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η-PSEUDOLINEARITY*

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The notion of η-pseudolinearity is introduced. First, some characterizations of an η-pseudolinear function are obtained. Then characterizations of the solution set of an η-pseudolinear program are derived. The paper generalizes various results on pseudolinear functions and programs.

KEYWORDS: η-pseudoconvex function, η-pseudolinear function, solution set.
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1. Introduction

In response to modeling needs in various disciplines, the classical notion of convexity has been generalized in many ways ([1]). Among others, pseudoconvexity introduced by Mangasarian (1965), proved to be very useful in economic theory and optimization, for example [1]. A real-valued differentiable function f defined on an open set D in $\mathbb{R}^n$ is called pseudolinear ([4]) if f and $-f$ are pseudoconvex. Hanson (1981) considered the class of functions f with the following property:

$$f(y) - f(x) \geq \nabla f(x)^\top \eta(y, x) \qquad \text{for all} \quad y, x \in D, \tag{1}$$

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for some given vector-valued function $\eta(y,x)$ defined on $D \times D$. Subsequently, Craven (1981a, 1981b) called the functions satisfying (1) "invex functions" while Kaul and Kaur (1985) called such functions "η-convex functions".

To generalize both η-convexity and pseudoconvexity, Hanson (1981) introduced a more general class of functions defined in the following way:

$$\nabla f(x)^\top \eta(y,x) \geq 0 \quad \text{implies} \quad f(y) \geq f(x) \quad \text{for all} \quad x, y \in D. \tag{2}$$

Later Kaul and Kaur (1985) called functions satisfying (2) "η-pseudoconvex functions" while Craven (1981a) called such functions "pseudoinvex functions". Ben-Israel and Mond (1986) pointed out, the class of pseudoinvex functions coincides with the class of invex functions. But it should be noted that a pseudoinvex function may not be invex with respect to the same vector function η. In order to avoid confusion, we will adopt the notion of η-pseudoconvexity in this paper.

Clearly, f is η-pseudoconvex on D if and only if $f(y) < f(x)$ implies $\nabla f(x)^\top \eta(y,x) < 0$ for all $x, y \in D$.

If $\eta(y,x) = y - x$ for all $x, y \in D$, then the definitions of η-convexity and η-pseudoconvexity reduce to the definitions of convexity and pseudoconvexity, respectively.

There is a sizable literature on pseudolinear functions; see for example [1, 3, 4, 8, 10, 11, 13] and the references therein. In this paper we introduce and study the following generalization.

DEFINITION 1. A differentiable function f defined on an open set D in $\mathbb{R}^n$ is called η-pseudolinear if f and $-f$ are η-pseudoconvex with respect to the same η.

We note that every pseudolinear function is η-pseudolinear with $\eta(x,y) = x - y$, but the converse is not true. The function in the following example is η-pseudolinear but not pseudolinear.

EXAMPLE 1. Let $D = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : x_1 > -1, \ -\frac{\pi}{2} < x_2 < \frac{\pi}{2}\}$ and $\eta : D \times D \to \mathbb{R}^2$ defined as follows

$$\eta(y,x) = \left(y_1 - x_1, \frac{\sin y_2 - \sin x_2}{\cos x_2}\right)^\top \quad \text{for all} \quad x = (x_1, x_2), \ y = (y_1, y_2) \in D.$$

Then the function $f : D \to \mathbb{R}$ defined by

$$f(x) = x_1 + \sin x_2 \quad \text{for all} \quad x = (x_1, x_2) \in D$$

is η-pseudolinear but not pseudolinear. To see the latter, take $x = (\frac{\pi}{6}, \frac{\pi}{3})$ and $y = (\frac{\pi}{3}, 0)$. Then $\nabla f(x)^\top (y - x) = 0$, but $f(y) < f(x)$.

DEFINITION 2. [14] For a given $\eta : K \times K \to \mathbb{R}^n$, a nonempty subset K of $\mathbb{R}^n$ is called η-convex (or invex), if for each $x, y \in K$, $0 \leq t \leq 1$, $x + t\eta(y,x) \in K$.

DEFINITION 3. [15] Let $\eta : K \times K \to \mathbb{R}^n$ be a given function and K be a nonempty η-convex subset of $\mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be pre-invex on K if

$$f(x + t\eta(y,x)) \leq tf(y) + (1-t)f(x) \quad \text{for all} \quad t \in [0,1].$$

Many results in this paper assume that the function $\eta : K \times K \to \mathbb{R}^n$ satisfies condition C in [14], i.e., for any $x, y \in K$

$$\eta(x, x + t\eta(y,x)) = -t\eta(y,x),$$

$$\eta(y, x + t\eta(y,x)) = (1-t)\eta(y,x)$$

for all $t \in [0,1]$.

