ON A STOCHASTIC MODEL FOR CONTINUOUS MASS BRANCHING POPULATION

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ABSTRACT

It is well known that the set of nonnegative integers is the state-space of usual branching stochastic processes. However in many applications one may have situations when it is difficult to count the number of individuals in the population, but some non-negative characteristic, such as volume, weight or product produced by the individuals can be measured. To model this kind of situation, branching stochastic processes with continuous state-space are introduced. In this paper two theorems which establish relationship between asymptotic behavior of processes continuous and discrete state-space and with immigration in varying environment will be proved.

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1 Introduction

The theory of branching stochastic processes is a rapidly developing part of the general theory of stochastic processes. During quite a long time the main object of investigation in the theory of branching processes was the number of individuals (particles) at a given time. So in classic models of branching processes the state-space is the set of nonnegative integers. However in many applications one may have situations when it is difficult to count the number of individuals in the population, but some non-negative characteristic, such as volume, weight or product produced by the individuals can be measured.

At the end of sixties M. Jirina [6], [7] defined a branching stochastic process with continuous state space as a homogeneous Markov process the transition probabilities of which satisfy some "branching condition". Later many papers were published to study this kind of processes (see [5], [10], [11], [14], [16]). A model of the branching process with continuous state space has appeared in [13] as limiting for branching processes with generalized immigration. Kallenberg (1979) introduced a branching model with continuous state-space and studied it under the assumption that the "offspring distribution" is infinitely divisible. Adke and Gadag (1995) defined a new model of continuous state-space process with immigration using in the "branching condition" a counting process with independent and stationary increments.

In the paper [1] the authors investigated some distributional properties (and the extinction probability) of the process in the case when the offspring and immigration distributions vary from generation to generation (so called varying environments). However the asymptotic results are obtained under the assumption of fixed environments. On the other hand in applications the immigration rate may usually be affected by seasonal and global changes of the environment. In this paper we will study asymptotic behavior of the continuous state-space branching stochastic process with immigration in varying environments.

It is convenient to define the process, which we are going to consider, as a family of non-negative random variables describing the amount of a product produced by individuals of the population. The initial state of the process is given by a non-negative random variable X_0 . The amount of the product X_1 of the first generation is defined as the sum of random products produced by $N_1(X_0)$ individuals and the product U_1 of immigrating to the first generation individuals. Similarly the amount X_2 of the product of the second generation is defined as the sum of products produced by $N_2(X_1)$ individuals and U_2 , and so on. Here $N_k(t), k \geq 1, t \in T$, are counting processes with independent stationary increments, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, 2, ...\}$ and $U_k, k \geq 1$, are non-negative random variables. In the paper [1] limit distributions are obtained for process X_n when $U_n, n \geq 1$ are i.i.d. random variables, which corresponds to the fixed environment. In particular when the process is critical it is shown that the linearly normalized process has a gamma limiting distribution. If one does not assume that $U_n, n \geq 1$ have a common distribution, several rather important questions appear: i) Under which conditions we may still use the linear normalization to get a non-degenerate limiting distribution? ii) When will the limiting gamma distribution be preserved? iii) How does a change of the rate of immigration affect the asymptotic behavior of the process ? In this paper we expect to obtain results which give answers for these kinds of questions.

2 Main theorems

We now give a detailed definition of the process which we are going to consider. Let $\{W_{in}, i, n \geq 1\}$ be a double array of independent and identically distributed non-negative random variables, $\{N_n(t), t \in T, n \geq 1\}$ be a family of nonnegative, integer-valued independent processes with independent stationary increments, with $N_n(0) = 0$ almost surely, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, ...\}$.

We define a new process $X_n, n \geq 0$, as following. Let the initial state of the process be X_0 which is an arbitrary non-negative random variable and for $n \geq 0$

$$X_{n+1} = \sum_{i=1}^{N_{n+1}(X_n)} W_{in+1} + U_{n+1}, \tag{1}$$

where $\{U_n, n \geq 1\}$ is e sequence of independent non-negative random variables. Assume that families of random variables $\{W_{in}, i, n \geq 1\}, \{U_n, n \geq 1\}$ of stochastic processes $\{N_n(t), t \in T, n \geq 1\}$ and random variable X_0 are independent. This process was first introduced in [1], where the authors have given a detailed comparisons of the process with classic models of Galton-Watson processes.

