Functional limit theorems for critical processes with immigration

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Abstract

We consider a critical discrete time branching process with generation dependent immigration. In the case when the mean number of immigrating individuals tends to infinity with the generation number we prove functional limit theorems for centered and normalized process. The limiting processes are deterministically time-changed Wiener with three different covariance functions depending on the behavior of the mean and variance of the number of immigrants. As an application we prove that the conditional least squares estimator of offspring mean is asymptotically normal, which demonstrates an alternative case of normality of the estimator for the process with non-degenerate offspring distribution. The norming factor is $n\sqrt{\alpha(n)}$ with $\alpha(n)$ being the mean number of immigrating individuals to nth generation.

Key Words: branching process, immigration, functional, martingale limit theorem, Skorokhod space, least squares estimator.

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1 Introduction

We consider a discrete time branching stochastic process $Z(n), n \geq 0, Z(0) = 0$. It can be defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, n, i \geq 1\}$ and $\{\xi_k, k \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1}$$

Assume that X_{ni} have a common distribution for all n and i, families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. Variables X_{ki} will be interpreted as the number of offspring of the ith individual in the $(k-1)$th generation and ξ_k is the number of immigrating individuals to the kth generation. Then $Z(n)$ can be considered as the size of nth generation of the population.

In this interpretation $A = EX_{ni}$ is the mean number of offspring of a single individual. Process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $A < 1, A = 1$ or $A > 1$ respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, in contrast of classical models, we do not assume that $\xi_n, n \geq 1$ are identically distributed, i. e. immigration rate may depend on the time of immigration.

Investigations show that asymptotic behavior of the process with immigration is very sensitive to any changes of the immigration process in time. For instance, in critical case change of the mean number of immigrating individuals in time leads to such fluctuations of the process, that one needs to use various functional normalization of the process to obtain non degenerate limit distribution for the process (see [17], Ch. III and references therein). Therefore description of processes which can be used as approximating in this situation is of interest. On the other hand this kind of functional limit theorems are useful in estimating parameters and in study of various functionals of the process.

In this article we prove functional limit theorems for critical processes in the case, when the immigration mean tends to infinity. It turns out that suitably normalized process may be approximated by a Gaussian process with independent increments and with three different covariance functions depending on behavior of the mean and variance of the number of immigrants. The limiting Gaussian process can be obtained from the Wiener process by a deterministic time-change. As an example of application of functional limit theorems we prove that conditional least squares estimator (CLSE) of offspring mean A is asymptotically normal. It is interesting to note that norming factor depends on the mean number of immigrants and can be chosen depending on the rate of immigration.

First approximation theorems of branching stochastic processes have appeared due to W. Feller [6], who demonstrated that branching stochastic process without immigration can be approximated by a diffusion process. Lamperti [12], [13] proved convergence of finite dimensional distributions of the process with large number of initial individuals to those of some diffusion processes with two different normalization. These results were extended to the functional form by T. Lindvall [14], [15]. Convergence of finite dimensional distributions of a sequence of Galton-Watson branching processes with stationary immigration has been investigated by Kawazu and Watanabe [11] and Aliev [1]. Wey and Winnicki [19] have shown that random step functions of a critical branching process with immigration converges in Skorohod metric to a nonnegative diffusion process. Fluctuation theorems for the sequence of nearly critical branching processes have been proved by Sriram [18], who obtained a diffusion approximation. In papers by Ispàny, Pap and Van Zuijlen [7], [8] the authors demonstrated that Sriram's result is also valid when offspring variance tends to zero and centralized process can be approximated by Ornstein-Uhlenbeck type processes. In addition, asymptotic normality of the mean-square estimator of the offspring mean was proved with normalizing factor $n^{3/2}$. Note that in the latter case reproduction process will approach to deterministic multiplication of individuals. The paper [9] of the same authors is also devoted to the critical branching process with varying offspring and immigration distributions. However, in contrast to our situation, in that paper the offspring variance tends to zero.

It was known that in critical or nearly critical case the CLSE of the offspring mean is not asymptotically normal (see [18] and [19]). Results of [7] and [8] have shown that, when the process is nearly critical and the offspring variance tends to zero, it has a normal limiting distribution. Our Theorem 4 demonstrates an alternative situation of asymptotic normality of CLSE for the process with non-degenerate offspring distribution.

It is known that (see Alzaid, Al-Osh [2], Dion, at all [5] and Franke, Seligmann [4]) in the case of Bernoulli offspring distribution process defined in equation (1) can be considered as an integer-valued, first order autoregressive

(INAR(1)) time series model with noise ξ_k. In this framework considered here process $Z(n)$ can be related to INAR(1) model with non stationary (rising) noise.

In proofs we follow the same scheme as in papers [7] and [8] and use some tricks of the proof of appropriate statements there. Namely we represent our process in the form of normalized martingale differences and use martingale limit theorems to derive our results.

Main results and examples will be provided in Section 2 of the paper. In Section 3 we prove several preliminary results which will be used in proofs of main theorems. Section 4 is devoted to proofs of main results of the paper.

