TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 3, pp. 703-716, August 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

GENERALIZED VARIATIONAL INCLUSIONS AND H-RESOLVENT EQUATIONS WITH H-ACCRETIVE OPERATORS

Rais Ahmad and Qamrul Hasan Ansari

Abstract. In this paper, we consider a more general form of variational inclusions, called generalized variational inclusion (for short, GVI). In connection with GVI, we also consider a generalized resolvent equation with H-resolvent operator, called H-resolvent equation (for short, H-RE). We suggest iterative algorithms to compute the approximate solutions of GVI and H-RE. The existence of a unique solution of GVI and H-RE and convergence of iterative sequences generated by the proposed algorithms are also studied. Several special cases are also discussed.

1. Introduction

In the last decade, variational inclusions, generalized forms of variational inequalities, have been extensively studied and generalized in various directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance and applied sciences, etc; See for example [1-12] and references therein. One of the most interesting and important aspects of the theory of variational inclusions is to develop an efficient and implementable iterative algorithm to compute the approximate solution of a variational inclusion. In the recent past, a significant work has been done in this direction; See for example [2-13] and references therein. The resolvent operator technique is interesting and important to study the existence of a solution and to develop iterative algorithms for different kind of variational inclusions. Very recently, Fang and Huang [8] introduced a new class of H-accretive operators in the setting of Banach spaces and

Received December 9, 2006.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 49J40, 47J20, 47H06, 49J53.
Key words and phrases: Generalized variational inclusions, H-resolvent equations, H-resolvent operators, H-accretive mappings, Iterative algorithms, Convergence results.
The second author is grateful to the Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia for providing excellent research facilities to carry out his part of this research work.

extended the concept of resolvent operators associated with the classical m-accretive operators to the new H-accretive operators. By using this new resolvent operator technique, they studied the approximate solutions of a class of variational inclusions with H-accretive operators in the setting of Banach spaces.

In this paper, we consider a more general form of variational inclusions, called *generalized variational inclusion* (for short, GVI), which contains the variational inclusions studied by Fang and Huang [8] and many known variational inclusions considered and studied in the literature. In connection with GVI, we also consider a generalized resolvent equation with H-resolvent operator, called *H-resolvent equation* (for short, H-RE). We suggest iterative algorithms to compute the approximate solutions of GVI and H-RE. The existence of a unique solution of GVI and H-RE and convergence of iterative sequences generated by the proposed algorithms are also studied. Our results are new and represent a significant improvement of previously known results. Some special cases are also discussed.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X is a real Banach space with its norm $\|\cdot\|$, X^* is the topological dual of X, $\langle .,. \rangle$ is the pairing between X and X^*, d is the metric induced by the norm $\|\cdot\|$, 2^X (respectively, $CB(X)$) is the family of all nonempty (respectively, nonempty closed and bounded) subsets of X, and $\mathcal{H}(.,.)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{H}(P,Q) = \max\left\{\sup_{x\in P} d(x,Q), \sup_{y\in Q} d(P,y)\right\},$$

where $d(x,Q) = \inf_{y\in Q} d(x,y)$ and $d(P,y) = \inf_{x\in P} d(x,y)$.

The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\} \quad \text{for all } x \in X,$$

where $q > 1$ is a constant. For $q = 2$, the generalized duality mapping coincides with the usual normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex.

The *modulus of smoothness* of X [14] is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 \ : \ \|x\| \le 1, \ \|y\| \le t\right\}.$$

A Banach space X is called *uniformly smooth* [14] if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called *q-uniformly smooth* [14] if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

Remark 2.1. All Hilbert spaces, L_p (or l_p) ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while, for $1 < p \leq 2$, L_p (or l_p) and W_m^p spaces are p-uniformly smooth; For further detail, we refer to [14] and references therein. We notice that J_q is single-valued if X is uniformly smooth.

The following result due to Xu [15] is very crucial in establishing our main results of this paper.