It is also shown in [1] that $Z(n) = N_n(X_{n-1})$ is a Galton-Watson process with an immigration component. We now provide a result establishing a relationship, in a sense of limiting behavior, between processes X_n and $Z(n)$. In order to do that we use the following Laplace transforms

$$G(\lambda) = Ee^{-\lambda W_{ni}}, H_n(\lambda) = Ee^{-\lambda U_n}.$$

We also denote

$$\Delta(n) = \frac{P\{Z(n) > 0\}}{P\{X_n > 0\}}, \delta(n, \lambda) = \frac{1 - H_n(\lambda)}{P\{Z(n) > 0\}}.$$

Let the sequences of positive numbers $\{k(n), n \geq 1\}$ and $\{a(n), n \geq 1\}$ be such that for each $\lambda > 0$ there exists

$$\lim_{n\to\infty} k(n)(1 - G(\frac{\lambda}{a(n)})) = b(\lambda) \in (0, \infty) \tag{2}$$

Theorem 1. *Let $\Delta(n) \to 1, n \to \infty$ and $\delta(n, \lambda/a(n)) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then as $n \to \infty$*

$$E[e^{-\lambda X_n/a(n)}|X_n > 0] \to \phi(b(\lambda)) \tag{3}$$

for $\lambda > 0$, if and only if as $n \to \infty$ for each $\lambda > 0$

$$E[e^{-\lambda Z(n)/k(n)}|Z(n) > 0] \to \phi(\lambda). \tag{4}$$

Proof. We consider the following obvious identity

$$E[e^{-\lambda X_n}|X_n > 0] = 1 - \frac{1 - Ee^{-\lambda X_n}}{P(X_n > 0)}. \tag{5}$$

It follows from definition (1) of the process X_n by total probability arguments that

$$Ee^{-\lambda X_n} = H_n(\lambda)EG^{Z(n)}(\lambda). \tag{6}$$

We obtain from (6) that

$$\frac{1 - Ee^{-\lambda X_n}}{P(Z_n > 0)} = 1 - E[G^{Z(n)}(\lambda)|Z(n) > 0] + \delta(n, \lambda)E[G^{Z(n)}(\lambda)].$$

Hence the ratio on the right side of (5) equals

$$\Delta(n)\frac{1 - Ee^{-\lambda X_n}}{P(Z_n > 0)} = -\Delta(n)E[G^{Z(n)}(\lambda)|Z(n) > 0] - \Delta(n)[1 + \delta(n, \lambda)EG^{Z(n)}(\lambda)].$$

If we use this in relation (5) we obtain

$$E[e^{-\lambda X_n}|X_n > 0] = \Delta(n)E[G^{Z(n)}(\lambda)|Z(n) > 0] + \varepsilon(n), \tag{7}$$

where

$$\varepsilon(n) = 1 - \Delta(n)(1 + \delta(n, \lambda))E[G^{Z(n)}(\lambda)].$$

Let (4) be satisfied for every $\lambda > 0$. Then, it clearly follows from continuity of the Laplace transform $\varphi(\lambda)$, that the convergence in (4) holds uniformly with respect to λ from arbitrary finite interval. Since $\ln x = -(1-x) + o(1-x), x \to 1$, we obtain from condition (2) that as $n \to \infty$

$$t_n = k(n) \ln G(\frac{\lambda}{a(n)}) \to b(\lambda). \tag{8}$$

Therefore for each fixed $\lambda > 0$ there is such a $T = T(\lambda)$, that $0 < t_n \leq T$ for any $n = 1, 2, ...$ Now we consider (7) replacing λ by $\lambda/a(n)$. It follows from (8) that

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n) > 0] = E[e^{-t_n Z(n)/k(n)}|Z(n) > 0]. \tag{9}$$

We show that the Laplace transform (9) as $n \to \infty$ approaches $\varphi(b(\lambda))$. In order to do it we consider the following relation:

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n) > 0] - \varphi(b(\lambda)) = I_1 + I_2, \tag{10}$$

where

$$I_1 = E[e^{-t_n Z(n)/k(n)}|Z(n) > 0] - \varphi(t_n), I_2 = \varphi(t_n) - \varphi(b(\lambda)).$$

