Controlled branching processes with continuous states

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Abstract

The controlled branching process with continuous space of states and time-dependent immigration introduced by Adke and Gadag (1995) is considered. The control function is a process with independent stationary increments. Theorems allowing to obtain limit theorems for this model from those of simple branching processes and vice versa are proved. Applying these results, limit distributions are obtained for critical processes in the case of decreasing and increasing rate of immigration when offspring distribution has infinite variance.

Key Words: counting process, branching process, time-dependent immigration, independent increment.

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1 Introduction

We consider a modification of the branching stochastic process which has a continuous space of states. It is convenient to define the process as a family of nonnegative random variables describing the amount of a product produced by individuals of some population. The initial state of the process is given by a nonnegative random variable $X(0)$. The amount of the product $X(1)$ of the first generation is defined as the sum of random products produced by $N_1(X(0))$ individuals and the product U_1 of immigrating to the first generation individuals. Similarly the amount $X(2)$ of the product of the second generation is defined as the sum of products produced by $N_2(X(1))$ individuals and U_2 , and so on. Here $N_k(t), k \geq 1, t \in T$, are counting processes with independent stationary increments, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, 2, ...\}$ and $U_k, k \geq 1$, are non-negative random variables. We also assume that processes $N_n(t), n \geq 1, t \in T$ have common one dimensional distributions. This modification of branching processes was considered by Adke and Gadag (1995).

The interest to this modification of the branching processes is connected with possibility of unified generating various non-Gaussian Markov time series models, as it was demonstrated in Adke and Gadag (1995). On the other hand described above process allow to model situations, when it is difficult to count the number of individuals in the population, but some non-negative characteristic, such as volume, weight or product produced by the individuals can be measured.

In the case when $X(0)$ and $U_k, k \geq 1$, are integer-valued, the process $X(n)$ can be considered as a special case of a controlled branching process introduced first in Sevastyanov and Zubkov (1974) and for random control functions in Yanev (1975). If we choose $\varphi_1(k, n) = N_k(n)$ and $\varphi_2(k, n) \equiv 1$ in so called Model 2 of φ- branching process, obtain a discrete-state version of the process $X(n)$. Further investigations of controlled branching processes with random control functions can be found in Gonzalez *et al.* (2005) and the references therein.

Using of counting processes $N_k(t), k \geq 1$ with independent and stationary increments in the definition of the process allows to obtain distributional properties of the process $X(n)$ that are similar to those of classic models. In particular, it was shown that $Z(n) = N_{n+1}(X(n))$ is usual $Bienaymé - Galton - Watson$ (BGW) process with immigration. The following question is interesting in connection with this situation. Is it possible to use this similarity in investigation of asymptotic behavior of the process? In particular can we obtain limit distributions of $X(n)$ directly from known limit theorems for BGW processes?

In this paper we prove certain theorems which establish relationship between these two processes in a sense of asymptotic behavior. These results allow to get limit theorems for $X(n)$ from those of $Z(n)$ and vice versa. We also demonstrate possibilities of these theorems in describing of the spectrum of limit distributions for critical processes $X(n)$, in the case of time-dependent immigration and infinite variance of offspring distribution. It will be seen later that these duality theorems are applicable to subcritical and supercritical processes and to the processes without immigration.

In Section 2 the duality theorems establishing relationship between processes $X(n)$ and $Z(n)$ are proved. Section 3 contains results on the first and second moments and the Laplace transform of the continuous-state process, which will be used in Sections 5 and 6. Section 4 is devoted to a study of the non-extinction probability of the process. The results of this section allow to verify conditions of the duality theorems in applications, though they of an independent interest as well. Usefulness of theorems of Section 2 in the case of a linear normalization, which is the case of the stationary immigration, is obvious. Therefore in Section 5 we demonstrate the applicability of those theorems in the case of a functional normalization, which appears in the process with decreasing immigration. In Section 6 we extend results on the convergence to infinitely divisible and stable distributions of the process with increasing immigration.

2 Three preliminary theorems

Let $\{W_{ni}, i, n \geq 1\}$ be a double array of independent and identically distributed non-negative random variables, $\{N_n(t), t \in T, n \geq 1\}$ be a family of nonnegative, integer-valued independent processes with independent stationary increments, with $N_n(0) = 0$ almost surely, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, ...\}$.

We define process $X(n), n \geq 0$, as following. Let the initial state of the process be $X(0)$ which is an arbitrary non-negative random variable and for

$n \geq 0$

$$X(n+1) = \sum_{i=1}^{N_{n+1}(X(n))} W_{n+1i} + U_{n+1}, \tag{1}$$

where $\{U_n, n \geq 1\}$ is a sequence of independent non-negative random variables not necessarily identically distributed. Assume that families of random variables $\{W_{ni}, i, n \geq 1\}, \{U_n, n \geq 1\}$, sequence of stochastic processes $\{N_n(t), t \in T, n \geq 1\}$ and random variable $X(0)$ are independent.