Theorem 2.1. [15] *Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.1. [8] Let $H : X \to X$ be an operator. The operator $T : X \to X$ is said to be

(i) *accretive* if

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq 0 \quad \text{for all } x, y \in X;$$

(ii) *strictly accretive* if

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq 0 \quad \text{for all } x, y \in X$$

and the equality holds if and only $x = y$;

(iii) *strongly accretive* if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq r\|x - y\|^q \quad \text{for all } x, y \in X;$$

(iv) *strongly accretive with respect to H* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), J_q(H(x) - H(y))\rangle \geq \gamma\|x - y\|^q \quad \text{for all } x, y \in X;$$

(v) *Lipschitz continuous* if there exists a constant $\sigma > 0$ such that

$$\|T(x) - T(y)\| \leq \sigma\|x - y\| \quad \text{for all } x, y \in X.$$

Definition 2.2. [8] A multivalued map $M : X \to 2^X$ is said to be

(i) *accretive* if for all $x, y \in X$,

$$\langle u - v, J_q(x - y)\rangle \geq 0 \quad \text{for all } u \in M(x) \text{ and } v \in M(y);$$

(ii) m-*accretive* if M is accretive and $(I + \lambda M)(X) = X$ for all $\lambda > 0$, where I denotes the identity mapping on X.

Definition 2.3. [8] Let $H : X \to X$ be an operator. A multivalued map $M : X \to 2^X$ is said to be H-*accretive* if M is accretive and $(H + \lambda M)(X) = X$ for all $\lambda > 0$.

Remark 2.2. If $H = I$, then Definition 2.3 reduces to the usual definition of m-accretive operator.

Theorem 2.2. [8] *Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be a H-accretive multivalued map. Then the operator $(H + \lambda M)^{-1}$ is single-valued, where $\lambda > 0$ is a constant.*

Based on Theorem 2.2, the following H-resolvent operator $R^H_{M,\lambda}$ associated with H and M is defined by Fang and Huang [8].

Definition 2.4. [8] Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be an H-accretive multivalued map. The H-*resolvent operator* $R^H_{M,\lambda} : X \to X$ associated with H and M is defined by

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u) \quad \text{for all } u \in H.$$

Theorem 2.3. [8] *Let $H : X \to X$ be a strongly accretive operator with constant r and $M : X \to 2^X$ be an H-accretive multivalued map. Then the H-resolvent operator $R^H_{M,\lambda} : X \to X$ associated with H and M is Lipschitz continuous with constant $1/r$, that is,*

$$\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\| \leq \frac{1}{r}\|u - v\| \quad \text{for all } u, v \in X.$$

3. Generalized Variational Inclusions

Let $N, W : X \times X \to X$ be two single-valued mappings and $A, B, C, D : X \to CB(X)$ be multivalued mappings. We consider the following *generalized variational inclusion problem*:

$$\text{(3.1)} \qquad \text{(GVIP)} \begin{cases} \text{Find } u \in X,\ x \in A(u),\ y \in B(u), z \in C(u) \\ \qquad \text{and } w \in D(u) \text{ such that} \\ 0 \in N(x, y) - W(z, w) + M(u). \end{cases}$$

A problem similar to (GVIP) is studied by Shi and Liu [12].

For C and D are single-valued identity mapping and $W(u,u) = -h(u)$, where $h : X \to X$, (GVIP) is introduced and studied by Ahmad et al. [3]. They developed the iterative algorithms to find the approximate solutions of their problem. They also studied the generalized resolvent operator equation for m-accretive mapping in the setting of real Banach spaces.

When $W \equiv 0$, then (GVIP) reduces to the problem of finding $u \in X,\ x \in A(u)$ and $y \in B(u)$ such that

$$0 \in N(x,y) + M(u). \tag{3.2}$$

A problem similar to (3.2) is considered and studied by Chang et al. [5] in the setting of Banach spaces.

If $W \equiv 0$, A and B are single-valued identity mappings and $N(x,y) = N(x)$ for all $x, y \in X$, then (GVIP) becomes the problem of finding $u \in X$ such that

$$0 \in N(u) + M(u). \tag{3.3}$$

This problem is considered and studied by Fang and Huang [8] in the setting of q-uniformly smooth Banach spaces with H-accretive operator M.