2 Main results and examples

From now on we assume that $A = EX_{ni}$ and $B = varX_{ni}$ are finite. We also assume that $\alpha(n) = E\xi_n < \infty$, $\beta(n) = var\xi_n < \infty$ for each $n \geq 1$ and regularly varying when $n \to \infty$ functions, i. e. have the following form

$$\alpha(n) = n^{\alpha} L_{\alpha}(n), \quad \beta(n) = n^{\beta} L_{\beta}(n), \tag{2}$$

where $\alpha, \beta \geq 0, L_{\alpha}(n)$ and $L_{\beta}(n)$ are slowly varying as $n \to \infty$ functions. Then $A(n) = EZ(n)$ and $B^2(n) = varZ(n)$ are finite for each $n \geq 1$ and, when $A = 1$,

$$A(n) = \sum_{k=1}^{n} \alpha(k), \quad B^2(n) = \Delta^2(n) + \sigma^2(n), \tag{3}$$

where

$$\Delta^2(n) = B \sum_{k=1}^{n} \alpha(k)(n-k), \quad \sigma^2(n) = \sum_{k=1}^{n} \beta(k).$$

For each $t \in R_+ = [0, \infty)$ we define sequence of step functions

$$Y_n(t) = \frac{Z([nt]) - A([nt])}{B(n)}.$$

Everywhere from now D, d and P denote convergence of random functions in Skorohkod topology and convergence of random variables in distribution and in probability respectively.

Theorem 1. *If $A = 1,\ B \in (0, \infty),\ \alpha(n) \to \infty$ and $\beta(n) = o(n\alpha(n))$, then $Y_n(t) \xrightarrow{D} W(t^{2+\alpha})$ as $n \to \infty$ weakly in Skorokhod space $D(R_+, R)$, where $(W(t), t \in R_+)$ is standard Brownian motion.*

Note that in Theorem 1 we do not require a Lindeberg type condition on offspring or immigration distribution. In fact, in this case it is satisfied for the immigration process due to normalization by $B^2(n)$. However in opposite case we need one on the sequence $\{\xi_n, n \geq 1\}$ which seems natural for the process with inhomogeneous immigration. Thus we denote for each $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{\sigma^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2; |\xi_k - \alpha(k)| > \varepsilon\sigma(n)]. \tag{4}$$

Theorem 2. *If $A = 1,\ B \in (0, \infty),\ \alpha(n) \to \infty,\ \alpha(n) = o(n^{-1}\beta(n))$ and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then $Y_n(t) \xrightarrow{D} W(t^{1+\beta})$ as $n \to \infty$ weakly in Skorokhod space $D(R_+, R)$.*

Next theorem is related to the case when $n\alpha(n)$ and $\beta(n)$ have the same rate.

Theorem 3. *If $A = 1,\ B \in (0, \infty),\ \alpha(n) \to \infty,\ \beta(n) \sim cn\alpha(n),\ c \in (0, \infty)$ and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then $Y_n(t) \xrightarrow{D} W(t^{1+\beta}) = W(t^{2+\alpha})$ as $n \to \infty$ weakly in Skorokhod space $D(R_+, R)$.*

Remarks. 1. Using Lemma 1 in Section 3, one can see that condition $\beta(n) = o(n\alpha(n))$ is equivalent to $\sigma^2(n) = o(\Delta^2(n))$, condition $\alpha(n) = o(n^{-1}\beta(n))$ is equivalent to $\Delta^2(n) = o(\sigma^2(n))$ and $\beta(n) \sim cn\alpha(n)$ as $n \to \infty$, if and only if $\sigma^2(n) \sim \theta B^2(n)$ with $\theta = d/(d + (1 + \beta)B)$, where $d = c(1 + \alpha)(2 + \alpha)$.

2. Since $\beta(n)/n\alpha(n)$ is regularly varying with exponent $\beta - 1 - \alpha$ and, when $\beta(n) \sim cn\alpha(n)$, as $n \to \infty$ has a positive finite limit, we conclude that $\beta = \alpha + 1$. This explains equality $W(t^{1+\beta}) = W(t^{2+\alpha})$ in Theorem 3.

Now we consider examples of the immigration process which satisfy conditions of provided theorems.

Example 1. Let $\xi_k, k \geq 1$ be Poisson with mean $\lambda(k) \to \infty, k \to \infty$ and regularly varies with exponent α. Then $\Delta^2(n) = B\sum_{k=1}^{n} \lambda(k)(n-k), \sigma^2(n) =$

$\sum_{k=1}^{n} \lambda(k)$ and clearly $\sigma^2(n) = o(\Delta^2(n))$. In this case we obtain the following result from Theorem 1.

Corollary 1. If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$ are Poisson with mean $\lambda(k) \to \infty, k \to \infty$ and $(\lambda(k))_{k=1}^{\infty}$ is regularly varying with exponent α, then $Y_n(t) \xrightarrow{D} W(t^{2+\alpha})$ as $n \to \infty$ weakly in Skorokhod space $D(R_+, R)$.

