Vector Variational Inequalities and Vector Equilibria

Mathematical Theories

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VECTOR EQUILIBRIUM PROBLEMS AND VECTOR VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we consider vector equilibrium problems and prove the existence of their solutions in the setting of Hausdorff topological vector spaces. We also derive some existence results for the scalar and vector variational inequalities.

KEY WORDS. Vector Equilibrium Problems, Vector Variational Inequalities, Vector Optimization, KKM-maps, Strongly Nonlinear Vector Variational Inequalities.

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1. INTRODUCTION

Let X and Y be two topological vector spaces and K be a nonempty and convex subset of X. Let $f : K \times K \to Y$ with $f(x,x) = 0$, $\forall x \in K$ and $\{C(x) : x \in K\}$ be a family of closed, pointed and convex cones in Y with apexes at the origin and with $\operatorname{int} C(x) \neq \emptyset$, $\forall x \in K$, where $\operatorname{int} C(x)$ denotes the interior of the set $C(x)$. We consider the problem of finding $y \in K$, such that:

$$(1.1) \qquad f(y,x) \not\geq_{\operatorname{int} C(y)} 0 \quad , \quad \forall x \in K,$$

where the inequality means that $f(y,x) \notin \operatorname{int} C(y)$. (1.1) is called *Vector Equilibrium Problem* (for short, VEP). For further details, we refer to [1, 10-11, 13]. The following problems are special cases of (1.1).

F. Giannessi (ed.), Vector Variational Inequalities and Vector Equilibria, 1-15.

(i) Let $T : K \to L(X,Y)$, where $L(X,Y)$ is the space of all continuous linear operators from X to Y. Then the Vector Variational Inequality (for short, VVI) introduced in [9] (see also [5-7, 15]) consists in finding $y \in K$, such that:

$$\langle T(y), x-y\rangle \not\leq_{\text{int}\, C(y)} 0 \quad , \quad \forall x \in K, \tag{1.2}$$

where $\langle T(y), x\rangle$ denotes the evaluation of the linear operator $T(y)$ at x.
We set $f(y,x) = \langle T(y), x-y\rangle$. Then (1.2) $\Leftrightarrow$ (1.1).

(ii) Let $\phi : K \to Y$. Then, the Vector Optimization Problem (for short, VOP) [16] consists in finding $y \in K$, such that:

$$\phi(x) - \phi(y) \not\leq_{\text{int}\, C(y)} 0 \quad , \quad \forall x \in K. \tag{1.3}$$

At $f(y,x) = \phi(x) - \phi(y)$, problem (1.3) coincides with (1.1).

(iii) Let $f : K \times K \to \mathbf{R}$ be a given function with $f(x,x) = 0, \forall x \in K$. Then the Equilibrium Problem (for short, EP) [2, 3] consists in finding $y \in K$, such that:

$$f(y,x) \geq 0 \quad , \quad \forall x \in K. \tag{1.4}$$

When $Y = \mathbf{R}$ and $C(x) = \mathbf{R}_-$ (the negative orthant), $\forall x \in K$, then $y \in K$ is a solution of (1.1) if and only if it is a solution of (1.4).

From the above examples, it is clear that our VEP (1.1) contains as special cases, for instance, VVI, VOP and EP.

In the next section, we present some preliminaries which will be used in rest of the paper. Sect.3 deals with the existence theorems for (1.1). In Sect.4, we apply a result of Sect.3 to prove the existence of solutions to the strongly nonlinear Variational Inequality (for short, VI) studied by Noor [12].

2. PRELIMINARIES

We denote by *conv*A, for all $A \subseteq X$, the convex hull of A. We need the following concepts and results.

Definition 1. Let K be a nonempty and convex subset of a topological vector space X and Y be another topological vector space with a closed and convex cone C, such that int $C \neq \emptyset$. A mapping $q : K \to Y$ is called *C–function*, iff

$$q(\alpha x + (1-\alpha)y) - \alpha q(x) - (1-\alpha)q(y) \in C \quad , \quad \forall x, y \in K, \quad \forall \alpha \in]0, 1[.$$

When C contains or is contained in the negative orthant (the positive orthant), then q is called *C–convex* (C–*concave*, respectively).