It is easy to see that (GVIP) includes many more known variational inclusions considered and studied in the literature.

To suggest the iterative algorithm for computing the approximate solutions of (GVIP), we establish the following equivalence between (GVIP) and a fixed point problem.

Lemma 3.1. *Let $H : X \to X$ be a strictly accretive operator and $M : X \to 2^X$ be an H-accretive multivalued map. Then (u,x,y,z,v), where $u \in X,\ x \in A(u),\ y \in B(u), z \in C(u)$ and $v \in D(u)$, is a solution of (GVIP) if and only if it satisfies*

$$u = R^H_{M,\lambda}[H(u) - \lambda\{N(x,y) - W(z,v)\}], \tag{3.4}$$

where $\lambda > 0$ is a constant.

Proof. Let u satisfy (3.4). Then by using the definition of H-resolvent operator $R^H_{M,\lambda}$, we have

$$\begin{aligned} u &= (H+\lambda M)^{-1}[H(u) - \lambda\{N(x,y) - W(z,v)\}] \\ &\Leftrightarrow H(u) - \lambda\{N(x,y) - W(z,v)\} \in H(u) + \lambda M(u) \\ &\Leftrightarrow -\{N(x,y) - W(z,v)\} \in M(u) \\ &\Leftrightarrow 0 \in N(x,y) - W(z,v) + M(u). \end{aligned}$$

■

In view of above lemma, we now propose the following iterative algorithm for computing the approximate solutions of (GVIP).

Algorithm 3.1. For any given $u_0 \in X$, $x_0 \in A(u_0)$, $y_0 \in B(u_0)$, $z_0 \in C(u_0)$ and $v_0 \in D(u_0)$, compute u_{n+1}, x_{n+1}, y_{n+1}, z_{n+1} and v_{n+1} by the following rules:

$$u_{n+1} = R^H_{M,\lambda}[H(u_n) - \lambda\{N(x_n, y_n) - W(z_n, v_n)\}], \tag{3.5}$$

$$x_{n+1} \in A(u_{n+1}),\ \|x_{n+1} - x_n\| \leq \mathcal{H}(A(u_{n+1}), A(u_n)), \tag{3.6}$$

$$y_{n+1} \in B(u_{n+1}),\ \|y_{n+1} - y_n\| \leq \mathcal{H}(B(u_{n+1}), B(u_n)), \tag{3.7}$$

$$z_{n+1} \in C(u_{n+1}),\ \|z_{n+1} - z_n\| \leq \mathcal{H}(C(u_{n+1}), C(u_n)), \tag{3.8}$$

$$v_{n+1} \in D(u_{n+1}),\ \|v_{n+1} - v_n\| \leq \mathcal{H}(D(u_{n+1}), D(u_n)), \tag{3.9}$$

$n = 0, 1, 2, 3, \ldots,$ where $\lambda > 0$ is a constant.

Remark 3.1. For suitable choices of the mappings involved in the formulation of (GVIP) we can easily derive the known iterative algorithms for different kinds of variational inclusion from Algorithm 3.1.

Next, we prove the existence of a unique solution of (GVIP) and study the convergence of iterative sequences generated by Algorithm 3.1.

Theorem 3.1. *Let X be a real q-uniformly smooth Banach space and let $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constant r and τ, respectively. Let N and W be both Lipschitz continuous in both arguments with constants λ_{N_1}, λ_{N_2} and λ_{W_1}, λ_{W_2}, respectively, also let A, B, C and D be $\mathcal{H}$-Lipschitz continuous with constant α, β, γ and μ, respectively. Suppose that $M : X \to 2^X$ is an H-accretive multivalued map and there exists $\lambda > 0$ such that*

$$0 < \frac{1}{r}\sqrt[q]{\tau^q - (q\lambda - \lambda^q c_q)[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]^q} < 1, \tag{3.10}$$

where c_q is the same as in Theorem 2.1. Then there exists a unique solution (u, x, y, z, v) of (GVIP) and the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 converge strongly to u, x, y, z and v, respectively.