Example 2. Let now $\xi_k, k \geq 1$ have positive geometric distributions with parameter $p_k = k^{-1}$ i. e. $P\{\xi_k = i\} = q_k^{i-1} p_k, i = 1, 2, ..., q_k = 1 - p_k$. In this case $\alpha(k) = k, \beta(k) = q_k p_k^{-2} = k^2(1 - k^{-1})$. Consequently we have $\Delta^2(n) \sim Bn^3/6$ and $\sigma^2(n) \sim n^3/3$. Therefore $\sigma^2(n) \sim 2B^2(n)/(B+2)$. Now we show fulfilment of the Lindeberg condition. Since $ES^{\xi_k} = (p_k S)(1 - q_k S)^{-1}$, we find that $(ES^{\xi_k})''' = 6p_k q_k^2 (1 - q_k S)^{-4}$. Therefore $E\xi_k(\xi_k - 1)(\xi_k - 2) = 6q_k^2 p_k^{-3}$. From this we conclude that $E|\xi_k - \alpha(k)|^3 = O(k^3), k \to \infty$ which leads to relation

$$C_n^3 =: \sum_{k=1}^{n} E|\xi_k - \alpha(k)|^3 = O(n^4), n \to \infty.$$

Thus $C_n^3/\sigma^3(n) = O(n^{-1/2}), n \to \infty$, i. e. Lyapunov condition is satisfied for $\xi_k, k \geq 1$. Now we obtain the following result from Theorem 3.

Corollary 2. If $A = 1, B \in (0, \infty)$ and $\xi_k, k \geq 1$, are geometric with parameter $p_k = k^{-1}$, then $Y_n(t) \xrightarrow{D} W(t^3)$ as $n \to \infty$ weakly in Skorokhod space $D(R_+, R)$.

Example 3. Let $\xi_k, k \geq 1$ be such that $p = P\{\xi_k = k^2\} = 1 - P\{\xi_k = 0\}, q = 1 - p$. Then simple calculations give that $\Delta^2(n) \sim Bpn^4/12, \sigma^2(n) \sim pqn^5/5$ as $n \to \infty$ and consequently $\Delta^2(n) = o(\sigma^2(n)$. Since in this case $C_n^3 \sim pq(p^2 + q^2)n^7/7$ and $\sigma^3(n) \sim (pq/5)^{3/2} n^{15/2}$ the Lyapunov condition is again satisfied. Thus we obtain from Theorem 2 that $Y_n(t)$ converges as $n \to \infty$ to $W(t^5)$ weakly in Skorokhod space $D(R+, R)$.

Now we consider one non trivial application of our theorems related to conditional least-squares estimator of offspring mean. Let $\Im(n)$ for each $n \geq 0$ be σ-algebra generated by $\{Z(k), k = 0, 1, ..., n\}$. We obtain from (1) that

$$E[Z(n)|\Im(n-1)] = AZ(n-1) + \alpha(n), \quad n \geq 1. \tag{5}$$

If we assume that the immigration mean $\alpha(n)$ is known, then CLSE $\hat{A}_n$ of A must minimize sum of squares error

$$\sum_{k=1}^{n}(Z(k) - AZ(k-1) - \alpha(k))^2.$$

By usual arguments we obtain that it has the form

$$\hat{A}_n = \frac{\sum_{k=1}^{n}(Z(k) - \alpha(k))Z(k-1)}{\sum_{k=1}^{n} Z^2(k-1)}. \tag{6}$$

Using Theorem 1 we shall prove the following result for $\hat{A}_n$.

Theorem 4. *If conditions of Theorem 1 are fulfilled, then*

$$n\sqrt{\alpha(n)}(\hat{A}_n - 1) \xrightarrow{d} N(0, a), \tag{7}$$

where $N(0, a)$ is normal random variable with mean 0 and variance

$$a^2 = \frac{(1+\alpha)(2\alpha+3)^2 B}{3\alpha+4}. \tag{8}$$

We note that, if $\alpha = 0$, i.e. the immigration mean tends to infinity as a slowly varying function, then the variance of the limiting distribution is 9B/4.

It will be seen further in the paper that one can treat the CLSE in the case when conditions of theorems 2 or 3 are satisfied. Note that in this case $\hat{A}_n$ may not necessarily be asymptotically normal.

3 Preliminary results

We start with two simple lemmas on regularly varying functions. If a sequence $(C_n)_{n=1}^{\infty}$ or function f is regularly varying with exponent ρ, we write $(C_n)_{n=1}^{\infty} \in R_\rho$ and $f \in R_\rho$. The following result is a discrete form of well known Karamata's theorem on regularly varying functions (see [3], Theorem 1.5.11).

Lemma 1. *If* $(C_n)_{n=1}^{\infty} \in R_\rho$*, then for any* $\theta \in (-\rho-1, \infty)$

$$\sum_{k=1}^{n} k^\theta C_k \sim \frac{n^{\theta+1} C_n}{\theta+\rho+1} \tag{9}$$

as $n \to \infty$ *and* $(\sum_{k=1}^{n} k^\theta C_k)_{n=1}^{\infty} \in R_{\theta+\rho+1}$.

