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NONLINEAR VARIATIONAL INEQUALITIES FOR PSEUDOMONOTONE OPERATORS WITH APPLICATIONS*

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Abstract. In this paper, we prove the existence of solutions to the variational and variational-like inequalities for pseudomonotone and pseudodissipative and, η-pseudomonotone and η-pseudodissipative operators, respectively. As applications of our results, we prove the existence of a unique solution of nonlinear equations, fixed point problems and eigenvalue problems.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a real locally convex Hausdorff topological vector space with topological dual X^* and K a non-empty subset of X. Let $T : K \to X^*$ be an operator and $\eta : K \times K \to X$ a bifunction. The *variational-like inequality problem* (for short, VLIP) is to find $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } \ y \in K,$$

where $\langle u, x\rangle$ denotes the pairing between $u \in X^*$ and $x \in X$. For further details on VLIP, we refere to [2, 5, 9-12, 16] and references therein.

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When $\eta(y,x) = y - x$, the VLIP reduces to the *variational inequality problem* (for short, VIP) [7] of finding $\bar{x} \in K$ such that

$$\langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all} \;\; y \in K.$$

In most of the results on the existence of solutions to the VIP and VLIP some kind of continuity assumption on the operator T is needed if it has some kind of monotonicity assumption, see for example [3-4, 6-8, 12-15, 17-18] and references therein.

The main object of this paper is to establish some existence results for VIP and VLIP in the setting of non-compact convex set K with pseudomonotone and pseudodissipative and, η-pseudomonotone and η-pseudodissipative operator T, respectively. As applications of our results, we prove the existence of a unique solution of nonlinear equations, fixed point problems and eigenvalue problems without any continuity assumption on the operator T.

We shall use the following notation and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. If A and B are non-empty subsets of a topological vector space Y such that $A \subseteq B$, we shall denote by $int_B A$ the interior of A in B.

The *inverse* F^{-1} of a multivalued map $F : X \to 2^Y$ is the multivalued map from $\mathcal{R}(F)$, the range of F, to X defined by

$$x \in F^{-1}(y) \quad \text{if and only if} \quad y \in F(x).$$

We shall use the following particular form of Corollary 1 in [1].

Lemma 1.1. *Let K be a non-empty and convex subset of a Hausdorff topological vector space E, and let $S : K \to 2^K$ be a multivalued map. Assume that the following conditions hold.*

(a) *For each $x \in K$, $S(x)$ is non-empty and convex.*

(b) $K = \bigcup \{int_K S^{-1}(y) : y \in K\}$.

(c) *If K is not compact, assume that there exists a non-empty, compact and convex subset C of K and a non-empty and compact subset D of K such that for each $x \in K \setminus D$, there exists $\tilde{y} \in C$ such that $x \in int_K S^{-1}(\tilde{y})$.*

Then S has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in S(x_0)$.

2. EXISTENCE RESULTS

For a given bifunction $\eta : K \times K \to X$, an operator $T : K \to X^*$ is called:

(i) η-*monotone* if,

$$\langle T(y) - T(x), \eta(y,x)\rangle \geq 0, \quad \text{for all} \;\; x, y \in K;$$

(i) η-*dissipative* if,

$$\langle T(y) - T(x), \eta(y,x)\rangle \leq 0, \quad \text{for all} \;\; x, y \in K;$$

(iii) η-*pseudomonotone* if,

$$\langle T(x), \eta(y,x)\rangle \geq 0 \;\text{ implies }\; \langle T(y), \eta(y,x)\rangle \geq 0, \quad \text{for all} \;\; x, y \in K,$$

or equivalently,

$$\langle T(y), \eta(y,x)\rangle < 0 \;\text{ implies }\; \langle T(x), \eta(y,x)\rangle < 0, \quad \text{for all} \;\; x, y \in K;$$

(iv) η-*pseudodissipative* if,

$$\langle T(y), \eta(y,x)\rangle \geq 0 \;\text{ implies }\; \langle T(x), \eta(y,x)\rangle \geq 0, \quad \text{for all} \;\; x, y \in K,$$

or equivalently,

$$\langle T(x), \eta(y,x)\rangle < 0 \;\text{ implies }\; \langle T(y), \eta(y,x)\rangle < 0, \quad \text{for all} \;\; x, y \in K.$$

When $\eta(y,x) = y - x$, the definitions of η-monotone, η-dissipative, η-pseudomonotone and η-pseudodissipative reduce to the definitions of monotone, dissipative [17], pseudomonotone and pseudodissipative, respectively.

EXAMPLE 2.1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined as

$$T(x) = \begin{cases} 1 & : \quad x \neq 1 \\ 2 & : \quad x = 1. \end{cases}$$

Then T is pseudomonotone as well as pseudodissipative but it is neither monotone nor hemicontinuous.

For $\eta(y,x) = y^2 - x^2$, T is also η-pseudomonotone as well as η-pseudodissipative but not η-monotone.