Proof. From Algorithm 3.1 and by using Lipschitz continuity of H-resolvent operator $R^H_{M,\lambda}$, we have

$$
\begin{aligned}
\|u_{n+1}-u_n\| &= \|R^H_{M,\lambda}\left[H(u_n)-\lambda\{N(x_n,y_n)-W(z_n,v_n)\}\right] \\
&\quad -R^H_{M,\lambda}\left[H(u_{n-1})-\lambda\{N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})\}\right]\| \\
&\le \frac{1}{r}\|H(u_n)-H(u_{n-1})-\lambda[\{N(x_n,y_n)-N(x_{n-1},y_{n-1})\} \\
&\quad -\{W(z_n,v_n)-W(z_{n-1},v_{n-1})\}]\|.
\end{aligned} \tag{3.11}
$$

By Theorem 2.1, we have

$$
\begin{aligned}
&\|H(u_n)-H(u_{n-1})-\lambda\left[\{N(x_n,y_n)-N(x_{n-1},y_{n-1})\}\right. \\
&\left.-\{W(z_n,v_n)-W(z_{n-1},v_{n-1})\}\right]\|^q \\
\le\ &\|H(u_n)-H(u_{n-1})\|^q-q\lambda\langle N(x_n,y_n)-N(x_{n-1},y_{n-1}) \\
&-\{W(z_n,v_n)-W(z_{n-1},v_{n-1})\}, J_q(H(u_n)-H(u_{n-1}))\rangle \\
&+\lambda^q c_q\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-(W(z_n,v_n)-W(z_{n-1},v_{n-1}))\|^q
\end{aligned} \tag{3.12}
$$

Again by using Theorem 2.1, we obtain

$$
\begin{aligned}
&\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-[W(z_n,v_n)-W((z_{n-1},v_{n-1})]\|^q \\
\le\ &\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^q \\
&-(q-c_q)\|W(z_n,v_n)-W(z_{n-1},v_{n-1})\|^q.
\end{aligned} \tag{3.13}
$$

By using Lipschitz continuity of N with constant λ_{N_1} for the first argument and λ_{N_2} for the second argument and $\mathcal{H}$-Lipschitz continuity of A and B with constants α and β, respectively, we have

$$
\begin{aligned}
&\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\| \\
=\ &\|N(x_n,y_n)-N(x_n,y_{n-1})+N(x_n,y_{n-1})-N(x_{n-1},y_{n-1})\| \\
\le\ &\|N(x_n,y_n)-N(x_n,y_{n-1})\|+\|N(x_n,y_{n-1})-N(x_{n-1},y_{n-1})\| \\
\le\ &\lambda_{N_2}\|y_n-y_{n-1}\|+\lambda_{N_1}\|x_n-x_{n-1}\| \\
\le\ &\lambda_{N_2}\mathcal{H}(B(u_n),B(u_{n-1}))+\lambda_{N_1}\mathcal{H}(A(u_n),A(u_{n-1})) \\
\le\ &\lambda_{N_2}\beta\|u_n-u_{n-1}\|+\lambda_{N_1}\alpha\|u_n-u_{n-1}\| \\
=\ &(\lambda_{N_1}\alpha+\lambda_{N_2}\beta)\|u_n-u_{n-1}\|
\end{aligned} \tag{3.14}
$$

Therefore,

$$
\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^q \le (\lambda_{N_1}\alpha+\lambda_{N_2}\beta)^q\|u_n-u_{n-1}\|^q. \tag{3.15}
$$

Similarly, by using Lipschitz continuity of W with constant λ_{W_1} for the first argument and λ_{W_2} for the second argument and $\mathcal{H}$-Lipschitz continuity of C and D with constants γ and μ, respectively, we obtain

$$\|W(z_n, v_n) - W(z_{n-1}, v_{n-1})\|^q \leq (\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q \|u_n - u_{n-1}\|^q. \tag{3.16}$$