Proof. We consider the function $f_c : [0, \infty) \mapsto R$ defined as $f_c(x) := C_{[x]}$ for each $x \in [0, \infty)$. Here $[x]$ denotes the integer part of x. Due to Theorem 1.9.5 in [3] $f_c \in R_\rho$ and locally bounded on $[0, \infty)$. Applying Theorem 1.5.11 in [3], we have

$$\int_0^x t^\theta f_c(t) dt \sim \frac{x^{\theta+1} f_c(x)}{\theta+\rho+1}$$

as $x \to \infty$, which implies that function $x \mapsto \int_0^x t^\theta f_c(t)dt$ is regularly varying with exponent $\theta+\rho+1$. Since $C_{n-1}/C_n \to \infty$, we obtain the statements of Lemma 1.

Lemma 2. *If* $A(n)$ *is a regularly varying function with exponent* $\alpha \geq 0$*, then*

$$\sup_{\varepsilon \leq t \leq a} |\frac{A(nt)}{A(n)} - t^\alpha| \to 0 \tag{10}$$

as $n \to \infty$ *for any* $0 < \varepsilon \leq a < \infty$.

The assertion of this lemma is a simple consequence of the uniform convergence theorem for slowly varying functions.

Lemma 3. *If* $\alpha(n) \to \infty$ *as* $n \to \infty$ *and regularly varies with exponent* α *and* $A(n) = \alpha(1) + ... + \alpha(n)$*, then as* $n \to \infty$
a)

$$\Delta^2(n) \sim \frac{B\alpha(n)n^2}{(\alpha+1)(\alpha+2)}, \quad \sigma^2(n) \sim \frac{n\beta(n)}{\beta+1}. \tag{11}$$

b) For each $\gamma \geq 0$

$$\sum_{k=1}^{n} A^\gamma(k) \sim \frac{n}{\gamma\alpha+\gamma+1} A^\gamma(n). \tag{12}$$

Proof. To prove Part (a) consider

$$\sum_{k=1}^{n} \alpha(k)(n-k) = nA(n) - \sum_{k=1}^{n} k\alpha(k).$$

If we apply Lemma 1 with $\theta = 0$ and $\theta = 1$, we obtain

$$A(n) \sim \frac{n\alpha(n)}{\alpha+1}, \quad \sum_{k=1}^{n} k\alpha(k) \sim \frac{n^2\alpha(n)}{\alpha+2} \tag{13}$$

as $n \to \infty$, which implies the first relation in (11). The second relation in (11) is a direct consequence of Lemma 1.

To prove Part (b) we take into account that $(A(n))_{n=1}^{\infty} \in R_{\alpha+1}$ and $((A(n))^{\gamma})_{n=1}^{\infty} \in R_{(\alpha+1)\gamma}$ for any $\gamma \in [0,\infty)$. Therefore, applying again Lemma 1 with $\theta = 0$, we obtain

$$\sum_{k=1}^{n} (A(k))^{\gamma} \sim \frac{n}{(\alpha+1)\gamma+1}(A(n))^{\gamma} \tag{14}$$

as $n \to \infty$. Lemma 3 is proved.

Lemma 4. *If $\alpha(n) \to \infty$ as $n \to \infty$, then for each $t \in R_+ = [0,\infty)$*
a)

$$B^{-4}(n)var\left(\sum_{k=1}^{[nt]} Z(k)\right) \to 0; \tag{15}$$

b)

$$B^{-4}(n)\sum_{k=1}^{[nt]} EZ^2(k) \to 0. \tag{16}$$

Proof. To prove Part (a) we consider

$$var\left(\sum_{k=1}^{[nt]} Z(k)\right) = I_1 + I_2,$$

where

$$I_1 = \sum_{k=1}^{[nt]} B^2(k), \quad I_2 = 2\sum_{i=1}^{[nt]-1} \sum_{j=i+1}^{[nt]} cov(Z(i), Z(j)).$$

It is easy to see that

$$B^{-4}(n)I_1 \le \frac{B^2([nt])[nt]}{B^4(n)}.$$

Due to Lemma 1 $B^2(n)$ is regularly varying and

$$B^2(n) \sim \frac{B\alpha(n)n^2}{(\alpha+1)(\alpha+2)} + \frac{n\beta(n)}{\beta+1} \tag{17}$$

as $n \to \infty$ and we obtain that $I_1/B^4(n) \to 0$ as $n \to \infty$. Since

$$cov(Z(t), Z(t+n)) = A^n varZ(t),$$

recalling that $B^2(n) = varZ(n)$, we have

$$I_2 = \sum_{k=1}^{[nt]-1} ([nt]-i)B^2(i) \le B^2([nt])([nt]-1)^2.$$

Again taking into account (17) and that $\alpha(n) \to \infty$, we conclude that $I_2/B^{-4}(n) \to 0$ as $n \to \infty$.

To prove Part (b) we observe that

$$\sum_{k=1}^{[nt]} EZ^2(k) = \sum_{k=1}^{[nt]} varZ(k) + \sum_{k=1}^{[nt]} A^2(k). \tag{18}$$

As it was just proved the first sum is $o(B^4(n))$ as $n \to \infty$. It follows from Lemma 1 and Part (b) of Lemma 3 that

$$\sum_{k=1}^{n} A^2(k) \sim \frac{n}{2\alpha+3}A^2(n), \quad A(n) \sim \frac{n\alpha(n)}{\alpha+1} \tag{19}$$

as $n \to \infty$. From (17) and (19) we obtain that the second sum in (18) is also $o(B^4(n))$ as $n \to \infty$. Lemma 4 is proved.