Now we provide first result establishing relationship between processes $X(n)$ and $Z(n)$ in a sense of limiting behavior. In order to do that we use the following Laplace transforms

$$G(\lambda) = Ee^{-\lambda W_{ni}}, H_n(\lambda) = Ee^{-\lambda U_n}.$$

We also denote ratios

$$\Delta(n) = \frac{P\{Z(n) > 0\}}{P\{X(n) > 0\}},\ \delta(n, \lambda) = \frac{1 - H_n(\lambda)}{P\{Z(n) > 0\}}.$$

Let the sequences of positive numbers $\{k(n), n \geq 1\}$ and $\{a(n), n \geq 1\}$ be such that $k(n), a(n) \to \infty$ and for each $\lambda > 0$ there exists $0 < b(\lambda) < \infty$ such that

$$\lim_{n\to\infty} k(n)(1 - G(\frac{\lambda}{a(n)})) = b(\lambda). \tag{2}$$

Existence of these sequences follows from monotonicity of the Laplace transform $G(\lambda)$. In fact one may choose

$$a(n) = \frac{\lambda}{G^{-1}(1 - \frac{b(\lambda)}{k(n)})}$$

for a given sequence $k(n)$, where G^{-1} stands for the inverse of $G(\lambda)$.

Theorem 2.1. *Let $\varphi(\lambda)$ be a Laplace transform, $\Delta(n) \to 1, n \to \infty$ and $\delta(n, \lambda/a(n)) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} E[e^{-\lambda X(n)/a(n)} | X(n) > 0] = \varphi(b(\lambda)) \tag{3}$$

for $\lambda > 0$, if and only if for each $\lambda > 0$

$$\lim_{n\to\infty} E[e^{-\lambda Z(n)/k(n)} | Z(n) > 0] = \varphi(\lambda). \tag{4}$$

Remark 1. Condition $\Delta(n) \to 1$ as $n \to \infty$ is trivially satisfied in the case of stationary or increasing ($EU_n \to \infty, n \to \infty$) immigration, because in this case the "non-extinction" probabilities of $X(n)$ and $Z(n)$ approach 1 as $n \to \infty$. In the case of decreasing immigration one may expect of its satisfaction in some natural assumptions on distributions of W_{ni} and U_n. For the supercritical process without immigration it may hold only when the extinction probabilities of $X(n)$ and $Z(n)$ are equal. The last may hold, for example, if $P\{W_{ni} = 0\} = 0$ (see Adke and Gadag (1995)).

Remark 2. Condition $\delta(n, \lambda_n) \to 0$ as $n \to \infty$ is trivially fulfilled for any sequence $\lambda_n, n \geq 1$ such that $\lambda_n \to 0$ as $n \to \infty$ in all cases, when the extinction probability of $Z(n)$ is less than one or the process does not allow immigration. For processes which become extinct it may hold under some restrictions on the rate of EU_n as $n \to \infty$.

Proof. We consider the following obvious identity

$$E[e^{-\lambda X(n)}|X(n) > 0] = 1 - \frac{1 - Ee^{-\lambda X(n)}}{P(X(n) > 0)}. \tag{5}$$

It follows from definition (1) of the process $X(n)$ by total probability arguments that

$$Ee^{-\lambda X(n)} = H_n(\lambda) EG^{Z(n)}(\lambda). \tag{6}$$

We obtain from (6) that

$$1 - Ee^{-\lambda X(n)} = (1 - H_n(\lambda)) EG^{Z(n)}(\lambda) + 1 - EG^{Z(n)}(\lambda).$$

Thus

$$\frac{1 - Ee^{-\lambda X(n)}}{P(Z(n) > 0)} = 1 - E[G^{Z(n)}(\lambda)|Z(n) > 0] + \delta(n, \lambda) E[G^{Z(n)}(\lambda)].$$

Hence the ratio on the right side of (5) equals

$$\Delta(n) \frac{1 - Ee^{-\lambda X(n)}}{P(Z(n) > 0)} = -\Delta(n) E[G^{Z(n)}(\lambda)|Z(n) > 0] + \Delta(n)[1 + \delta(n, \lambda) EG^{Z(n)}(\lambda)].$$

If we use this in relation (5) we obtain

$$E[e^{-\lambda X(n)}|X(n)>0]=\Delta(n)E[G^{Z(n)}(\lambda)|Z(n)>0]+\varepsilon(n), \tag{7}$$

where

$$\varepsilon(n)=1-\Delta(n)(1+\delta(n,\lambda)E[G^{Z(n)}(\lambda)]).$$

Let (4) be satisfied for every $\lambda>0$. Then, it clearly follows from continuity of the Laplace transform $\varphi(\lambda)$, that the convergence in (4) holds uniformly with respect to λ from arbitrary bounded interval. Since $\ln x = -(1-x)+o(1-x), x\to 1$, we obtain from condition (2) that as $n\to\infty$

$$t_n =: -k(n)\ln G(\frac{\lambda}{a(n)})\to b(\lambda). \tag{8}$$

Therefore for each fixed $\lambda>0$ there is such a $T=T(\lambda)$, with $0<t_n\leq T$ for any $n=1,2,...$ Now we consider (7) replacing λ by $\lambda/a(n)$. It follows from the definition of t_n that