In view of (3.15) and (3.16), (3.13) becomes
$\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - [W(z_n, v_n) - W(z_{n-1}, v_{n-1})]\|^q$

$$\leq [(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]\|u_n - u_{n-1}\|^q. \tag{3.17}$$

Therefore, (3.12) becomes

$$\begin{aligned}
&\|H(u_n) - H(u_{n-1}) - \lambda\,[\{N(x_n, y_n) - N(x_{n-1}, y_{n-1})\} \\
&-\{W(z_n, v_n) - W(z_{n-1}, v_{n-1})\}]\,\|^q \\
\leq\ & \tau^q\|u_n - u_{n-1}\| - (q\lambda - \lambda^q c_q)[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q \\
&-(q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]\|u_n - u_{n-1}\|^q \\
=\ & [\tau^q - (q\lambda - \lambda^q c_q)[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q \\
&-(q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]]\,\|u_n - u_{n-1}\|^q.
\end{aligned} \tag{3.18}$$

By using above inequality (3.18), (3.11) becomes

$$\|u_{n+1} - u_n\| \leq k\|u_n - u_{n-1}\|, \tag{3.19}$$

where

$$k = \frac{1}{r}\sqrt[q]{\tau^q - (q\lambda - \lambda^q c_q)[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]^q}. \tag{3.20}$$

By condition (3.10), $0 \leq k < 1$ and hence $\{u_n\}$ is a Cauchy sequence in a Banach space X. So, there exists $u \in X$ such that $u_n \to u$ as $n \to \infty$. By (3.6)–(3.9) and Lipschitz continuity of A, B, C and D it follows that there exist x, y, z and v in X such that $x_n \to x$, $y_n \to y$, $z_n \to z$, and $v_n \to v$.

From Algorithm 3.1, we have

$$u = R^H_{M,\lambda}[H(u) - \lambda\{N(x, y) - W(z, v)\}].$$

and therefore by Lemma 3.1, (u, x, y, z, v) is a solution of (GVIP).

It is remain to show that $x \in A(u)$, $y \in B(u)$, $z \in C(u)$, and $v \in D(u)$. In fact, since $x_n \in A(u_n)$ and

$$\begin{aligned}
d(x_n, A(u)) &\leq \max\left\{d(x_n, A(u)),\ \sup_{b\in A(u)} d(A(u_n), b)\right\} \\
&\leq \max\left\{\sup_{a\in A(u_n)} d(a, A(u)), \sup_{b\in A(u)} d(A(u_n), b)\right\} \\
&= \mathcal{H}(A(u_n), A(u)),
\end{aligned}$$

we have

$$\begin{aligned} d(x, A(u)) &\leq \|x - x_n\| + d(x_n, A(u)) \\ &\leq \|x - x_n\| + \mathcal{H}(A(u_n), A(u)) \\ &\leq \|x - x_n\| + \alpha\|u_n - u\| \to 0 \quad \text{as } n \to +\infty, \end{aligned}$$

which implies that $d(x, A(u)) = 0$. Since $A(u) \in CB(E)$ [16], it follows that $x \in A(u)$. Similarly, we can prove that $y \in B(u)$, $z \in C(u)$, and $v \in D(u)$.

Let $(u^*, x^*, y^*, z^*, v^*)$ be another solution of (GVIP). Then by Lemma 3.1, we have

$$u^* = R^H_{M,\lambda}[H(u^*) - \lambda\{N(x^*, y^*) - W(z^*, v^*)\}].$$

From above two equations and by using the similar arguments as above, we obtain

$$\|u - u^*\| \leq k\|u - u^*\|,$$

where k is the same as defined in (3.20). Since $0 \leq k < 1$, we get $u = u^*$, and so by Algorithm 3.1 $(u^*, x^*, y^*, z^*, v^*)$ is a unique solution of (GVIP). This completes the proof. ■

4. H-Resolvent Equations

In connection with (GVIP), we consider the following *H-resolvent equation problem:*

$$(4.1) \qquad (H\text{-REP}) \begin{cases} \text{Find } s, u \in X,\ x \in A(u),\ y \in B(u), z \in C(u) \\ \qquad \text{and } v \in D(u) \text{ such that} \\ N(x, y) - W(z, v) + \lambda^{-1} J^H_{M,\lambda}(s) = 0, \end{cases}$$

where $J^H_{M,\lambda} = I - H(R^H_{M,\lambda})$, I is the identity operator, $R^H_{M,\lambda}$ is the H-resolvent operator and $\lambda > 0$ is a constant. The equation (4.1) is called *H-resolvent equation*.