Lemma 5. *Let X_{ki} be random variables from (1), $\bar{X}_{ki} = X_{ki} - 1$ and $T(k) = \sum_{i=1}^{Z(k-1)} \bar{X}_{ki}$. Then*
a)

$$E[(T(k))^2|\Im(k-1)] = BZ(k-1), \tag{20}$$

b)

$$E[(\Sigma' \bar{X}_{ki}\bar{X}_{kj})^2|\Im(k-1)] = 2B^2 Z(k-1)(Z(k-1)-1), \tag{21}$$

where $\sum'$ *means summation for* $i,j = 1,2,...,Z(k-1)$ *such that* $i \neq j$.

Proof. The relations (20) and (21) are direct consequence of independence of random variables $\bar{X}_{ki}, \bar{X}_{kj}, i \neq j$ and of a simple property of the conditional expectation.

The following technical result is vital in proofs of main theorems.

Lemma 6. *For any* $\theta > 0$

$$E[(T(k))^2\chi(|T(k)| > \theta)|\Im(k-1)] \leq I_1 + I_2, \tag{22}$$

where

$$I_1 = Z(k-1)E[\bar{X}_{11}^2\chi(|\bar{X}_{11}| > \theta/2)], \quad I_2 = \frac{4B^2}{\theta^2}(Z(k-1))^2 + \frac{2^{1/2}B^{3/2}}{\theta}(Z(k-1))^{3/2}.$$

Proof. If we use simple inequality

$$\chi(|\xi+\eta| > \varepsilon) \leq \chi(|\xi| > \varepsilon/2) + \chi(|\eta| > \varepsilon/2), \tag{23}$$

the left side of (22) can be estimated by $R_1 + R_2 + R_3$, with

$$R_1 = E[\sum_{i=1}^{Z(k-1)} \bar{X}_{ki}^2\chi(|\bar{X}_{ki}| > \theta/2)|\Im(k-1)] = Z(k-1)E[\bar{X}_{11}^2\chi(|\bar{X}_{11}| > \theta/2)],$$

$$R_2 = E[\sum_{i=1}^{Z(k-1)} \bar{X}_{ki}^2\chi(|\tau_{ki}| > \theta/2)|\Im(k-1)],$$

$$R_3 = E[\Sigma' \bar{X}_{ki}\bar{X}_{kj}\chi(|T(k)| > \theta)|\Im(k-1)],$$

where Σ' is the same as in Lemma 5 and $\tau_{ki} = T(k) - \bar{X}_{ki}$. To estimate R_2 we use independence of $\bar{X}_{ki}$ and τ_{ki}, Chebishev inequality, relation (20), and obtain

$$R_2 \leq \frac{4B}{\theta^2}\sum_{i=1}^{Z(k-1)} E[\tau_{ki}^2|\Im(k-1)] = \frac{4B^2}{\theta^2}Z(k-1)(Z(k-1)-1)$$

Using Cauchy-Schwarz and Chebishev inequalities and relations (20) and (21) we see that R_3 is dominated by

$$\frac{1}{\theta}(E[(\Sigma'\bar{X}_{ki}\bar{X}_{kj})^2|\Im(k-1)]E[(T(k))^2|\Im(k-1)])^{1/2} = \frac{2^{1/2}B^{3/2}Z(k-1)}{\theta}\sqrt{Z(k-1)-1}$$

Lemma is proved.

4 Proofs of theorems

We represent our process $Y_n(t)$ in the form of a sum of normalized martingale differences and deduce our results from a martingale limit theorem. Let $M(k) = Z(k) - E[Z(k)|\Im(k-1)]$. Then it follows from (1) and (5) that

$$Z(k) - E[Z(k)] = Z(k-1) - E[Z(k-1)] + M(k).$$

Consequent application of this identity leads to relation

$$Y_n(t) = \sum_{k=1}^{[nt]} \frac{M(k)}{B(n)}. \tag{24}$$

Proof of Theorem 1. We use the following version of martingale central limit theorem from [10] (see [10], Theorem VIII, 3.33).

Theorem A. *Let $\{U_k^n, k \geq 1\}$ for each $n \geq 1$ be a sequence of martingale differences with respect to some filtration $\{\Im_k^n, k \geq 1\}$, such that the conditional Lindeberg condition*

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2 \chi(|U_k^n| > \varepsilon)|\Im_{k-1}^n] \xrightarrow{P} 0 \tag{25}$$

holds as $n \to \infty$ for all $\varepsilon > 0$ and $t \in R_+$. Then

$$\sum_{k=1}^{[nt]} U_k^n \xrightarrow{D} U(t) \tag{26}$$

as $n \to \infty$ weakly, where $U(t)$ is a continuous Gaussian martingale with mean zero and covariance function $C(t), t \in R_+$, if and only if

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2|\Im_{k-1}^n] \xrightarrow{P} C(t) \tag{27}$$

as $n \to \infty$ for each $t \in R_+$.