An example of a pseudomonotone hemicontinuous operator is given in [15] which is not continuous on finite dimensional spaces.

THEOREM 2.1. *Let K be a non-empty and convex subset of a locally convex Hausdorff topological vector space X and let $\eta : K \times K \to X$ be a bifunction such that $\eta(x,x) = 0$, for all $x \in K$. Assume that*

(i) $T : K \to X^*$ *is η-pseudomonotone and η-pseudodissipative;*

(ii) *for each fixed $y \in K$, the map $x \mapsto \langle T(y), \eta(y, x)\rangle$ is upper semicontinuous on K;*

(iii) *for each fixed $x \in K$, the map $y \mapsto \langle T(x), \eta(y, x)\rangle$ is quasi-convex;*

(iv) *there exists a non-empty, compact and convex subset C of K and a non-empty and compact subset D of K such that for each $x \in K\backslash D$, there exists $\tilde{y} \in C$ such that $\langle T(x), \eta(\tilde{y}, x)\rangle < 0$.*

Then the VLIP *has a solution.*

Proof. Assume that the VLIP has no solution. Then for each $x \in K$,

$$\{y \in K : \langle T(x), \eta(y, x)\rangle < 0\} \neq \emptyset.$$

We define a multivalued map $S : K \to 2^K$ by

$$S(x) = \{y \in K : \langle T(x), \eta(y, x)\rangle < 0\}, \quad \text{for all } \ x \in K.$$

Then clearly for all $x \in K$, $S(x) \neq \emptyset$. From assumption (iii), it is easy to see that $S(x)$ is convex, for all $x \in K$. Now

$$S^{-1}(y) = \{x \in K : \langle T(x), \eta(y, x)\rangle < 0\}.$$

For each $y \in K$, we denote by $[S^{-1}(y)]^c$ the complement of $S^{-1}(y)$ in K. From the η- pseudomonotonicity of T, we have

$$\begin{aligned}
[S^{-1}(y)]^c &= \{x \in K : \langle T(x), \eta(y, x)\rangle \geq 0\} \\
&\subseteq \{x \in K : \langle T(y), \eta(y, x)\rangle \geq 0\} \\
&= H(y)(\text{say}).
\end{aligned}$$

From condition (ii), it is easy to show that for all $y \in K$, $H(y)$ is closed in K.

From the η-pseudodissipativeness of T, we have

$$\begin{aligned}
S^{-1}(y) &= \{x \in K : \langle T(x), \eta(y, x)\rangle < 0\} \\
&\subseteq \{x \in K : \langle T(y), \eta(y, x)\rangle < 0\} \\
&= [H(y)]^c, \text{ the complement of } H(y) \text{ in } K.
\end{aligned}$$

Hence $S^{-1}(y) = [H(y)]^c$ and $S^{-1}(y)$ is open in K. Since $S(x) \neq \emptyset$, we have

$$K = \bigcup_{y \in K} S^{-1}(y) = \bigcup_{y \in K} int_K S^{-1}(y).$$

By assumption (iv), for each $x \in K \setminus D$, there exists $\tilde{y} \in C$ such that $\langle T(x), \eta(\tilde{y}, x)\rangle < 0$, we have $x \in int_K S^{-1}(\tilde{y})$. Then S satisfies all the conditions of Lemma 1.1, hence there exists $x_0 \in K$ such that $x_0 \in S(x_0)$, that is,

$$\langle T(x_0), \eta(x_0, x_0)\rangle < 0.$$

Since $\eta(x_0, x_0) = 0$, we have

$$0 = \langle T(x_0), \eta(x_0, x_0)\rangle < 0,$$

a contradiction. Hence the result is proved. □

Remark 2.1. If X is a reflexive Banach space equipped with weak topology, then the assumption (iv) in Theorem 2.1 can be replaced by the following condition:

(iv)′ There exists $\tilde{y} \in K$ such that $\liminf_{||x|| \to \infty,\ x \in K} \langle T(x), \eta(\tilde{y}, x)\rangle < 0$.

Proof. By (iv)′, there exists $r > 0$ such that $||\tilde{y}|| < r$ and if $x \in K$ with $||x|| \geq r$, we have $\langle T(x), \eta(\tilde{y}, x)\rangle < 0$. Define $B_r = \{x \in K : ||x|| \leq r\}$. Then B_r is a non-empty weakly compact and convex subset of X. By taking $C = D = B_r$ in assumption (iv) of Theorem 2.1, we get the conclusion. □

In view of Remark 2.1, we have the following result.