It follows from (4), due to the uniform convergence, that

$$|I_1| \leq \sup_{0 < t_n < T} |E[e^{-t_n Z(n)/k(n)}|Z(n) > 0] - \varphi(t_n)| \to 0 \tag{11}$$

as $n \to \infty$. On the other hand $I_2 \to 0$ as $n \to \infty$ due to continuity of the Laplace transform $\varphi(\lambda)$, for $\lambda > 0$. Thus we conclude that as $n \to \infty$

$$E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n) > 0] \to \varphi(b(\lambda)). \tag{12}$$

Since $\Delta(n) \to 1$ and $\delta(n, \frac{\lambda}{a(n)}) \to 0$ as $n \to \infty$, we obtain that $\varepsilon(n) \to 0$ as $n \to \infty$. The assertion (3) now follows from relations (7) and (12). The first part of Theorem 1 is proved.

Let now (3) be satisfied. It follows from condition (2) that $\tau_n = t_n/b(\lambda) \to 1$ as $n \to \infty$ for each $\lambda > 0$ (recall that $t_n = -k(n) \ln G(\frac{\lambda}{a(n)})$). We consider the following Laplace transform:

$$E[e^{-Z(n)b(\lambda)\tau_n/k(n)}|Z(n) > 0] = E[G^{Z(n)}(\frac{\lambda}{a(n)})|Z(n) > 0]. \tag{13}$$

It follows from relations (3), (7) and (13), due to continuity of $\varphi(\lambda)$, that

$$\lim_{n\to\infty} E[e^{-Z(n)b(\lambda)\tau_n/k(n)}|Z(n) > 0] = \varphi(b(\lambda)). \tag{14}$$

Due to continuity theorem for Laplace transforms (14) means that

$$\left\{ \frac{Z(n)\tau_n}{k(n)} |Z(n) > 0 \right\} \xrightarrow{\mathrm{D}} \xi$$

as $n \to \infty$, with $Ee^{-\lambda\xi} = \varphi(\lambda)$. Since $\tau_n \to 1, n \to \infty$, we have that $Z(n)/k(n)$ given $Z(n) > 0$, as $n \to \infty$ converges to ξ in distribution. If we write this in terms of Laplace transforms, we get assertion of (4). Theorem 1 is proved completely.

Now we provide similar duality result for unconditional distributions of processes $Z(n)$ and X_n.

Theorem 2. *Let for sequences $\{a(n), n \geq 1\}$ and $\{k(n), n \geq 1\}$ condition (2) be satisfied. Then*

$$Ee^{-\lambda X_n/a(n)} \to \varphi(b(\lambda)) \tag{15}$$

if and only if for each $\lambda > 0$ as $n \to \infty$

$$Ee^{-\lambda Z(n)/k(n)} \to \varphi(\lambda). \tag{16}$$

3 The Foster-Williamson Theorem

As it was indicated before process $Z(n) = N_n(X_{n-1})$ is a Galton - Watson process with immigration. The offspring distribution and the distribution of the number of immigrating "individuals" have Laplace transforms

$G(f(\lambda)) = Ee^{-\lambda\xi_n}$ and $H_n(f(\lambda)) = Ee^{-\lambda\eta_n}$, respectively (see [1]). Here $\xi_n = N_n(W_{n-1}), \eta_n = N_n(U_{n-1})$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

We obtain the moments of offspring distribution by standard arguments as following:

$$m = E\xi_n = \frac{d}{d\lambda}G(f(\lambda))_{\lambda=0} = EWEN,$$

where $N = N_1(1), W = W_1$ and for $B = E\xi_n(\xi_n - 1)$ we obtain

$$B = EW[varN - EN] + EW^2(EN)^2.$$

Now we consider applicability of Theorem 2 to obtain a version of well known result by Foster and Williamson (1971). They assume convergence in distribution of the normalized immigration process (the partial sum of the number of immigrating individuals) to a random variable ξ. Since ξ is nonnegative and has an infinitely divisible distribution its Laplace transform has the form (see Feller [3], page 426)

$$Ee^{-\lambda\xi} = \exp\left\{-\int_0^\infty \frac{1 - e^{-\lambda x}}{x} dP(x)\right\},$$

where $P(x)$ is a measure such that $\int_0^\infty x^{-1} dP(x) < \infty$. First we provide the theorem for the process $Z(n) = N_n(X_{n-1})$ from [4].