First we prove fulfilment of condition (27) for the sum in (24) with $U_k^n = M(k)/B(n)$ and $\Im_k^n = \Im(k)$ for all $n \geq 1$. Since

$$M(k) = \sum_{i=1}^{Z(k-1)} (X_{ki} - 1) + \xi_k - \alpha(k), \tag{28}$$

we easily obtain that

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2 | \Im_{k-1}^n] = \frac{\sigma^2([nt])}{B^2(n)} + \frac{B}{B^2(n)} \sum_{k=1}^{[nt]} Z(k-1). \tag{29}$$

It follows from Lemma 4 that the variance of the second sum in (29) tends to zero as $n \to \infty$. Therefore we just consider

$$\frac{B}{B^2(n)} \sum_{k=1}^{[nt]} EZ(k-1) = \frac{B}{B^2(n)} \sum_{k=1}^{[nt]} A(k-1).$$

Since $B^2(n) \sim \Delta^2(n)$ as $n \to \infty$ under conditions of Theorem 1, appealing to Lemma 3 we see that the last sum converges in probability to $t^{2+\alpha}$ as $n \to \infty$. Hence condition (27) is satisfied with $C(t) = t^{2+\alpha}$.

In our case the Lindeberg condition will be satisfied, if for each $\varepsilon > 0$ as $n \to \infty$

$$I(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[(M(k))^2 \chi(|M(k)| > \varepsilon B(n)) | \Im(k-1)] \xrightarrow{P} 0. \tag{30}$$

Taking into account (28) and independence of the immigration and reproduction processes we have

$$I(n) = I_1(n) + I_2(n), \tag{31}$$

where

$$I_1(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[(T(k))^2 \chi(|M(k)| > \varepsilon B(n)) | \Im(k-1)]$$

$$I_2(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[(\xi_k - \alpha(k))^2 \chi(|M(k)| > \varepsilon B(n)) | \Im(k-1)].$$

and $T(k)$ is defined in Lemma 5.

Consider $I_1(n)$. Using inequality (23) we see that it can be estimated by sum $I_{11}(n) + I_{12}(n)$, where

$$I_{11}(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[(T(k))^2 \chi(|T(k)| > \frac{\varepsilon B(n)}{2}) | \Im(k-1)],$$

$$I_{12}(n) = \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[(T(k))^2 \chi(|\xi_k - \alpha(k)| > \frac{\varepsilon B(n)}{2}) | \Im(k-1)].$$

Due to Lemma 6 we obtain that

$$I_{11}(n) \leq \frac{\Upsilon(\frac{\varepsilon B(n)}{4})}{B^2(n)} \sum_{k=1}^{[nt]} Z(k-1) + \frac{16B^2}{\varepsilon^2 B^4(n)} \sum_{k=1}^{[nt]} (Z(k-1))^2 + \frac{2B}{\varepsilon B^3(n)} \sum_{k=1}^{[nt]} (Z(k-1))^{3/2},$$

where $\Upsilon_n(\varepsilon) = E[\bar{X}_{11}^2 \chi(|\bar{X}_{11}| > \varepsilon)]$. As it was proved, the sum in the first term divided by $B^2(n)$ as $n \to \infty$ converges in probability to $t^{2+\alpha}$. The second term as $n \to \infty$ converges in probability to zero due to (16). To estimate the last term we use inequality

$$\sum_{k=1}^{n} a_k b_k \leq \sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2} \tag{32}$$

and obtain that it is not greater than

$$\frac{2B}{\varepsilon} \sqrt{\frac{1}{B^4(n)} \sum_{k=1}^{[nt]} (Z(k-1))^2} \sqrt{\frac{1}{B^2(n)} \sum_{k=1}^{[nt]} Z(k-1)}$$

and therefore also converges to zero in probability as $n \to \infty$. Hence we proved that $I_{11}(n) \xrightarrow{P} 0$ as $n \to \infty$ for any $\varepsilon > 0$.

Consider $I_{12}(n)$. Using independence of the immigration and reproduction processes, Chebishev inequality and (20), we have

$$I_{12}(n) \leq \frac{4B}{\varepsilon^2 B^4(n)} \sum_{k=1}^{[nt]} Z(k-1)\beta(k).$$

Taking into account inequality

$$\sum_{k=1}^{[nt]} EZ(k-1)\beta(k) \leq A([nt])\sigma^2([nt]) \tag{33}$$

we derive that $I_{12}(n) \xrightarrow{P} 0$ as $n \to \infty$.

To estimate $I_2(n)$ we take into account that $I_2(n) \le \sigma^2([nt])/B^2(n)$ and due to condition $\sigma^2(n) = o(\Delta^2(n))$ conclude that $I_2(n) \xrightarrow{P} 0$ as $n \to \infty$. Theorem is proved.