Remark 1. $q : K \to Y$ is a C–function iff $\forall x_i \in K$, for $i = 1, ..., n$ and $\alpha_i \geq 0$, such that $\sum_{i=1}^n \alpha_i = 1$, we have:

$$q(\sum_{i=1}^{n} \alpha_i x_i) - \sum_{i=1}^{n} \alpha_i q(x_i) \in C.$$

Definition 2. A point-to-set map $T : X \rightrightarrows Y$ is called *upper semicontinuous* (for short, u.s.c.) at $x \in X$, iff for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x$ in X and for any net $\{y_\lambda\}$ in Y with $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y$ in Y, we have $y \in T(x)$. T is called u.s.c. on X, iff it is u.s.c. at each $x \in X$.

Definition 3. A point-to-set map $T : X \rightrightarrows X$ is called $KKM - map$, iff for every finite subset $\{x_1, \ldots, x_n\}$ of X, $conv\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n T(x_i)$.

Lemma 1 [8]. Let K be a nonempty and convex subset of a Hausdorff topological vector space X. Let $T : K \rightrightarrows X$ be a KKM-map, such that $\forall x \in K$, $T(x)$ is closed and $T(x^*)$ is contained in a compact set $D \subseteq X$ for some $x^* \in K$. Then $\exists y \in D$ such that $y \in T(x)$ $\forall x \in K$.

Lemma 2. Let Y be a topological vector space with a closed, pointed and convex cone C such that int $C \neq \emptyset$. Then $\forall x, y, z \in Y$, we have

(i) $x - y \in \text{int } C$ and $x \notin \text{int } C \Rightarrow y \notin \text{int } C$;

(ii) $x + y \in C$ and $x + z \notin \text{int } C \Rightarrow z - y \notin \text{int } C$;

(iii) $x + z - y \notin \text{int } C$ and $-y \in C \Rightarrow x + z \notin \text{int } C$;

(iv) $x + y \notin \text{int } C$ and $y - z \in C \Rightarrow x + z \notin \text{int } C$.

Proof. (i) Let $y \in \text{int } C$ and $x - y \in \text{int } C$. Then $x - y + y \in \text{int } C + \text{int } C \subseteq \text{int } C \Rightarrow x \in \text{int } C$, a contradiction of our assumption. (ii) Let $z - y \in \text{int } C$ and $x + y \in C$. Then $z - y + x + y \in \text{int } C + C \subseteq \text{int } C \Rightarrow z + x \in \text{int } C$, a contradiction of our assumption. (iii) Let $x + z \in \text{int } C$ and $-y \in C$. Then we have $x + z - y \in \text{int } C + C \subseteq \text{int } C \Rightarrow x + z - y \in \text{int } C$, a contradiction of our assumption. Similarly, we can prove (iv). □

3. EXISTENCE RESULTS

We first prove the following existence theorem.

Theorem 1. Let K be a nonempty and convex subset of a Hausdorff topological vector space X, Y be a topological vector space, assume that:

1^o $C : K \rightrightarrows Y$ is a point-to-set map such that $\forall x \in K$, $C(x)$ is a closed, pointed and convex cone with apex at the origin and with $\text{int } C(x) \neq \emptyset$;

2^o the point-to-set map $W : K \rightrightarrows Y$, defined by $W(x) := Y \setminus \{\text{int } C(x)\}$ $\forall x \in K$, is upper semicontinuous on K;

3^o $f(\cdot, x)$ is continuous, $\forall x \in K$;

4^o there exists a function $p : K \times K \to Y$, such that:

(a) $p(z, x) - f(z, x) \in \text{int } C(z)$, $\forall x, z \in K$;

(b) the set $\{x \in K : p(z, x) \in \text{int } C(z)\}$ is convex, $\forall z \in K$;

(c) $p(x, x) \notin \text{int } C(x)$, $\forall x \in K$;

(d) there exists a nonempty, compact and convex subset $D \subset K$, such that $\forall z \in K \setminus D$, $\exists \tilde{x} \in D$ such that $f(z, \tilde{x}) \in \text{int } C(z)$.