When H is the identity map, (4.1) is called *resolvent equation*. In this case, for C and D are single-valued identity mappings and $W(u, u) = -h(u)$, where $h : X \to X$, then the above problem is introduced and studied by Ahmad et al. [3].

Now we present an equivalence between (GVIP) and (H-REP).

Proposition 4.1. *The (GVIP) has a solution (u, x, y, z, v) with $u \in X$, $x \in A(u)$, $y \in B(u)$, $z \in C(u)$ and $v \in D(u)$ if and only if (H-REP) has a solution*

(s, u, x, y, z, v) *with* $s, u \in X$, $x \in A(u)$, $y \in B(u)$, $z \in C(u)$ *and* $v \in D(u)$, *where*

$$u = R^H_{M,\lambda}(s) \tag{4.2}$$

and

$$s = H(u) - \lambda\{N(x, y) - W(z, v)\}, \tag{4.3}$$

$\lambda > 0$ *is a constant.*

Proof. Let (u, x, y, z, v) be a solution of (GVIP). Then by Lemma 3.1, it is a solution of the following equation

$$u = R^H_{M,\lambda}[H(u) - \lambda\{N(x, y) - W(z, v)\}]. \tag{4.4}$$

Let $s = H(u) - \lambda\{N(x, y) - W(z, v)\}$, then from (4.4), we have $u = R^H_{M,\lambda}(s)$. By using the fact that $J^H_{M,\lambda} = I - H(R^H_{M,\lambda})$, we obtain

$$\begin{aligned} s &= H(R^H_{M,\lambda}(s)) - \lambda\{N(x, y) - W(z, v)\} \\ \Leftrightarrow\ s - H(R^H_{M,\lambda}(s)) &= -\lambda\{N(x, y) - W(z, v)\} \\ \Leftrightarrow\ \left[I - H(R^H_{M,\lambda})\right](s) &= -\lambda\{N(x, y) - W(z, v)\} \\ \Leftrightarrow\ J^H_{M,\lambda}(s) &= -\lambda\{N(x, y) - W(z, v)\}. \end{aligned}$$

Hence $N(x, y) - W(z, v) + \lambda^{-1} J^H_{M,\lambda}(s) = 0$. ∎

Based on Proposition 4.1, we suggest the following iterative method to compute the approximate solution of (H-REP).

Algorithm 4.1. For any given s_0, $u_0 \in X$, $x_0 \in A(u_0)$, $y_0 \in B(u_0)$, $z_0 \in C(u_0)$ and $v_0 \in D(u_0)$, compute $\{s_n\}$, $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ by the iterative schemes

$$u_{n+1} = R^H_{M,\lambda}(s_{n+1}) \tag{4.5}$$

$$x_{n+1} \in A(u_{n+1})\ :\ \|x_{n+1} - x_n\| \le \mathcal{H}(A(u_{n+1}), A(u_n)) \tag{4.6}$$

$$y_{n+1} \in B(u_{n+1})\ :\ \|y_{n+1} - y_n\| \le \mathcal{H}(B(u_{n+1}), B(u_n)) \tag{4.7}$$

$$z_{n+1} \in C(u_{n+1})\ :\ \|z_{n+1} - z_n\| \le \mathcal{H}(C(u_{n+1}), C(u_n)) \tag{4.8}$$

$$v_{n+1} \in D(u_{n+1}) \ : \ \|v_{n+1} - v_n\| \leq \mathcal{H}(D(u_{n+1}), D(u_n)) \tag{4.9}$$

$$s_{n+1} = H(u_n) - \lambda\{N(x_n, y_n) - W(z_n, v_n)\}, \tag{4.10}$$

$n = 0, 1, 2, 3, \ldots$, where $\lambda > 0$ is a constant.