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n)>0]=E[e^{-t_n Z(n)/k(n)}|Z(n)>0]. \tag{9}$$

We show that the Laplace transform (9) as $n\to\infty$ approaches $\varphi(b(\lambda))$. In order to do it we consider the following relation:

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n)>0]-\varphi(b(\lambda))=I_1+I_2, \tag{10}$$

where

$$I_1=E[e^{-t_n Z(n)/k(n)}|Z(n)>0]-\varphi(t_n), I_2=\varphi(t_n)-\varphi(b(\lambda)).$$

It follows from (4), due to the uniform convergence, that

$$|I_1|\leq \sup_{0<t_n<T}|E[e^{-t_n Z(n)/k(n)}|Z(n)>0]-\varphi(t_n)|\to 0 \tag{11}$$

as $n\to\infty$. On the other hand $I_2\to 0$ as $n\to\infty$ due to continuity of the Laplace transform $\varphi(\lambda)$, for $\lambda>0$. Thus we conclude that as $n\to\infty$

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n)>0]\to\varphi(b(\lambda)). \tag{12}$$

Since $\Delta(n) \to 1$ and $\delta(n, \frac{\lambda}{a(n)}) \to 0$ as $n \to \infty$, we obtain that $\varepsilon(n) \to 0$ as $n \to \infty$. The assertion (3) now follows from relations (7) and (12). The first part of Theorem 2.1 is proved.

Let now (3) be satisfied. It follows from condition (2) that $\tau_n = t_n/b(\lambda) \to 1$ as $n \to \infty$ for each $\lambda > 0$ (recall that $t_n = -k(n)\ln G(\lambda/a(n))$). We consider the following Laplace transform:

$$E[e^{-Z(n)b(\lambda)\tau_n/k(n)}|Z(n) > 0] = E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n) > 0]. \qquad (13)$$

It follows from relations (3), (7) and (13), due to continuity of $\varphi(\lambda)$, that

$$\lim_{n\to\infty} E[e^{-Z(n)b(\lambda)\tau_n/k(n)}|Z(n) > 0] = \varphi(b(\lambda)). \qquad (14)$$

Due to continuity theorem for Laplace transforms (14) means that

$$\left\{\frac{Z(n)\tau_n}{k(n)}|Z(n) > 0\right\} \xrightarrow{\mathrm{d}} \xi$$

as $n \to \infty$, with $Ee^{-\lambda\xi} = \varphi(\lambda)$. Since $\tau_n \to 1, n \to \infty$, we have that $Z(n)/k(n)$ given $Z(n) > 0$, as $n \to \infty$ converges to ξ in distribution. If we write this in terms of Laplace transforms, we get assertion of (4). Theorem 2.1 is proved completely.

Next theorem relates to the situation when the limit distribution of $Z(n)$ is discrete.

Theorem 2.2. *Let $\varphi(\lambda)$ be a Laplace transform of a discrete random variable, $\Delta(n) \to 1$ and $\delta(n, \lambda) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} E[e^{-\lambda X(n)}|X(n) > 0] = \varphi(-\log(G(\lambda))) \qquad (15)$$

for each $\lambda > 0$, if and only if for each $u > 0$

$$\lim_{n\to\infty} E[e^{-uZ(n)}|Z(n) > 0] = \varphi(u). \qquad (16)$$

Proof. Let (16) be satisfied. Making substitution $u = -\log G(\lambda), \lambda > 0$, we get that as $n \to \infty$

$$E[G^{Z(n)}(\lambda)|Z(n) > 0] \to \varphi(-\log(G(\lambda))).$$

Taking this into account in relation (7) we obtain first part of the theorem.

Let now (15) holds. Then it follows from relation (7) that as $n \to \infty$

$$E[e^{Z(n)\log G(\lambda)}|Z(n) > 0] \to \varphi(-\log(G(\lambda)))$$

which shows (16) by the same substitution $u = -\log G(\lambda)$. Theorem 2.2 is proved.

Now we obtain a similar duality result for unconditional distributions of processes $Z(n)$ and $X(n)$. It will also be formulated in terms of Laplace transforms.

Theorem 2.3. *Let $\varphi(\lambda)$ be a Laplace transform and for sequences $\{a(n), n \geq 1\}$ and $\{k(n), n \geq 1\}$ condition (2) be satisfied. Then*

$$\lim_{n\to\infty} Ee^{-\lambda X(n)/a(n)} = \varphi(b(\lambda)) \qquad (17)$$

if and only if for each $\lambda > 0$

$$\lim_{n\to\infty} Ee^{-\lambda Z(n)/k(n)} = \varphi(\lambda). \qquad (18)$$

Proof. Now we use equation (6) directly. Let (18) be satisfied for each $\lambda > 0$. Then it holds uniformly with respect to $\lambda > 0$ from each bounded interval. Again taking into account relation (9) we present difference $E[G^{Z(n)}(\lambda/a(n))]-\varphi(b(\lambda))$ as $I_1 + I_2$ and, as in the proof of Theorem 1, show that both I_1 and I_2 approach to zero as $n \to \infty$. This leads assertion of (17) due to relation (6).