Then $\exists y \in D \subset K$, such that:

$$f(y,x) \not\leq_{\text{int}\, C(y)} 0 \quad , \quad \forall x \in K. \tag{3.1}$$

Proof. We define

$$G(x) := \{z \in D : f(z,x) \notin \text{int } C(z)\} \quad , \quad \forall x \in K.$$

We first prove that $\forall x \in K$, $G(x)$ is closed. Let $\{z_\lambda\}$ be a net in $G(x)$ such that $z_\lambda \to z$. Then $z \in D$ because D is compact. Since $z_\lambda \in G(x)$, we have

$$f(z_\lambda, x) \notin \text{ int } C(z_\lambda) \ \Rightarrow \ f(z_\lambda, x) \in W(z_\lambda) = Y \setminus \{\text{int } C(z_\lambda)\}.$$

Since $f(\cdot, x)$ is continuous, we have $f(z_\lambda, x) \to f(z,x)$. Because of the upper semicontinuity of W we have that $f(z,x) \in W(z) \Rightarrow f(z,x) \notin \text{ int } C(z)$ and hence $G(x)$ is closed. Since every element $y \in \bigcap_{x\in K} G(x)$ is a solution of (3.1), we have to prove that $\bigcap_{x\in K} G(x) \neq \emptyset$. Since D is compact, it is sufficient to show that the family $\{G(x)\}_{x\in K}$ has the finite intersection property. Let $\{x_1, \ldots, x_m\} \in K$ be a finite subset of K. We note that $A := conv(D \cup \{x_1, \ldots, x_m\})$ is a compact and convex subset of K (see for example [4]).

We now consider the point–to–set maps $F_1, F_2 : K \rightrightarrows A$, defined by

$$F_1(x) = \{z \in A : f(z,x) \notin \text{int } C(z)\}$$

and

$$F_2(x) = \{z \in A : p(z,x) \notin \text{int } C(z)\} \quad , \quad \forall x \in K.$$

From assumptions $4^o(a)$ and $4^o(c)$, we have

$$p(x,x) - f(x,x) \in \text{int } C(x) \quad \text{and} \quad p(x,x) \notin \text{int } C(x).$$

Then by Lemma 2(i) we have:

$$f(x,x) \notin \text{int } C(x)$$

and hence $F_1(x)$ is nonempty. $F_1(x)$ is compact, since it is a closed subset of a compact set A. Now we will prove that F_2 is a KKM–map. Suppose that there exists a finite subset $\{v_1, \ldots, v_n\}$ of A and $\alpha_i \geq 0, i = 1, \ldots, n$, with $\sum_{i=1}^n \alpha_i = 1$, such that:

$$\hat{v} = \sum_{i=1}^n \alpha_i v_i \notin \bigcup_{j=1}^n F_2(v_j);$$

then we have:

$$p(\hat{v}, v_j) \in \text{int } C(\hat{v}) \quad , \quad \text{for } 1 \leq j \leq n.$$

By assumption $4^o(b)$, we have:

$$p(\hat{v}, \hat{v}) \in \text{int } C(\hat{v}),$$

which contradicts to assumption $4^o(c)$. Hence F_2 is a KKM-map. From assumption $4^o(a)$ and Lemma 2(i), we have $F_2(x) \subseteq F_1(x)$, $\forall x \in K$. Indeed, let $z \in F_2(x)$; then $p(z, x) \notin \text{int } C(z)$ and by assumption $4^o(a)$, we have

$$p(z, x) - f(z, x) \in \text{int } C(z).$$

By Lemma 2(i), we get:

$$f(z, x) \notin \text{int } C(z).$$

This implies that F_1 is also a KKM-map. By Lemma 1, $\exists y \in A$ such that $y \in F_1(x)$, $\forall x \in K$. Hence

$$\exists y \in A \ \text{ such that } \ f(y, x) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

By assumption $4^o(d)$, we find that $y \in D$ and moreover $y \in G(x_i)$, for every $1 \leq i \leq m$. Hence $\{G(x)\}_{x \in K}$ has the finite intersection property. □

Let K be nonempty and convex subset of a Hausdorff topological vector space X, and Y be a topological vector space. Suppose that the bilinear form $\langle \cdot, \cdot \rangle$ is continuous. As a consequence of Theorem 1, we have the following result.

