pi(\theta)\lambda)}{\pi(\theta)}(1+\alpha_j(k,t)), \tag{13}$$

where $\alpha_j(k,t) \to 0$ for $j = 1,2,...,n$, as $t \to \infty$ uniformly with respect to $1 \le k \le [ta]$.

It is not difficult to see that the integral

$$(14)\quad \sum_{j=1}^{n}\sum_{k=1}^{[ta]} \frac{1-\varphi(\pi(\theta)\lambda)}{\pi(\theta)} Q_j(t)Y_j(k,t) = \int_0^{1-\varepsilon} \frac{1-\varphi(\pi(1-x)\lambda)}{\pi(1-x)} d\xi_t(x),$$

where

$$\xi_t(x) = \sum_{k=1}^{[tx]}\sum_{j=1}^{n} Q_j(t)Y_j(k,t),$$

conveges in probability to

$$B(\lambda,\varepsilon) = \int_0^{1-\varepsilon} \frac{1-\varphi(\pi(1-x)\lambda)}{\pi(1-x)} dT(x)$$

due to condition (6). Since $T(x)$ is stochastically contiuous at $x = 1$ and $\varphi(\lambda)$ is the Laplace transform of a random vector having finite expectation, one can show by standard arguments that the last integral converges in probability to the $B(\lambda, 0)$ as $\varepsilon \to 0$. Hence it follows from (13) that A_1 converges in probability to $-B(\lambda, 0)$ as $t \to \infty$ and $\varepsilon \to 0$.

We now consider

$$A_2 = \sum_{j=1}^{n}\sum_{k=[ta]}^{t} Y_j(k,t) \log F_j(t-k, \mathbf{e}^{-\lambda\oplus\mathbf{Q}(t)}).$$

Using the simple inequality $\log(1-x) \geq -x - x^2/(1-x), 0 \leq x < 1$, we get

$$0 > A_2 \geq -\sum_{j=1}^{n}\sum_{k=[ta]}^{t} Y_j(k,t)(R_{kj} + \frac{R_{kj}^2}{1-R_{kj}}),$$

where

$$R_{kj} = R_{kj}(t,\lambda) = 1 - F_j(t-k, \mathbf{e}^{-\lambda\oplus\mathbf{Q}(t)}).$$

It is not difficult to see that

$$\sum_{j=1}^{n}\sum_{k=[ta]}^{t} Y_j(k,t)R_{kj} \leq C_0 \sum_{j=1}^{n} \lambda_j[\xi_t(1) - \xi_t(1-\varepsilon)]$$

and the last difference in probability converges to $T(1) - T(1-\varepsilon)$ as $t \to \infty$. Using again the stochastic continuity of $T(x)$ at $x = 1$ we obtain that this difference in probability converges to 0 as $\varepsilon \to 0$, and, therefore, A_2 converges in probability to 0 as $t \to \infty$ and $\varepsilon \to 0$. The lemma is proved.

Let $\mathbf{X}_i(n) = (X_{i1}(n), X_{i2}(n), ..., X_{ip}(n)), i = 1, 2, ...$ be p-dimensional random vectors such that $\mathbf{X}_i(n)$ is $F_i(n)$-measurable, where $F_i(n)$ are some σ-algebras such that $F_i(n) \subseteq F_{i+1}(n), i = 0, 1, 2, ...$ for any n. We consider the following sum

$$S_n = \sum_{i=1}^{n} \nu_i(n)\mathbf{X}_i(n), \tag{15}$$

where $\nu_i(n)$ are random variables taking values 0 and 1 and $F_{i-1}(n)$-measurable. Denote

$$f_j^{(n)}(\lambda) = E[e^{-(\lambda, \mathbf{X}_j(n))}|F_{j-1}(n)], \lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in R_+^p.$$

The following lemma for $p = 1$ was proved in Rahimov(1995, p.29) using a semimartingale technique. A similar result for simple random sums can be seen in Beska et. al.(1982).

Lemma 2. Let for any $j = 1, 2, ..., n$ random variable $\nu_j(n)$ is $F_{j-1}(n)$ -measurable and

$$\prod_{j=1}^{n} \left\{ f_j^{(n)}(\lambda) \right\}^{\nu_j(n)} \xrightarrow{P} \varphi(\lambda), n \to \infty, \tag{16}$$

where $\varphi(\lambda)$ is some $\mathcal{F}_0$-measurable random variable, such that $\varphi(\lambda) > 0$ almost everywhere, $\mathcal{F}_0 \subset F_0(n)$ for all $n = 1, 2, ...$ Then

$$E\left[e^{-(\lambda, \mathbf{S}_n)}|\mathcal{F}_0\right] \xrightarrow{P} \varphi(\lambda), n \to \infty. \tag{17}$$