Proof of Theorem 2. We again prove fulfilment of conditions of Theorem A. Since $B^2(n) \sim \sigma^2(n)$ as $n \to \infty$ under conditions of Theorem 2, first term in (29) tends to $t^{1+\beta}$ as $n \to \infty$. Due to Lemma 3

$$\sum_{k=1}^{[nt]} EZ(k-1) \sim \frac{constant}{n}\Delta^2(n) \tag{34}$$

as $n \to \infty$ and appealing to Lemma 4 we derive that the second term in (29) converges to zero in probability as $n \to \infty$. Hence condition (27) is satisfied with $C(t) = t^{1+\beta}$.

In the proof of fulfilment of Lindeberg condition, convergence to zero in probability of term $I_1(n)$ in (31) will remain true under the conditions of Theorem 2. Consider $I_2(n)$. Using inequality (23) we obtain that

$$I_2(n) \le \delta_n(\frac{\varepsilon}{2}) + \frac{1}{B^2(n)} \sum_{k=1}^{[nt]} \beta(k) P\{|T(k)| > \frac{\varepsilon B(n)}{2} | \Im(k-1)\}. \tag{35}$$

It follows from Chebishev inequality and (20) that the second term in (35) is dominated by

$$\frac{4B}{\varepsilon^2 B^4(n)} \sum_{k=1}^{n} \beta(k) Z(k-1).$$

Using (33) one can prove that the second term converges to zero in probability as $n \to \infty$. Consequently $I_2(n) \xrightarrow{P} 0$ as $n \to \infty$ for each $\varepsilon > 0$ under conditions of Theorem 2. Theorem is proved.

Proof of Theorem 3. Under conditions of Theorem 3 first term in (29) tends to $\theta t^{1+\beta}$. Using (34), Lemma 4 and taking into account Remark 1, we obtain that the second term converges to $(1-\theta)t^{2+\alpha}$ in probability as $n \to \infty$. Consequently condition (27) is satisfied with $C(t) = \theta t^{1+\beta} + (1-\theta)t^{2+\alpha} = t^{2+\alpha}$.

The fulfilment of the Lindeberg condition can be proved similarly as in the proof of previous theorems. Therefore its proof is omitted. The assertion of Theorem 3 again follows from Theorem A.

Proof of Theorem 4. We obtain from (6) that

$$\hat{A}_n - 1 = \frac{\sum_{k=1}^n Z(k-1)M(k)}{\sum_{k=1}^n Z^2(k-1)} =: \frac{D(n)}{Q(n)}. \tag{36}$$

Rewrite $D(n)$ as $D(n) = D_1(n) + D_2(n)$, where

$$D_1(n) = \sum_{k=2}^n \sum_{i=1}^{k-1} M(i)M(k), \quad D_2(n) = \sum_{k=2}^n A(k-1)M(k).$$

Consider $D_1(n)$. Since

$$D_1(n) = \sum_{k=1}^n \sum_{i=1}^{k} M(i)M(k) - \sum_{k=1}^n M^2(k),$$

using simple identity

$$(\sum_{k=1}^n M(k))^2 = D_1(n) + \sum_{k=1}^n \sum_{i=1}^{k} M(i)M(k),$$

we obtain that

$$D_1(n) = \frac{B^2(n)}{2} Y_n^2(1) - \frac{1}{2} \sum_{k=1}^n M^2(k). \tag{37}$$

It was shown in the proof of Theorem 1 that

$$\frac{1}{B^2(n)} \sum_{k=1}^{[nt]} E[M^2(k)|\Im(k-1)] \xrightarrow{P} t^{2+\alpha}$$

as $n \to \infty$ for each $t \in R_+$. Using this and that $B^2(n) \sim \Delta^2(n), A(n) \sim C\Delta(n)\sqrt{\alpha(n)}$ as $n \to \infty$, where C is a positive constant, one can prove

$$\frac{1}{K(n)} \sum_{k=1}^n M^2(k) \xrightarrow{P} 0 \tag{38}$$

as $n \to \infty$, where $K(n) = A(n)\Delta(n)$. It follows from Theorem 1 that $Y_n^2(1) \xrightarrow{D} W^2(1)$ as $n \to \infty$. Taking this into account in (37) we have that as $n \to \infty$

$$\frac{1}{K(n)} D_1(n) \xrightarrow{P} 0. \tag{39}$$

Now we consider $D_2(n)$. It is not difficult to see that

$$D_2(n) = \sum_{k=2}^{n} \sum_{i=1}^{k-1} \alpha(i) M(k) = \sum_{i=1}^{n-1} \alpha(i) \sum_{k=i+1}^{n} M(k).$$

Therefore, taking into account (24), it can be written as

$$\frac{1}{K(n)} D_2(n) = \int_0^1 [Y_n(1) - Y_n(t)] dA_n(t), \tag{40}$$

where $A_n(t) = A([nt])/A(n), n \geq 1$, are non-decreasing functions of t.