Corollary 1 [14]. Assume that

1^o $C : K \rightrightarrows Y$ is a point-to-set map such that $\forall x \in K$, $C(x)$ is a closed, pointed and convex cone with apex at the origin and with int $C(x) \neq \emptyset$;

2^o the point-to-set map $W : K \rightrightarrows Y$ defined as $W(x) = Y \setminus \{\text{int } C(x)\}$, $\forall x \in K$ is upper semicontinuous on K;

3^o $T : K \to L(X, Y)$ is continuous;

4^o $g : K \to K$ is continuous;

5^o there exists a function $p : K \times K \to Y$, such that

(a) $p(z,x) - \langle T(z), x - g(z)\rangle \in \text{int } C(z)$, $\forall x, z \in K$;

(b) the set $\{x \in K : p(z,x) \in \text{int } C(z)\}$ is convex, $\forall z \in K$;

(c) $p(x,x) \notin \text{int } C(x)$, $\forall x \in K$;

(d) there exists a nonempty, compact and convex subset $D \subset K$ such that $\forall z \in K \setminus D, \exists \tilde{x} \in D$, such that $\langle T(z), \tilde{x} - g(z)\rangle \in \text{int } C(z)$.

Then $\exists y \in D \subset K$ such that:

$$\langle T(y), x - g(y)\rangle \not\geq_{\text{int } C(y)} 0 \quad , \quad \forall x \in K.$$

The proof of this corollary follows by setting $f(z,x) = \langle T(z), x - g(z)\rangle$, $\forall x, z \in K$ and using Theorem 1.

Let $g, h : K \times K \to Y$ be two given functions such that:

$$g(x,x) = h(x,x) = 0 \quad , \quad \forall x \in K.$$

We now prove the existence result for the VEP (1.1) in the case where

$$f(z,x) = g(z,x) + h(z,x).$$

Theorem 2. Let K be a nonempty and convex subset of a Hausdorff topological vector space X, and Y be a topological vector space. Assume that

1^o $C : K \rightrightarrows Y$ is a point–to–set map, such that $\forall x \in K$, $C(x)$ is a closed, pointed and convex cone with apex at the origin and with int $C(x) \neq \emptyset$, and $P := \bigcap_{x \in K} C(x)$ such that int $P \neq \emptyset$;

2^o the point-to-set map $W : K \rightrightarrows Y$ defined by $W(x) := Y \setminus \{\text{int } C(x)\}$, $\forall x \in K$ is upper semicontinuous on K;

3^o the given function $g : K \times K \to Y$ has the following properties:

(i) $g(x,x) = 0$, $\forall x \in K$,

(ii) $g(z,x) + g(x,z) \in (C(z) \bigcap C(x))$, $\forall x, z \in K$,

(iii) $g(\cdot,\cdot)$ is continuous in the second argument and $\forall x, z \in K$, the function $t : [0,1] \mapsto g(tx + (1-t)z, x)$ is upper semicontinuous at $t = 0$ (hemicontinuity),

(iv) $g(z, \cdot)$ is P-function, $\forall z \in K$;

4^o the given function $h : K \times K \to Y$ has the following properties:

(i) $h(x, x) = 0$, $\forall x \in K$,

(ii) $h(\cdot, x)$ is continuous, $\forall x \in K$,

(iii) $h(z, \cdot)$ is P-function, $\forall z \in K$;

5^o there exists a nonempty, compact and convex subset $D \subset K$, such that $\forall z \in K \setminus D$, $\exists \tilde{x} \in D$ such that

$$g(z, \tilde{x}) + h(z, \tilde{x}) \in \text{int } C(z).$$

Then, $\exists y \in D \subset K$ such that:

$$g(y, x) + h(y, x) \not\geq_{\text{int } C(y)} 0 \quad , \quad \forall x \in K.$$

For the proof of above theorem we need the following two lemmas, for which the hypotheses remain the same as for Theorem 2.