Proof. Let $\widetilde{\mathbf{X}}_i\ (n) = \mathbf{X}_i(n)\chi(\mathcal{A}_i(n))$, where

$$\mathcal{A}_i(n) = \left\{ \prod_{j=1}^{i} \left\{ f_j^{(n)}(\lambda) \right\}^{\nu_j(n)} \geq \frac{1}{2}\varphi(\lambda) \right\},$$

and $\chi(.)$ is the indicator function. It is clear that $\mathcal{A}_i(n) \in F_{i-1}(n)$. First we shall prove that for any n and $m \leq n$

$$E\left[\prod_{k=1}^{m} Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right] = 1, \tag{18}$$

where

$$Z_k(n) = \frac{e^{-(\lambda, \widetilde{\mathbf{X}}_k(n))}}{\overset{\sim(n)}{f_k}(\lambda)}, \quad \overset{\sim(n)}{f_k}(\lambda) = E[e^{-(\lambda, \widetilde{\mathbf{X}}_k(n))}|F_{k-1}(n)].$$

If $m = 1$, then

$$E\left[\prod_{k=1}^{1} Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right] = E[E[Z_1^{\nu_1(n)}(n)|F_0(n)]|\mathcal{F}_0].$$

Using the simple equality

$$e^{-(\lambda, \tilde{\mathbf{X}}_k(n))\nu_k(n)} = \chi(\nu_k(n) = 1)e^{-(\lambda, \tilde{\mathbf{X}}_k(n))} + \chi(\nu_k(n) = 0) \tag{19}$$

we have

$$E\left[\prod_{k=1}^{1} Z_k^{\nu_k(n)}(n)|\mathcal{F}_0\right] = 1$$

Assuming that (18) holds for $m = i$ one can obtain by similar arguments that it is true for $m = i + 1$ also. Therefore (18) holds for any $m = 1, 2, ...$

Using induction on m we can prove that

$$\prod_{i=1}^{m} \left\{\tilde{f}_i^{(n)}(\lambda)\right\}^{\nu_i(n)} \geq \frac{1}{2}\varphi(\lambda). \tag{20}$$

almost everywhere, for all n and $m \leq n$.

Putting $\tilde{\mathbf{S}}_n = \sum_{k=1}^{n} \tilde{\mathbf{X}}_k(n)\nu_k(n)$ and using (18) and (20) we have

$$|E[e^{-(\lambda, \tilde{\mathbf{S}}_n)}|\mathcal{F}_0] - \varphi(\lambda)| \leq \frac{2}{\varphi(\lambda)} E[W_n(\lambda)|\mathcal{F}_0], \tag{21}$$

where

$$W_n(\lambda) = \left|\prod_{k=1}^{n} \left\{\tilde{f}_k^{(n)}(\lambda)\right\}^{\nu_k(n)} - \varphi(\lambda)\right|.$$

The estimate (21) shows that in order to be

$$E\left[e^{-(\lambda, \tilde{\mathbf{S}}_n)}|\mathcal{F}_0\right] \xrightarrow{P} \varphi(\lambda), n \to \infty, \tag{22}$$

it is sufficient that $W_n(\lambda)$ converges in probability to zero as $n \to \infty$. In fact, then we obtain from the dominated convergence theorem that $EW_n(\lambda)$ tends to zero and, since

$$P\left\{E[W_n(\lambda)|\mathcal{F}_0] > \varepsilon\right\} \leq \frac{1}{\varepsilon} EW_n(\lambda)$$

we have that $E[W_n(\lambda)|\mathcal{F}_0]$ converges to zero in probability as $n \to \infty$.

For any $\varepsilon > 0$ we have

$$P\{W_n(\lambda) > \varepsilon\} \leq P\{V_n(\lambda) > \varepsilon\} + P\{\bigcup_{k=1}^{n} R_k(n) \bigcap \{\nu_k(n) = 1\}\}, \tag{23}$$

where

$$V_n(\lambda) = \left| \prod_{k=1}^{n} f_k^{(n)}(\lambda)^{\nu_k(n)} - \varphi(\lambda) \right|, R_k(n) = \left\{ f_k^{(n)}(\lambda) \neq \tilde{f}_k^{(n)}(\lambda) \right\}.$$

Since $R_k(n) = \bar{\mathcal{A}}_k(n)$, we have to show that

$$P\{T_n\} \to 0, n \to \infty, \tag{24}$$

where

$$T_n = \bigcup_{k=1}^{n} \bar{\mathcal{A}}_k(n) \bigcap \{\nu_k(n) = 1\}.$$

We have from the definition of $\mathcal{A}_k(n)$, that

$$\bar{\mathcal{A}}_k(n) \subseteq \left\{ \prod_{j=1}^{n} \left\{ f_j^{(n)}(\lambda) \right\}^{\nu_j(n)} < \frac{1}{2}\varphi(\lambda) \right\}, k \leq n.$$

Therefore we have

$$P\{T_n\} \leq P\left\{ \varphi(\lambda) - \prod_{j=1}^{n} \{f_j^{(n)}(\lambda)\}^{\nu_j(n)} > \frac{1}{2}\varphi(\lambda) \right\}.$$

Choosing ε such that $\varepsilon < \varphi(\lambda)$, we obtain due to condition (16)

$$P\{T_n\} \leq P\{V_n(\lambda) > \frac{\varepsilon}{2}\} \to 0, n \to \infty. \tag{25}$$

Thus (24) holds. It follows from (23) and (24) that $W_n(\lambda)$ converges to zero in probability.