Now we study the existence of a solution of (H-REP) and the convergence of iterative sequences generated by the above algorithm to the exact solution of (H-REP).

Theorem 4.1. *Let X be a real q-uniformly smooth Banach space and $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constant r and τ, respectively. Let N and W be both Lipschitz continuous in both arguments with constants λ_{N_1}, λ_{N_2} and λ_{W_1}, λ_{W_2}, respectively, also let A, B, C and D be $\mathcal{H}$-Lipschitz continuous with constant α, β, γ and μ, respectively. Suppose that $M : X \to 2^X$ is an H-accretive multivalued map and there exists $\lambda > 0$ such that condition (3.10) holds. Then there exists a unique solution (s, u, x, y, z, v) of (H-REP) with $s, u \in X$, $x \in A(u)$, $y \in B(u)$, $z \in C(u)$ and $v \in D(u)$, and the iterative sequences $\{s_n\}$, $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ generated by Algorithm 4.1 converge to s, u, x, y, z and v strongly in X, respectively.*

Proof. From Algorithm 4.1 and (3.18), we have

$$\begin{aligned} &\|s_{n+1} - s_n\| \\ = \ &\|H(u_n) - \lambda\{N(x_n, y_n) - W(z_n, v_n)\} - (H(u_{n-1}) \\ &-\lambda\{N(x_{n-1}, y_{n-1}) - W(z_{n-1}, v_{n-1})\})\| \\ \leq \ &\sqrt[q]{(\tau^q - (q\lambda - \lambda^q c_q))\,[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q]^q} \\ &\|u_n - u_{n-1}\| \end{aligned} \tag{4.11}$$

By (4.5), we obtain

$$\begin{aligned} &\|u_n - u_{n-1}\| = \|u_n - u_{n-1} + u_n - u_{n-1} - R^H_{M,\lambda}(s_n) + R^H_{M,\lambda}(s_{n-1})\| \\ \leq \ &2\|u_n - u_{n-1}\| - \|R^H_{M,\lambda}(s_n) - R^H_{M,\lambda}(s_{n-1})\| \\ \leq \ &2\|u_n - u_{n-1}\| - \frac{1}{r}\|s_n - s_{n-1}\| \text{ since } R^H_{M,\lambda} \text{ is } \tfrac{1}{r}\text{-Lipschitz continuous.} \end{aligned}$$

Therefore,

$$\|u_n - u_{n-1}\| \leq \frac{1}{r}\|s_n - s_{n-1}\|. \tag{4.12}$$

By combining (4.11) and (4,12), we get

$$\|s_{n+1} - s_n\| \leq k\|s_n - s_{n-1}\|, \tag{4.13}$$

where

$$k = \frac{1}{r}\sqrt[q]{(\tau^q - (q\lambda - \lambda^q c_q))\left[(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - c_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q\right]^q}.$$

From (3.10), it follows that $0 \leq k < 1$. Consequently, from (4.13), we see that the sequence $\{s_n\}$ is a Cauchy sequence in a Banach space X. So there exists $s \in X$ such that $s_n \to s$ as $n \to \infty$. From (4.12), we know that the sequence $\{u_n\}$ is a Cauchy sequence in X, so there exists $u \in X$ such that $u_n \to u$. Also from Algorithm 4.1, we have

$$\|x_{n+1} - x_n\| \leq \mathcal{H}(A(u_{n+1}), A(u_n)) \leq \alpha\|u_{n+1} - u_n\|$$
$$\|y_{n+1} - y_n\| \leq \mathcal{H}(B(u_{n+1}), B(u_n)) \leq \beta\|u_{n+1} - u_n\|$$
$$\|z_{n+1} - z_n\| \leq \mathcal{H}(C(u_{n+1}), C(u_n)) \leq \gamma\|u_{n+1} - u_n\|$$
$$\|v_{n+1} - v_n\| \leq \mathcal{H}(D(u_{n+1}), D(u_n)) \leq \mu\|u_{n+1} - u_n\|$$

and hence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{v_n\}$ are also Cauchy sequence in X, so that there exist x, y, z and v in X such that $x_n \to x$, $y_n \to y$, $z_n \to z$, and $v_n \to v$. By using the same argument as in the proof of Theorem 3.1, it is easy to see that $x \in A(u),\ y \in B(u),\ z \in C(u)$ and $v \in D(u)$