The proof of the necessity of (18) for (17) is similar to the proof of the second part of previous theorem. One just needs to consider unconditional Laplace transforms instead of conditional ones. Theorem 2.3 is proved.

3 Moments and regularly varying tails

The offspring distribution and the distribution of the number of immigrating "individuals" have Laplace transforms $G(f(\lambda)) = Ee^{-\lambda\xi_n}$ and $H_n(f(\lambda)) = Ee^{-\lambda\eta_n}$, respectively (Adke and Gadag (1995)). Here $\xi_n = N_n(W_{n-11}), \eta_n = N_n(U_{n-1})$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

We obtain the moments of offspring and immigration distributions by standard arguments. It is easy to see that

$$m = E\xi_n = -\frac{d}{d\lambda}G(f(\lambda))_{\lambda=0} = EWEN,$$

where $N = N_1(1), W = W_{11}$ and

$$\alpha(n) = E\eta_n = -\frac{d}{d\lambda}H_n(f(\lambda))_{\lambda=0} = EU_nEN.$$

By similar arguments we obtain that

$$E\eta_n^2 = EU_n varN + EU_n^2(EN)^2$$

and for the factorial moment $\beta(n) = E\eta_n(\eta_n - 1)$ we have

$$\beta(n) = (EN)^2 EU_n(U_n - 1) + EN(N-1)EU_n.$$

We assume that Laplace transforms of random variables W and N can be represented in the form

$$Ee^{-\lambda W} = e^{-a\lambda} + (1 - e^{-a\lambda})^{1+\alpha} L_\alpha(1 - e^{-\lambda}), \tag{19}$$

and

$$Ee^{-\lambda N} = e^{-b\lambda} + (1 - e^{-b\lambda})^{1+\beta} L_\beta(1 - e^{-\lambda}), \tag{20}$$

where a, b are fixed positive numbers $0 < \alpha, \beta \leq 1, L_\alpha(s)$ and $L_\beta(s)$ are slowly varying functions as $s \uparrow 1$. It is not difficult to see that in this case $EW = a$ and $EN = b$ are finite but second moments may not be finite. Note that in the case of finite variances relations (19) and (20) are satisfied with $\alpha = \beta = 1$ and $L_\alpha(s)$ and $L_\beta(s)$ having finite limits.

Proposition. *If (19) and (20) are satisfied and $ab = 1$, then $Z(n)$ is critical and the offspring distribution has Laplace transform*

$$G(f(\lambda)) = e^{-\lambda} + (1 - e^{-\lambda})^{1+\theta} L(1 - e^{-\lambda}), \tag{21}$$

where $\theta = \min(\alpha, \beta)$ and $L(x)$ is slowly varying function such that

$$L(x) \sim \begin{cases} L_\alpha(x), & \text{if } \alpha < \beta \\ L_\beta(x)b^\beta, & \text{if } \alpha > \beta \\ L_\alpha(x) + L_\beta(x)b^\beta, & \text{if } \alpha = \beta \end{cases}$$

for $0 < \alpha, \beta < 1$ *and*

$$L(x) \sim L_\alpha(x) + bL_\beta(x) + \frac{b-1}{2}$$

for $\alpha = \beta = 1$.

Proof. We obtain the proof of the proposition easily using Taylor expansion of function $(1-x)^a$ and simple properties of slowly varying functions.

From now on we assume throughout that (21) is satisfied with $0 < \theta \leq 1$ and with some slowly varying function $L(x)$. We define by $V(n)$ a BGW process with offspring distribution defined by Laplace transform $G(f(\lambda))$. It is known (Harris (1966)) that, if $0 < G(f(\infty)) < 1$, then process $V(n)$ has a stationary measure $\{\mu_k, k \geq 1\}$ whose generating function $U(s)$ is analytic in the disk $|s| < q$, where q is the extinction probability, and satisfies Abel's equation

$$U(G(f(-\log s))) = 1 + U(s) \tag{22}$$

with initial condition $U(G(f(\infty))) = 1, U(0) = 0, U(1) = \infty$.

If $G(f(\lambda))$ satisfies (21), then it is not difficult to see (Slack(1968)), that

$$U(s) = \frac{1 + o(1)}{\theta(1-s)^\theta L(1-s)}, s \uparrow 1 \tag{23}$$

solves equation (22). On the other hand $U(1-s)$ is invertible and its inverse $g(x), x > 0$, has the form

$$g(x) = \frac{M(x)}{x^{1/\theta}}, \tag{24}$$

where $M(x)$ varies slowly at infinity and $\theta M^\theta(x)L(g(x)) \to 1$ as $x \to \infty$.

4 The probability of non-extinction

In the case of stationary immigration (fixed environment) $P(X(n) > 0)$ approaches 1 as $n \to \infty$. However, if the immigration rate depends on the environment, this probability may approach any number in [0, 1] including 0 and 1. Moreover, the asymptotic behavior of the process strongly depends on the behavior of this probability. Here we provide some results for $P(X(n) > 0)$ in the case when the immigration rate approaches zero as $n \to \infty$. It turns out that asymptotic behavior of this probability depends on partial sum $d(n) = \sum_{k=0}^{n} P\{V(k) > 0\}$.