Theorem A. *If* $m = 1, B \in (0, \infty)$ *and*

$$\frac{1}{n}\sum_{k=1}^{n} N_k(U_{k-1}) \xrightarrow{\mathrm{D}} \xi, \tag{17}$$

then $Z(n)/n \xrightarrow{\mathrm{D}} W$, *with*

$$Ee^{\lambda W} = \exp\left\{-\int_0^\infty \frac{1 - e^{-\lambda x}}{x} dQ(x)\right\},$$

where $Q(x) = R * P(x), R(x) = 1 - \exp\{-2x/B\}$.

Now we formulate Foster-Williamson result for process X_n.

Theorem 3. *If $m = 1, B \in (0, \infty)$ and*

$$\frac{EN}{n} \sum_{k=1}^{n} U_k \xrightarrow{\mathrm{D}} \xi, \tag{18}$$

then $X_n/n \xrightarrow{\mathrm{D}} X$, with

$$Ee^{\lambda X} = \exp\left\{-\int_0^\infty \frac{1 - e^{-\lambda x EW}}{x} dQ(x)\right\},$$

and $Q(x)$ is the same as in Theorem A.

Proof First we show that, if condition (18) is fulfilled, then (17) holds. In fact, in terms of Laplace transforms (18) is

$$\prod_{k=1}^{n} H_k(\frac{\lambda EN}{n}) \to Ee^{-\lambda\xi}. \tag{19}$$

If we denote the sum in (17) by S_n, we have

$$Ee^{-\lambda S_n/n} = \prod_{k=1}^{n} H_k(f(\frac{\lambda}{n})).$$

Using relation $\log x = -(1-x) + o(1-x), x \downarrow 1$, we obtain that

$$nf(\frac{\lambda}{n}) = -n \log Ee^{-\lambda N/n} \sim n(1 - Ee^{-\lambda N/n})$$

as $n \to \infty$, consequently $nf(\frac{\lambda}{n}) \to \lambda EN, n \to \infty$. Thus, due to continuity of the Laplace transform, we conclude that

$$\prod_{k=1}^{n} H_k(f(\frac{\lambda}{n})) \sim \prod_{k=1}^{n} H_k(\frac{\lambda EN}{n})$$

and this together with (19) gives (17).

It follows from the above that, if condition (18) is satisfied, then Theorem A holds, i. e. $Z(n)/n \xrightarrow{\mathrm{D}} W, n \to \infty$. This can be written in terms of Laplace transforms as $Ee^{-\lambda Z(n)/n} \to Ee^{-\lambda W}, n \to \infty$. Now we appeal to Theorem 2. If we choose $k(n) = a(n) = n$, then as $n \to \infty$

$$n(1 - G(\frac{\lambda}{n})) \to \lambda EW.$$

Thus condition (1) is fulfilled with $b(\lambda) = \lambda EW$. The assertion of Theorem 3 now follows from Theorem 2.

Example. Let the immigration process be stationary, i. e. $\{U_k, k \geq 1\}$ have a common distribution and $a = EU_k$ is finite. Then, due to weak law of large numbers, condition (18) is satisfied with $\xi = aEN$. Thus we obtain from Theorem 3 the following result.

Corollary. *If $m = 1, B \in (0, \infty)$ and immigration is stationary with $a = EU_k < \infty$, then X_n/n as $n \to \infty$ has a gamma limit distribution with density function*

$$\frac{1}{\Gamma(\frac{2aE(N)}{B})} \left(\frac{2}{E(W)B} \right)^{\frac{2aE(N)}{B}} x^{\frac{2aE(N)}{B} - 1} e^{-\frac{2x}{E(W)B}}.$$

In conclusion we note that Theorem 3 and its corollary answer questions concerning linear normalization and gamma limit distribution, stated in the introduction. To investigate affect of the change of immigration rate to asymptotic behavior of the process, one needs to consider cases of decreasing and increasing immigration separately. In this study one still may use Theorems 1 and 2 of this paper.

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