Now we consider sequence of functionals $\Psi_n : D(R_+, R) \mapsto R, n \geq 1$ defined by

$$\Psi_n(x) = \int_0^1 [x(1) - x(t)] dA_n(t).$$

Since $A_n(t) \to t^{1+\alpha}$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$, for all $x, x_n \in D(R_+, R)$ such that $\sup |x_n - x| \to 0, n \to \infty$ we have $|\Psi_n(x_n) - \Psi(x)| \to 0$ as $n \to \infty$, where

$$\Psi(x) = (1 + \alpha) \int_0^1 [x(1) - x(t)] t^\alpha dt.$$

It follows from Theorem 1 and Lemma 4.1 in [7] that $\Psi_n(Y_n) \xrightarrow{D} \Psi(W(t^{2+\alpha}))$ as $n \to \infty$. Hence we conclude that

$$\frac{1}{K(n)} D_2(n) \xrightarrow{D} \eta \tag{41}$$

as $n \to \infty$, where

$$\eta = W(1) - (1 + \alpha) \int_0^1 W(t^{2+\alpha}) t^\alpha dt.$$

Now we consider $Q(n)$. It can be written as $Q(n) = Q_1(n) + 2Q_2(n) + Q_3(n)$, where

$$Q_1(n) = \sum_{k=1}^{n} A^2(k-1), \quad Q_2(n) = \sum_{k=1}^{n} A(k-1)(Z(k-1) - A(k-1)),$$

$$Q_3(n) = \sum_{k=1}^{n} (Z(k-1) - A(k-1))^2.$$

It follows from Lemma 3 that

$$\lim_{n\to\infty} \frac{Q_1(n)}{nA^2(n)} = (2\alpha+3)^{-1}. \tag{42}$$

To estimate $Q_2(n)$ we consider

$$\frac{Q_2(n)}{nK(n)} = \sum_{k=1}^{n} A_n(\frac{k}{n}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} Y_n(t)dt.$$

Now we define functionals $\Phi_n : D(R_+, R) \mapsto R, n \geq 1$ by

$$\Phi_n(x) = \sum_{k=1}^{n-1} A_n(\frac{k}{n}) \int_{\frac{k}{n}}^{\frac{k+1}{n}} x(t)dt.$$

It is easy to see that for any $x, x_n \in D(R_+, R)$ such that $\sup |x_n - x| \to 0, n \to \infty$ we have $|\Phi_n(x_n) - \Phi(x)| \to 0$ as $n \to \infty$, where

$$\Phi(x) = \int_0^1 t^{1+\alpha} x(t)dt.$$

We have

$$\frac{Q_2(n)}{nK(n)} = \int_0^1 A_n(t)Y_n(t)dt = \Psi_n(Y_n),$$

where $\Psi_n(x) := \int_0^1 A_n(t)x(t)dt$. Therefore again using Theorem 1 and Lemma 4.1 in [7] we conclude that

$$\frac{Q_2(n)}{nK(n)} \xrightarrow{D} \int_0^1 t^{1+\alpha} W(t^{2+\alpha})dt. \tag{43}$$

Since $A^2(n)/K(n) \sim \sqrt{(2+\alpha)\alpha(n)}/\sqrt{B(1+\alpha)}$ as $n \to \infty$ we have

$$\frac{Q_2(n)}{nA^2(n)} \xrightarrow{P} 0 \tag{44}$$

as $n \to \infty$. Now we consider

$$\frac{Q_3(n)}{nB^2(n)} = \int_0^1 Y_n^2(t)dt.$$

It follows from Theorem 1 and continuous mapping theorem that right side of the last equality as $n \to \infty$ converges to $\int_0^1 W^2(t^{2+\alpha})dt$ in distribution. Therefore, taking into account (17) and the second relation in (19), we conclude that

$$\frac{Q_3(n)}{nA^2(n)} \xrightarrow{P} 0 \tag{45}$$

as $n \to \infty$.

Now we obtain from relations (36), (39), (41) and (42)-(45) that as $n \to \infty$

$$n\sqrt{\alpha(n)}(\hat{A}_n - 1) \xrightarrow{D} (2\alpha + 3)\sqrt{\frac{B(1+\alpha)}{2+\alpha}}\eta. \tag{46}$$

Since η can be written as $\eta = (1+\alpha)\int_0^1 t^\alpha[W(1) - W(t^{2+\alpha})]dt$, we have

$$E\eta^2 = \int_0^1 \int_0^1 s^\alpha t^\alpha R(t,s)dsdt,$$

where

$$R(t,s) = (1+\alpha)^2 E[(W(1) - W(s^{2+\alpha}))(W(1) - W(t^{2+\alpha}))].$$

We consider

$$E\eta^2 = \int_0^1 \int_0^t s^\alpha t^\alpha R(t,s)dsdt + \int_0^1 \int_t^1 s^\alpha t^\alpha R(t,s)dsdt. \tag{47}$$

By a standard technique we obtain that the first term on the right of (47) is equal to

$$\int_0^1 \int_0^t s^\alpha t^\alpha E[(W(1) - W(t^{2+\alpha}))^2]dsdt = \frac{\alpha+2}{2(3\alpha+4)}.$$

In a similar way we derive that the second term on the right of (47) is also equal to $(\alpha+2)/(2(3\alpha+4))$. Hence we have

$$E\eta^2 = \frac{\alpha+2}{3\alpha+4},$$

which implies the desired result. Theorem 4 is proved.

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