We assume that $\alpha(n) < \infty, \beta(n) < \infty$ for each $n \geq 1, \alpha(n)$ varies regularly at infinity and as $n \to \infty$

$$P\{U_n > 0\} = O(EU_n) \tag{25}$$

Theorem 4.1. *Let (21) and (25) be satisfied and $\alpha(n) \to 0, n \to \infty$.*
a) If $\alpha(n)d(n) \to \infty$, then $P\{X(n) > 0\} \to 1$;
b) If $\alpha(n)d(n) \to C \in (0, \infty)$, then $P\{X(n) > 0\} \to 1 - e^{-C}$;
c) If $\alpha(n)d(n) \to 0$, then $P\{X(n) > 0\} \to 0$.

Remark. If $U_n, n \geq 1$ takes nonnegative integer values, condition (25) is obviously satisfied. In general (25) may hold, for instance, if distribution of U_n has an atom at zero which seems natural in the case of vanishing immigration. Let, for example, $U_n, n \geq 1$ has the following cumulative distribution function

$$P\{U_n \leq x\} = \frac{a_n + 1 - e^{-x/b_n}}{1 + a_n}, x \geq 0,$$

where a_n and b_n are some positive numbers. We see that in this case $P\{U_n > 0\} = (1 + a_n)^{-1}$ and $EU_n = b_n(1 + a_n)^{-1}$ and condition (25) is satisfied, if $\liminf_{n\to\infty} b_n > 0$.

Proof of Theorem 4.1. Putting $\lambda \to \infty$ in relation (6), we obtain equation

$$P\{X(n) = 0\} = P\{U_n = 0\}\Psi(n, P_0),$$

where $P_0 = P\{W = 0\}$ and $\Psi(n, s) = Es^{Z(n)}, 0 \leq s \leq 1$. If $P_0 = 0$, it is clear that $P\{X(n) = 0\} \sim P\{Z(n) = 0\}, n \to \infty$, when $\alpha(n) \to 0$. Assume that $0 < P_0 < 1$. Let $f^*(s) = G(f(-\log s))$ be generating function of offspring distribution of $V(n)$ and $f_n^*(s)$ its nth functional iteration. It is known that, if (21) is satisfied (see Rahimov 1995, p. 107, for example), then

$$1 - f_n^*(s) = g(n + U(s)). \tag{26}$$

Since $g(x)$ is regularly varying as $x \to \infty$ we obtain that $1 - f_n^*(P_0) \sim 1 - f_n^*(0), n \to \infty$ for each $0 < P_0 < 1$ (recall that $U(0) = 0$). Using this fact we obtain by standard analysis that $\Psi(n, P_0) \sim \Psi(n, 0), n \to \infty$ and consequently we again have $P\{X(n)=0\} \sim P\{Z(n)=0\}$ as $n \to \infty$. The assertion of Theorem 4.1 now follows from Theorem 1.1 in Rahimov (1986), where asymptotic behavior of the last probability is studied.

Now we provide a result which gives decreasing rate of the non extinction probability when $\alpha(n)d(n) \to 0$. We denote $Q_1(n) = \alpha(n)d(n)$ and $Q_2(n) = P\{V(n) > 0\} \sum_{k=1}^{n} \alpha(k)$.

Theorem 4.2. *If (21) and (25) are satisfied,* $Q_1(n) \to 0, d(n) \to \infty$ *and* $\beta(n) = o(Q_1(n) + Q_2(n))$ *then as* $n \to \infty$

$$P\{X(n) > 0\} \sim Q_1(n) + Q_2(n) \tag{27}$$

Proof. We obtain using (6) that

$$P(X(n) > 0) = 1 - \Psi(n, P_0) + P(U_n > 0)\Psi(n, P_0). \tag{28}$$

By the same arguments as in the proof of previous theorem we have that $1 - \Psi(n, P_0) \sim 1 - \Psi(n, 0) = P\{Z(n) > 0\}$ as $n \to \infty$ for each $0 \le P_0 < 1$. If conditions of Theorem 4.2 are fulfilled , then $P\{Z(n) > 0\} \sim Q_1(n) + Q_2(n), n \to \infty$, due to Theorem 1.2 in Rahimov (1986). Consequently, it is sufficient to show that last summand on the right side of (28) is $o(Q_1(n) + Q_2(n))$. Since $P\{U_n > 0\} = O(\alpha(n))$ and $d(n) \to \infty$, we obtain that $P\{U_n > 0\} = o(Q_1(n))$ which means that the last assertion holds. Theorem is proved.

Results of this section will further be used when we apply theorems from Section 2 to obtain limit distributions for process $X(n)$. However these results are of independent interest as well. For instance Theorem 4.2 shows that event $\{X(n) > 0\}$ may occur, roughly speaking, either because of descendants of "recent immigrants" or because of the individuals immigrated in the beginning of the process.