It remains to show that the variable $\tilde{\mathbf{S}}_n$ in (22) can be replaced by S_n. It is not difficult to see that

$$E\left[e^{-(\lambda,\tilde{\mathbf{S}}_n)}|\mathcal{F}_0\right] = E\left[e^{-(\lambda,\mathbf{S}_n)}|\mathcal{F}_0\right] + E\left[\left(e^{-(\lambda,\tilde{\mathbf{S}}_n)} - e^{-(\lambda,\mathbf{S}_n)}\right)\chi(T_n)|\mathcal{F}_0\right]$$

and the variable $\chi(T_n)$ converges to zero in probability due to (25).

The lemma is proved.

Proof of Theorem. It follows from (2) that $\mathbf{Z}(t)$ can be written in the form

$$\mathbf{Z}(t) = \sum_{k=0}^{t}\sum_{r=1}^{n}\sum_{i=1}^{\infty}\chi(i \le Y_r(k,t))\mathbf{X}_{ki}^{r}(t-k). \tag{26}$$

Let $\{N_t^r\}, t \ge 0$ be a sequence of integers such that

$$P\{\bigvee_{k=1}^{t} Y_r(k,t) > N_t^r\} \to 0 \tag{27}$$

as $t \to \infty$ for $r = 1, 2, ..., n$. Putting $N_t = \bigvee_{r=1}^{n} N_t^r$, we define process $\mathbf{W}^\star(t)$ by the following relation

$$\mathbf{W}^\star(t) = \sum_{l=1}^{k(t)} \nu_l(t)\mathbf{V}_l(t) \tag{28}$$

where $k(t) = nN_t t$,

$$\nu_l(t) = \chi(i \le Y_r(k,t)), \quad \mathbf{V}_l(t) = \mathbf{X}_{ki}^{r}(t-k)$$

for such l that

$$l = N_t n(k-1) + j, \quad j = N_t(r-1) + i \tag{29}$$

and $1 \le j \le nN_t, 1 \le i \le N_t, 1 \le k \le t, 1 \le r \le n$. Then we have from (26) that

$$T_t\mathbf{Z}(t) = T_t\mathbf{W}^\star(t), \tag{30}$$

where

$$T_t = \chi\{\bigcap_{r=1}^{n}\{\bigvee_{k=1}^{t} Y_r(k,t) \le N_t\}\}.$$

It is not difficult to see that for any l satisfying (29) the vector $\mathbf{V}_l(t)$ is $\mathcal{F}_{ki}(t)$-measurable. On the other hand it follows from condition (3) that $\nu_l(t)$ is $\mathcal{F}_{ki-1}(t)$-measurable. Thus Lemma 2 is applicable to $\mathbf{W}^\star(t)$. According to that lemma, in order to be

$$E\left[e^{-(\lambda_t, \mathbf{W}^\star(t))}|\mathcal{F}_0\right] \xrightarrow{P} H(\lambda) \tag{31}$$

with $\lambda_t = \lambda \oplus \mathbf{Q}(t)$ for any $\lambda \in R_+^n$ it is sufficient that

$$D(t,\lambda) \xrightarrow{P} H(\lambda), t \to \infty, \tag{32}$$

where

$$D(t,\lambda) = \prod_{k=1}^{t} \prod_{j=1}^{n} \prod_{i=1}^{N_t} \left\{ F_j(t-k, \mathbf{e}^{-\lambda_t}) \right\}^{\chi(i \le Y_j(k,t))}.$$

Since

$$D(t,\lambda) = T_t H(t,\lambda) + (1-T_t) D(t,\lambda)$$

we have from Lemma 1 and (27) that (32) holds for any $\lambda \in R_+^n$. Thus, due to the dominated convergence theorem, it follows from (31) that the Laplace transform of $\mathbf{Q}(t) \oplus \mathbf{W}^\star(t)$ tends as $t \to \infty$ to $EH(t,\lambda)$, that is

$$\mathbf{Q}(t) \oplus \mathbf{W}^\star(t) \xrightarrow{\mathcal{D}} \mathbf{W}, t \to \infty. \tag{33}$$

It is not difficult to see that the inequality

$$|P\{\chi \mathbf{X} \le \mathbf{x}\} - P\{\mathbf{X} \le \mathbf{x}\}| \le P\{\chi = 0\} \tag{34}$$

is true for any random vector $\mathbf{X}$, indicator χ and $\mathbf{x} \in R^n$.

The assertion of the theorem follows from (30), (33), (34) and the choice of N_t. The theorem is proved.

In conclusion note that assumptions (4) and (5) are fulfilled, if the limit theorem holds for the reproduction process. Therefore, using arguments of the proof of this theorem, one can obtain results sort of our main theorem for generalized multitype (Bellman-Harris or Crump-Mode-Jagers) models of branching processes, when the limit theorem holds for the corresponding process without immigration.

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