Lemma 3. There exists $y \in D$, such that:

$$h(y, x) - g(x, y) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

Proof. Consider the set

$$G(x) = \{z \in D : h(z, x) - g(x, z) \notin \text{int } C(z)\} \quad , \quad \forall x \in K.$$

Then $\forall x \in K$, $G(x)$ is closed. Indeed, let $\{z_\lambda\}$ be a net in $G(x)$ such that $z_\lambda \to z$. Then $z \in D$ because D is compact and

$$h(z_\lambda, x) - g(x, z_\lambda) \notin \text{int } C(z_\lambda) \quad , \quad \forall \lambda$$

$$\Rightarrow \quad h(z_\lambda, x) - g(x, z_\lambda) \in W(z_\lambda) = Y \setminus \{\text{int } C(z_\lambda)\}.$$

Since $h(\cdot, x)$ and $g(x, \cdot)$ are continuous, we have:

$$h(z_\lambda, x) - g(x, z_\lambda) \to h(z, x) - g(x, z).$$

The upper semicontinuity of point-to-set map W implies that $h(z, x) - g(x, z) \in W(z)$ and hence $h(z, x) - g(x, z) \notin \text{int } C(z)$. Hence $z \in G(x)$

and thus $G(x)$ is closed. Now, we will prove that G is a KKM-Map. Let $\{z_1, \ldots, z_n\}$ be a finite subset of D and $\alpha_i \geq 0, i = 1, \ldots, n$, such that $\sum_{i=1}^n \alpha_i = 1$. Assume that

$$\hat{z} = \sum_{i=1}^n \alpha_i z_i \notin \bigcup_{j=1}^n G(z_j).$$

Then

$$h(\hat{z}, z_j) - g(z_j, \hat{z}) \in \text{int } C(\hat{z}) \quad , \quad \forall j = 1, \cdots, n. \tag{3.2}$$

From the assumption 3^o(ii), we have:

$$g(z_j, \hat{z}) + g(\hat{z}, z_j) \in C(\hat{z}). \tag{3.3}$$

By adding (3.2) and (3.3), we obtain:

$$h(\hat{z}, z_j) + g(\hat{z}, z_j) \in \text{int } C(\hat{z}) + C(\hat{z}) \subseteq \text{int } C(\hat{z}) \quad , \quad \forall j.$$

Since $C(\hat{z})$ is the convex cone, we have:

$$\sum_{j=1}^n \alpha_j h(\hat{z}, z_j) + \sum_{j=1}^n \alpha_j g(\hat{z}, z_j) \in \text{int } C(\hat{z}). \tag{3.4}$$

Since $h(\hat{z}, \cdot)$ and $g(\hat{z}, \cdot)$ are P-function, we have $h(\hat{z}, \cdot) + g(\hat{z}, \cdot)$ is also P-function and hence

$$h(\hat{z}, \hat{z}) - \sum_{j=1}^n \alpha_j h(\hat{z}, z_j) + g(\hat{z}, \hat{z}) - \sum_{j=1}^n \alpha_j g(\hat{z}, z_j) \in P. \tag{3.5}$$

From (3.4) and (3.5), we have:

$$h(\hat{z}, \hat{z}) + g(\hat{z}, \hat{z}) \in \text{int } C(\hat{z}) + \text{int } P \subseteq \text{int } C(\hat{z}),$$

a contradiction with $g(\hat{z}, \hat{z}) = 0$ and $h(\hat{z}, \hat{z}) = 0$. Hence G is a KKM-map. Since $G(x)$ is contained in a compact set D, by Lemma 1, $\exists y \in D$ such that $y \in G(x)$, $\forall x \in K$. Hence $\exists y \in D$ such that:

$$h(y, x) - g(x, y) \notin \text{int } C(y) \quad , \quad \forall x \in K. \qquad \square$$

Lemma 4. The following statements are equivalent:

(1) $y \in D \ : \ h(y,x) - g(x,y) \notin \text{int } C(y) \quad , \quad \forall x \in K.$