5 Limit theorems

Here we show how limit theorems for $X(n)$ can be deduced from those of $Z(n)$ in the case of functional normalization. We use the following normalizing functions:

$$T(x) = \exp\{\int_0^x g(u)du\},\ \Omega(x) = T(U(1 - x^{-1})).$$

As it was noted before, relation (21) is satisfied in the case of finite variance, if $\theta = 1$ and $L(s) \to C_1 > 0, s \uparrow 1$. We exclude here the situation of $C_1 = 0$, as in this case the offspring variance is zero. From here it follows that $M(x)$ from equality (24) has a finite limit $C_2 \geq 0$ as $x \to \infty$. Therefore $xT'(x)/T(x) = M(x)$ also has finite limit, which means that $T(x)$ is a regularly varying function. From here and relation (23) we conclude that $\Omega(x)$ also varies regularly as $x \to \infty$.

It follows from Theorem 2.1 in Rahimov (1986) that, if (21) is satisfied, $\alpha(n) \to 0, \alpha(n)d(n) \to \infty$ and $\beta(n) \to 0$, then

$$\left(\frac{\Omega(Z(n))}{\Omega(1/g(n))}\right)^{\alpha(n)} \to \xi \tag{29}$$

as $n \to \infty$ in distribution, where ξ has the uniform distribution on $[0,1]$. Since $\Omega(x)$ varies regularly $\Omega(x/y) \sim y^{-C_3}\Omega(x)$ for each $y > 0$ and some $C_3 > 0$ and it follows from (29) that

$$\lim_{n\to\infty} P\{(\frac{\Omega(Z(n)/y)}{\Omega(1/g(n))})^{\alpha(n)} \leq x\} = x, 0 \leq x \leq 1. \tag{30}$$

Taking into account trivial equalities

$$\Omega(1/g(n)) = T(U(1-g(n))) = T(n), T^{-1}(\Omega(x)) = U(1-x^{-1})$$

and facts that $T(x)$ is increasing and $g(x)$ is decreasing functions, we obtain from (30) that $P\{Z(n)g(t(n)) \leq y\} \to x, n \to \infty$, where $t(n) = T^{-1}(T(n)x^{1/\alpha}), y > 0$.

Thus condition (18) of Theorem 2.3 is fulfilled with $k(n) = 1/g(t(n))$ and $\varphi(\lambda) = Ee^{-\lambda\eta} = x$, where$P\{\eta = 0\} = 1 - P\{\eta = \infty\} = x$. Since $1 - G(\lambda) \sim \lambda a, \lambda \to 0$ condition (2) is also satisfied for $a(n) = k(n)$ and $b(\lambda) = \lambda a$. Therefore from Theorem 2.3 we obtain

$$Ee^{-\lambda X(n)g(t(n))} \to Ee^{-\eta\lambda a} \equiv x,$$

which implies that $P\{X(n)g(t(n)) \leq y\} \to P\{\eta \leq y\} = x$ as $n \to \infty$ for each $y > 0$. Putting $y = 1$ we obtain the following result.

Theorem 5.1. *If (21) is satisfied, $\alpha(n) \to 0, \alpha(n)d(n) \to \infty$ and $\beta(n) \to 0$, then*

$$\lim_{n\to\infty} P\left\{(\frac{\Omega(X(n))}{\Omega(1/g(n))})^{\alpha(n)} \leq x\right\} = x, 0 \leq x \leq 1.$$

Now we provide results concerning the situation when $\alpha(n)$ approaches zero faster.

Theorem 5.2. *If (21) and (25) are satisfied,* $\alpha(n) \to 0, \alpha(n)d(n) \to C \in (0,\infty)$*, then*

$$\lim_{n\to\infty} P\left\{\frac{(\Omega(X(n)))^{\alpha(n)}-1}{(\Omega(1/g(n)))^{\alpha(n)}-1} \le x | X(n) > 0\right\} = x, 0 \le x \le 1.$$

Note that when conditions of Theorem 5.2 are fulfilled $\Omega^{\alpha(n)}(1/g(n)) = T^{\alpha(n)}(n) \to e^C$ as $n \to \infty$. When $\alpha(n) \to 0$ faster than $1/d(n)$, the behavior of the process is effected by new parameter $\gamma(n) = Q_1(n)/Q_2(n)$.

Theorem 5.3. *If (21) and (25) are satisfied,* $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n))$ *and* $\gamma(n) \to \infty$*, then*

$$\lim_{n\to\infty} P\left\{\frac{\log\Omega(X(n))}{\log\Omega(1/g(n))} \le x | X(n) > 0\right\} = x, 0 \le x \le 1.$$

When $\gamma(n) \to 0, n \to \infty$ we eventually come to the situation when process $X(n)$ is not effected by immigration component at all.