Corollary 2.1. *Let K be a non-empty and convex subset of a reflexive Banach space X equipped with weak topology and let $\eta : K \times K \to X$ be a bifunction such that it is affine in the first argument, weakly continuous in the second argument and $\eta(x, x) = 0$, for all $x \in K$. Assume that $T : K \to X^*$ is η-pseudomonotone, η-pseudodissipative and there exists $\tilde{y} \in K$ such that $\liminf_{||x|| \to \infty,\ x \in K} \langle T(x), \eta(\tilde{y}, x)\rangle < 0$. Then the* VLIP *has a solution.*

Corollary 2.2. *Let K be a non-empty and convex subset of a locally convex Hausdorff topological vector space X and let $T : K \to X^*$ be pseudomonotone and pseudodissipative. Assume that there exists a non-empty, compact and convex subset C of K and a non-empty and compact subset D of K such that for each $x \in K \setminus D$, there exists $\tilde{y} \in C$ such that $\langle T(x), \tilde{y} - x\rangle < 0$. Then the* VIP *has a solution.*

Remark 2.2. In the results of Browder [3-4], Hartman and Stampacchia [6] (Theorem 1.1), Tarafdar [13] (Theorem 2 and Corollary), Verma [14] (Theorem 2.2) and Yao [18] (Theorem 3.3), we need continuity/hemicontinuity/continuity on finite dimensional spaces. But in Corollary 2.2 we do not assume any kind of continuity assumption.

Corollary 2.3. *Let K be a non-empty and convex subset of a reflexive Banach space X equipped with weak topology and let $T : K \to X^*$ be pseudomonotone, pseudodissipative and has the property that there exists $\tilde{y} \in K$ such that $\liminf_{||x|| \to \infty,\ x \in K} \langle T(x), \tilde{y} - x \rangle < 0$. Then the* VIP *has a solution. Moreover, it T is strongly pseudomonotone then the solution is unique.*

Remark 2.3. Corollary 2.3 is different from Theorems 3.1 and 3.2 in [17] in the following ways:

(a) X need not be a Hilbert space,

(b) K need not be closed,

(c) T need not be continuous on finite-dimensional subspaces,

(d) T need not be hemicontinuous,

(e) T is assumed only pseudomonotone and pseudodissipative, need not be monotone.

3. APPLICATIONS

Throughout this section, we will assume that H is a real Hilbert space with its inner product denoted by $(.,.)$.

Let K be a non-empty subset of H. An operator $T : K \to K$ is called:

(i) *strongly monotone* if, there exists a constant $\alpha > 0$ such that

$$(T(y) - T(x), y - x) \geq \alpha ||y - x||^2, \quad \text{for all} \ \ x, y \in K;$$

(ii) *relaxed strongly monotone* if, there exists a constant $\beta < 1$ such that

$$(T(y) - T(x), y - x) \leq \beta ||y - x||^2, \quad \text{for all} \ \ x, y \in K;$$

(iii) *relaxed strongly dissipative* if, there exists a constant $\nu < 1$ such that

$$(T(y) - T(x), y - x) \geq \nu ||y - x||^2, \quad \text{for all} \ \ x, y \in K;$$

(iv) *strongly pseudomonotone* if, there exists a constant $\gamma > 0$ such that

$$(T(x), y - x) \geq 0 \ \ \text{implies} \ \ (T(y), y - x) \geq \gamma ||y - x||^2, \quad \text{for all} \ \ x, y \in K.$$

We now give the following result concerning the existence of a unique solution of a nonlinear equation.

THEOREM 3.1. *Let $T : H \to H$ be pseudomonotone, pseudodissipative and assume that there exists $\tilde{y} \in H$ such that $\liminf_{||x|| \to \infty} (T(x), \tilde{y} - x) < 0$. Then there exists $\bar{x} \in H$ such that $T(\bar{x}) = 0$. Moreover, if T is strongly pseudomonotone then the solution is unique.*

PROOF. It is similar to the proof of Theorem 3.3 in [17].

REMARK 3.1. Theorem 3.1 is different from Theorem 3.3 in [17] in the following ways:

(a) T need not be hemicontinuous,

(b) T is assumed only pseudomonotone and pseudodissipative, need not be monotone.

By using the results of Section 2, we establish the following fixed point theorem.

THEOREM 3.2. *Let K be a non-empty and convex subset of H and $T : K \to K$ be relaxed strongly monotone and relaxed strongly dissipative. Then T has a unique fixed point.*

PROOF. It is similar to the proof of Theorem 3.4 in [17].

REMARK 3.2. Theorem 3.2 is different from Theorem 3.4 in [17] in the following ways:

(a) K need not be closed,

(b) T is assumed relaxed strongly dissipative, need not be hemicontinuous.

Finally, we derive the following existence results for solutions to the eigenvalue problem.

COROLLARY 3.1. *Let K be a non-empty convex cone of H and $T : K \to K$ be monotone and dissipative. Then for any nonnegative real number λ and any $z \in K$, there exists a unique $\bar{x} \in K$ such that $\lambda T(\bar{x}) + z = \bar{x}$.*

PROOF. It is similar to the proof of Corollary 3.7 in [17].

REMARK 3.3. Corollary 3.1 is different from Corollary 3.7 in [17] in the following ways:

(a) K need not be closed,

(b) T is assumed monotone, need not be hemicontinuous.

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