(2) $y \in D \ : \ h(y,x) + g(y,x) \notin \text{int } C(y) \quad , \quad \forall x \in K.$

Proof. Let (2) hold. Then

$$y \in D \ : \ h(y,x) + g(y,x) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

From assumption $3^o(ii)$, we have

$$g(y,x) + g(x,y) \in (C(y) \cap C(x)) \subseteq C(y).$$

Then by Lemma 2(ii), we find:

$$y \in D \text{ such that } h(y,x) - g(x,y) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

Conversely, let (1) hold, then

$$y \in D \ : \ h(y,x) - g(x,y) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

Let $y_t = tx + (1-t)y \in K$, $0 < t \leq 1$ and since $C(y)$ is the convex cone, we have

(3.6) $$tg(y_t,x) - (1-t)g(y_t,y) - tg(y_t,x) + (1-t)h(y,y_t) \notin \text{int } C(y).$$

Since $g(y_t,\cdot)$ is the P-function, we find

$$g(y_t,y_t) - tg(y_t,x) - (1-t)g(y_t,y) \in P.$$

Since $g(y_t,y_t) = 0$, then we see that

(3.7) $$-tg(y_t,x) - (1-t)g(y_t,y) \in P = \bigcap_{z \in K} C(z) \subseteq C(y).$$

By (3.6) and (3.7), and Lemma 2(iii), we have:

(3.8) $$tg(y_t,x) + (1-t)h(y,y_t) \notin \text{int } C(y).$$

Since $h(y,\cdot)$ is P-function and $h(y,y) = 0$, we have

$$h(y,y_t) - th(y,x) \in P = \bigcap_{z \in K} C(z) \subseteq C(y).$$

Since $C(y)$ is the convex cone, we have:

$$(1-t)h(y,y_t) - t(1-t)h(y,x) \in C(y). \tag{3.9}$$

From (3.8), (3.9) and Lemma 2(iv), we have:

$$tg(y_t,x) + t(1-t)h(y,x) \notin \text{int } C(y).$$

Dividing by t, we get

$$g(y_t,x) + (1-t)h(y,x) \notin \text{int } C(y)$$

and therefore

$$g(y_t,x) + (1-t)h(y,x) \in W(y).$$

Letting $t \searrow 0$ and thereby $y_t \to y$. Since $W(y)$ is closed and g is hemicontinuous in the first argument, we have

$$g(y,x) + h(y,x) \in W(y)$$

and therefore

$$g(y,x) + h(y,x) \notin \text{int } C(y). \qquad \square$$

Proof of Theorem 2. Let $\{x_1,\ldots,x_n\}$ be a finite subset of K and $B = conv(D \bigcup \{x_1,\ldots,x_n\})$. Then B is a compact and convex subset of K. Then by Lemma 3, $\exists \bar{y} \in B$ such that

$$h(\bar{y},x) - g(x,\bar{y}) \notin \text{int } C(\bar{y}) \quad , \quad \forall x \in K,$$

in particular,

$$h(\bar{y},x_i) - g(x_i,\bar{y}) \notin \text{int } C(\bar{y}) \quad , \quad \forall i = 1,\ldots,n.$$

So every finite subfamily of the family of closed sets

$$H(x) = \{z \in B : h(z,x) - g(x,z) \notin \text{int } C(z)\} \quad , \quad \forall x \in K$$

has nonempty intersection and since B is compact, $\bigcap_{x\in K} H(x) \neq \emptyset$. From Lemma 4, we obtain $\bigcap_{x\in K} G(x) \neq \emptyset$. Hence $\exists y \in B$ such that

$$h(y,x) + g(y,x) \notin \text{int } C(y) \quad , \quad \forall x \in K.$$

From assumption 5°, we have $y \in D$ such that:

$$h(y,x) + g(y,x) \notin \text{int } C(y) \quad , \quad \forall x \in K$$

and the proof is completed. □

4. STRONGLY NONLINEAR VECTOR VARIATIONAL INEQUALITIES

Let X be a Hausdorff topological vector space and Y be a topological vector space. Let K be a nonempty subset of X and $\{C(x) : x \in K\}$ be a family of closed, pointed and convex cones in Y with apexes at the origin and with int $C(x) \neq \emptyset$, $\forall x \in K$. Then we consider the problem of finding $y \in K$ such that:

$$\text{(4.1)} \qquad \langle T(y), x - y \rangle - \langle A(y), x - y \rangle \not\leq_{\text{int } C(y)} 0 \quad , \quad \forall x \in K.$$

where $T, A : K \to L(X,Y)$ are nonlinear operators. (4.1) shall be called Strongly Nonlinear Vector Variational Inequality (for short, SNVVI).

If $Y = \mathbf{R}$ and $C(x) = \mathbf{R}_-$, $\forall x \in X$, then the SNVVI becomes the problem of finding $y \in K$, such that:

$$\text{(4.2)} \qquad \langle T(y), x - y \rangle \geq \langle A(y), x - y \rangle \quad , \quad \forall x \in K,$$

where $T, A : K \to X^*$ (the topological dual of X) are nonlinear operators. (4.2) is called Strongly Nonlinear Variational Inequality (for short, SNVI). It has been introduced and studied by Noor [12] in the setting of Hilbert spaces.

Definition 4. A map $T : K \to L(X,Y)$ is said to be $C-$*operator*, iff

$$\langle T(x) - T(z), x - z \rangle \in (C(x) \cap C(z)) \quad , \quad \forall x, z \in K.$$

When C contains or is contained in the positive orthant, then T is called *C–monotone.*

Now we present an application of Theorem 3. The bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous.

Theorem 3. Let K be a nonempty and convex subset of a Hausdorff topological vector space X, and let Y be a topological vector space. Assume that

1° $C : K \rightrightarrows Y$ is a point-to-set map such that $\forall x \in K$, $C(x)$ is a closed, pointed and convex cone with apex at the origin and with int $C(x) \neq \emptyset$, and $P := \bigcap_{x \in K} C(x)$ such that int $P \neq \emptyset$;

2° the point-to-set map $W : K \rightrightarrows Y$ defined as $W(x) = Y \setminus \{\text{int } C(x)\}$, $\forall x \in K$ is upper semicontinuous on K;

3° $T : K \to L(X, Y)$ is C-operator and hemicontinuous;

4° $A : K \to L(X, Y)$ is continuous on K;

5° there exists a nonempty, compact and convex subset $D \subset K$, such that $\forall z \in K \setminus D$, $\exists \tilde{x} \in D$ such that:

$$\langle T(z), \tilde{x} - z \rangle - \langle A(z), \tilde{x} - z \rangle \in \text{int } C(z).$$

Then, $\exists y \in D \subset K$ such that:

$$\langle T(y), x - y \rangle - \langle A(y), x - y \rangle \not\geq_{\text{int } C(y)} 0 \quad , \quad \forall x \in K.$$

Proof. Let $g(z, x) = \langle T(z), x - z \rangle$ and $h(z, x) = -\langle A(z), x - z \rangle$. Since $h(z, x) = -\langle A(z), x - z \rangle$ is affine in the second argument, it follows that $h(z, \cdot)$ is P-function. Then, all the assumptions of Theoerm 2 are satisfied. □

If $Y = \mathbb{R}$ and $C(x) = \mathbb{R}_-, \forall x \in K$, then Theorem 3 reduces to the following result.

Corollary 2. Let K be a nonempty and convex subset of a Hausdorff topological vector space X, and Y be a topological vector space. Assume that

1° $T : K \to X^*$ is monotone and hemicontinuous;

2° $A : K \to X^*$ is continuous on K;

3^o there exists a nonempty, compact and convex subset $D \subset K$, such that: $\forall z \in K \setminus D$, $\exists \tilde{x} \in D$ such that;

$$\langle T(z), \tilde{x} - z \rangle < \langle A(z), \tilde{x} - z \rangle.$$

Then, $\exists y \in D \subset K$ such that:

$$\langle T(y), x - y \rangle \geq \langle A(y), x - y \rangle \quad , \quad \forall x \in K.$$

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