Theorem 5.4. *If (21) and (25) are satisfied,* $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n))$ *and* $\gamma(n) \to 0$*, then*

$$\lim_{n\to\infty} P\{g(n)X(n) \le x | X(n) > 0\} = 1 - e^{-x}, x \ge 0.$$

Theorem 5.5. *If (21) and (25) are fulfilled,* $d(n) \to \infty, \alpha(n)d(n) \to 0, \beta(n) = o(Q_1(n) + Q_2(n))$ *and* $\gamma(n) \to \gamma \in (0,\infty)$*,as* $n \to \infty$*, then the following two assertions hold*

$$i) \lim_{n\to\infty} P\left\{\frac{\log\Omega(X(n))}{\log\Omega(1/g(n))} \le x | X(n) > 0\right\} = \frac{x\gamma}{1+\gamma}, 0 \le x \le 1;$$

$$ii) \lim_{n\to\infty} P\{g(n)X(n) \le x | X(n) > 0\} = \frac{1+\gamma-e^{-x}}{1+\gamma}, x \ge 0.$$

Remarks. 1. It is not difficult to see that limit distribution in last theorem has an atom of the mass $(1+\gamma)^{-1}$ at $x=1$ in the case (i) and has atom of the mass $\gamma(1+\gamma)^{-1}$ at zero in the case (ii).

2. Theorems 5.1-5.5 extend results of the paper [4] obtained for BGW processes to the process $X(n)$.

Proof of Theorem 5.2. This time we use Theorem 2.1. By the same arguments as in the proof of previous theorem we show that, when conditions of our theorem are fulfilled,

$$E[e^{-\lambda Z(n)g(t(n))}|Z(n)>0] \to x$$

as $n\to\infty$, where $t(n)=T^{-1}([x(T^{\alpha(n)}(n)-1)+1]^{1/\alpha(n)})$. Thus condition (4) is satisfied with $k(n)=1/g(t(n))$ and $\varphi(\lambda)=Ee^{-\lambda a\eta}=x$. Condition (2) is also satisfied with $a(n)=k(n)$ and $b(\lambda)=a\lambda$. It follows from Theorem 4.1 that non extinction probabilities of processes $X(n)$ and $Z(n)$ have the same limit and consequently $\Delta(n)\to 1$ as $n\to\infty$.

Since $1-H_n(\lambda)\le \lambda EU_n$ and $P\{Z(n)>0\}\to 1-e^{-C}$, there is a constant $K_1>0$ such that for sufficiently large n

$$\delta(n,\lambda g(t(n)))\le K_1 g(t(n))\alpha(n)$$

and thus $\delta(n,\lambda g(t(n)))\to 0$ as $n\to\infty$. Hence all conditions of Theorem 2.1 are satisfied and we conclude that

$$P\{X(n)g(t(n))\le x|X(n)>0\}\to x$$

as $n\to\infty$ for each $y>0$. If we put $y=1$ here, we obtain the assertion of Theorem 5.2.

Proof of Theorem 5.3. The same arguments as in the proof of previous theorem lead that condition (4) is satisfied with $k(n)=1/g(t(n))$ and $t(n)=T^{-1}(T^x(n)), 0<x<1$. When $\gamma(n)\to\infty$ the non extinction probabilities of both processes $X(n)$ and $Z(n)$ behave as $Q_1(n)$ and consequently $\Delta(n)\to 1, n\to\infty$.

Since $1 - H_n(\lambda) \leq \lambda EU_n$ and $P\{Z(n) > 0\} \sim Q_1(n)$, there is a constant $K_2 > 0$ such that

$$\delta(n, \lambda g(t(n))) \leq K_2 \frac{g(t(n))\alpha(n)}{Q_1(n)}.$$

Taking into account that $Q_1(n) = \alpha(n)d(n)$ we conclude that $\delta(n, \lambda g(t(n))) \to 0$ as $n \to \infty$. Thus all conditions of Theorem 2.1 are fulfilled and assertion of Theorem 5.3 follows as in the proof of previous result. Theorem 5.3 is proved.

Proofs of remaining theorems follow the same scheme as proofs of previous theorems. Namely we show fulfillment of conditions of Theorem 2.1, using corresponding results for BGW processes and results from Section 4 of present paper.

We also note that similar results can be obtained when $d(n)$ has a finite limit as $n \to \infty$. In particular when $\gamma(n) \to \infty, n \to \infty$ conditioned process $\{X(n)|X(n) > 0\}$ has a discrete limit distribution. It is clear that in this case the proof will be based on Theorem 2.2.

6 Increasing immigration

In this section we consider the case $\alpha(n) \to \infty$ as $n \to \infty$. We demonstrate that Theorem 2.3 allows to extend results for BGW processes with increasing immigration on convergence to infinitely divisible and stable distributions obtained in [5], to the continuous-state process $X(n)$.

Let $h(n) = ng(n) = M(n)/n^{1/\theta - 1}, B(n) = \sum_{k=1}^{n} \beta(k)$.