Now by using the continuity of the operators $H, N, W, A, B, C, D, R^H_{M,\lambda}$ and Algorithm 4.1, we have

$$s = H(u) - \{N(x, y) - W(z, v)\}.$$

Finally, by using the arguments as in the proof of Theorem 3.1, we obtain (s, u, x, y, z, v) is a unique solution of (H-REP). ■

REFERENCES

1. S. Adly, Perturbed algorithm and sensitivity analysis for a general class of variational inclusions, *J. Math. Anal. Appl.*, **201** (1996), 609-630.

2. R. Ahmad and Q. H. Ansari, An iterative algorithm for generalized nonlinear variational inclusions, *Appl. Math. Lett.*, **13** (2000), 23-26.

3. R. Ahmad, Q. H. Ansari and S. S. Irfan, Generalized variational inclusions and generalized resolvent equations in Banach spaces, *Computers Math. Appl.*, **29(11-12)** (2005), 1825-1835.

4. S. S. Chang, Existence and approximation of solutions for set-valued variational inclusions in Banach Spaces, *Nonlinear Analysis*, **47** (2001), 583-594.

5. S. S. Chang, Y. J. Cho, B. S. Lee and I. H. Jung, Generalized set-valued variational inclusions in Banach spaces, *J. Math. Anal. Appl.*, **246** (2000), 409-422.

6. S. S. Chang, J. K. Kim and K. H. Kim, On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Bananch spaces, *J. Math. Anal. Appl.*, **268** (2002), 89-108.

7. X. P. Ding, Perturbed proximal point algorithms for generalized quasi-variational inclusions, *J. Math. Anal. Appl.*, **210** (1997), 88-101.

8. Y.-P. Fang and N.-J. Hunag, H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces, *Appl. Math. Lett.*, **17** (2004), 647-653.

9. A. Hassouni, and A. Moudafi, A perturbed algorithm for variational inclusions, *J. Math. Anal. Appl.*, **185** (1994), 706-712.

10. N. J. Huang, A new completely general class of variational inclusions with noncompact valued mappings, *Computers Math. Appl.*, **35(10)** (1998), 9-14.

11. N. J. Huang, A new class of generalized set-valued implicit variational inclusions in Banach spaces with an applications, *Computers Math. Appl.*, **41** (2001), 937-943.

12. C.-F. Shi and S.-Y. Liu, Generalized set-valued variational inclusions in q-uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **296** (2004), 553-562.

13. Ya. Alber and J. C. Yao, Algorithm for generalized multi-valued co-variational inequalities in Banach Spaces, *Funct. Diff. Equ.*, **7** (2000), 5-13.

14. R. P. Agarwal, Y. J. Cho and N.-J. Hunag, Stability of iterative procedures with errors approximating common fixed points for a couple of quasi-contractive mappings in q-uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **272** (2002), 435-447.

15. H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.*, **16**(12) (1991), 1127-1138.

16. S. B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.*, **30** (1969), 475-488.

Rais Ahmad
Department of Mathematics,
Aligarh Muslim University,
Aligarh 202 002,
India
E-mail: raisain@lycos.com

Qamrul Hasan Ansari
Department of Mathematical Science,
King Fahd University of Petroleum & Minerals,
P.O. Box 1169, Dhahran, 31261,
Saudi Arabia
and
Department of Mathematics,
Aligarh Muslim University,
Aligarh 202 002,
India
E-mail: qhansari@kfupm.edu.sa