Theorem 6.1. *If (21) is fulfilled, $\alpha(n)h(n) \to C \in (0, \infty)$ and $B(n)g^2(n) \to 0$ as $n \to \infty$, then $g(n)X(n)$ converges in distribution to a random variable $Z(\theta, C)$ which has an infinitely divisible distribution with Laplace transform*

$$\Psi(\theta, C, \lambda) = \exp\left\{-C \int_0^1 (\frac{x^{1-\theta}}{1 - x + (\lambda a)^{-\theta}})^{1/\theta} dx\right\}, \lambda > 0. \qquad (31)$$

Remark. It is not difficult to see that, if $\theta = 1$ the limit distribution is gamma with density function

$$\frac{a^{-C}}{\Gamma(C)} x^{C-1} e^{-x/a}, x \geq 0.$$

If $\theta = 1/2$, then the Laplace transform in (31) is

$$(1+\sqrt{a\lambda})^{-C} e^{-C\sqrt{a\lambda}}$$

and in general for each natural k

$$\Psi(1/2k, C, \lambda) = \exp\left\{-\frac{C}{2k(1+a\lambda)^{2k}} F_1(2k, 2k, 2k+1, \frac{a\lambda}{1+a\lambda})\right\},$$

where $F_1(a,b,c,y)$ is Gauss' hypergeometric function.

Now we consider the case $\alpha(n)h(n) \to 0$. In this case there exists positive sequence $m(n), n \geq 1$ such that $\alpha(n)h(m(n))$ has a finite limit as $n \to \infty$ and we have the following result.

Theorem 6.2. *If (21) is fulfilled with* $0 < \theta < 1, \alpha(n)h(n) \to 0, \alpha(n)h(m(n)) \to C \in (0,\infty)$ *and* $B(n)g^2(m(n)) \to 0$ *as* $n \to \infty$, *then* $X(n)g(m(n))$ *converges in distribution to a random variable* $W(\theta, C)$ *which has a stable distribution with Laplace transform*

$$Ee^{-\lambda W(\theta,C)} = \exp\left\{-\frac{a^{1-\theta}C\theta}{1-\theta}\lambda^{1-\theta}\right\}, \lambda > 0.$$

Example. Let in relation (21) $0 < \theta < 1$ and $L(s) \to C_0 \in (0,\infty), s \uparrow 1$. Then it is clear that in (24) $M(x) \to C_1 = (C_0\theta)^{-1/\theta}$ as $x \to \infty$. If we take $m(n) = (\alpha(n))^{r\theta}$ where $r = 1/(1-\theta)$, then $\alpha(n)h(m(n)) \to C_1$ and $g(m(n)) \sim C_1/(\alpha(n))^r$. Hence we obtain the following result from Theorem 6.2.

Corollary. If conditions of Theorem 6.2 are satisfied and $L(s) \to C_0, S \uparrow 1$, then $X(n)(\alpha(n))^{-r}$ as $n \to \infty$ converges in distribution to random variable $W(\theta, C_0)$ such that

$$Ee^{-\lambda W(\theta,C_0)} = \exp\left\{-\frac{a^{1-\theta}}{C_0(1-\theta)}\lambda^{1-\theta}\right\}.$$

Proof of Theorem 6.1. Proof will use Theorem 2.3. We obtain from Theorem 3 in Rahimov (1993) that when conditions of Theorem 6.1 are satisfied $g(n)Z(n) \to Z^*$ as $n \to \infty$ in distribution , where Z^* has Laplace transform $\Psi^*(\lambda) = \Psi(\theta, C, \lambda/a)$. This means that condition (18) is satisfied with $k(n) = 1/g(n)$ and $\varphi(\lambda) = \Psi^*(\lambda)$. Condition (2) is also satisfied for $a(n) = k(n)$ and $b(\lambda) = a\lambda$. Hence we obtain from our Theorem 2.3 that $g(n)X(n) \to Z(\theta, C)$ in distribution and $Z(\theta, C)$ has Laplace transform $\Psi(\theta, C, \lambda)$.

Proof of Theorem 6.2. It follows from Theorem 2 in Rahimov (1993) that under conditions of Theorem 6.2 $Z(n)g(m(n)) \to W^*$ in distribution as $n \to \infty$, where sequence $m(n), n \geq 1$ is such that $\alpha(n)h(m(n)) \to C$ and the Laplace transform of W^* is $\exp\{-C\theta\lambda^{1-\theta}/(1-\theta)\}$. Consequently condition (18) is fulfilled with $k(n) = 1/g(m(n))$ and $\varphi(\lambda) = E^{-\lambda W^*}$. Since condition (2) is satisfied again with $a(n) = k(n)$ and $b(\lambda) = a\lambda$, the assertion of the theorem follows from Theorem 2.3.

In conclusion we note that, if $\alpha(n)h(n) \to 0, n \to \infty$ and $\theta = 1$, one can obtain a limit theorem with functional normalization similar to Theorem 5.1.

Concluding remarks.

Now we may conclude that Theorems 2.1-2.3 are also applicable to subcritical and supercritical processes. Conditions on process $Z(n)$ are usually satisfied in cases, when corresponding limit theorems for BGW processes hold. Assumption $\Delta(n) \to 1, n \to \infty$, is trivially fulfilled for processes which do not extinct. To apply these theorems for processes without immigration one needs just assume that $U_k = 0, k \geq 1$, almost surely. In this case $\delta(n, \lambda) = 0$ for each $n \geq 1$ and $\lambda > 0$.

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