This condition ensures that an η-convex (invex) function is also pre-invex ([14]). For an example of a function η which satisfies condition C, see [14, Example 2.4].

2. Characterizations of η-Pseudolinear Functions

In this section, we provide some characterizations of η-pseudolinear functions.

PROPOSITION 1. *Let f be a differentiable function defined on an open set D in $\mathbb{R}^n$ and K be an η-convex subset of D such that $\eta : K \times K \to \mathbb{R}^n$ satisfies condition C. Suppose that f is η-pseudolinear on K. Then for all $x, y \in K$, $\nabla f(x)^\top \eta(y,x) = 0$ if and only if $f(x) = f(y)$.*

Proof. Suppose that f is η-pseudolinear on K. Then for all $x, y \in K$, we have

$$\nabla f(x)^\top \eta(y,x) \geq 0 \quad \text{implies} \quad f(y) \geq f(x)$$

and

$$\nabla f(x)^\top \eta(y,x) \leq 0 \quad \text{implies} \quad f(y) \leq f(x).$$

Combining these two inequalities, we obtain

$$\nabla f(x)^\top \eta(y,x) = 0 \quad \text{implies} \quad f(x) = f(y) \qquad \text{for all } x, y \in K.$$

Now we prove that $f(x) = f(y)$ implies $\nabla f(x)^\top \eta(y,x) = 0$ for all $x, y \in K$. For that, we show that for any $x, y \in K$ such that $f(x) = f(y)$ implies that $f(x + t\eta(y,x)) = f(x)$ for all $t \in (0,1)$.

If $f(x + t\eta(y,x)) > f(x)$, then by the definition of η-pseudoconvexity of f we have

$$\nabla f(z)^\top \eta(x,z) < 0 \tag{3}$$

where $z = x + t\eta(y,x)$.

We show that $\eta(x,z) = \frac{-t}{1-t}\eta(y,z)$. From condition C, we have

$$\begin{aligned} \eta(x,z) = \eta(x, x + t\eta(y,x)) &= -t\eta(y,x) \\ &= \frac{-t}{1-t}\eta(y,z). \end{aligned}$$

Therefore from (3), we obtain

$$\nabla f(z)^\top \left(\frac{-t}{1-t}\right)\eta(y,z) < 0$$

and hence $\nabla f(z)^\top \eta(y,z) > 0$. By η-pseudoconvexity of f, we have

$$f(y) \geq f(z).$$

This contradicts the assumption that

$$f(z) > f(x) = f(y).$$

Similarly, we can also show that $f(x + t\eta(y,x)) < f(x)$ leads to a contradiction, using η-pseudoconvexity of $-f$. This proves the claim that $f(x+t\eta(y,x)) = f(x)$ for all $t \in (0,1)$. Thus

$$\nabla f(x)^\top \eta(y,x) = \lim_{t \to 0^+} \frac{f(x + t\eta(y,x)) - f(x)}{t} = 0. \qquad \blacksquare$$

Now we give an example where the converse of Proposition 1 is not true, that is, if for all $x, y \in K$, $\nabla f(x)^\top \eta(y,x) = 0$ if and only if $f(x) = f(y)$, then f need not be η-pseudolinear.

EXAMPLE 2. Let $D = K = (-\infty, +\infty)$ and $f : D \to \mathbb{R}$, $\eta : D \times D \to \mathbb{R}$ be defined as follows

$$f(x) = e^x, \quad \eta(y,x) = e^{-y} - e^{-x}.$$

Then $\nabla f(x)^\top \eta(y,x) = 0 \Leftrightarrow y = x \Leftrightarrow f(x) = f(y)$. But for $x = 2$ and $y = 1$, we have

$$\nabla f(x)^\top \eta(y,x) = e^2(e^{-1} - e^{-2}) = e - 1 > 0$$

and $f(y) = e < e^2 = f(x)$. Hence f is not η-pseudoconvex on D.

PROPOSITION 2. *Let f be a differentiable function defined on an open set D in $\mathbb{R}^n$ and K be an η-convex subset of D. Then f is η-pseudolinear on K if and only if there exists a function p defined on $K \times K$ such that $p(x,y) > 0$ and $f(y) = f(x) + p(x,y)\nabla f(x)^\top \eta(y,x)$ for all $x, y \in K$.*

Proof. Let f be an η-pseudolinear function. We have to construct a function p on $K \times K$ such that $p(x,y) > 0$ and $f(y) = f(x) + p(x,y)\nabla f(x)^\top \eta(y,x)$ for all $x, y \in K$.

If $\nabla f(x)^\top \eta(y,x) = 0$ for $x, y \in K$, then we define $p(x,y) = 1$. In this case we have $f(y) = f(x)$, due to Proposition 1. On the other hand, if $\nabla f(x)^\top \eta(y,x) \neq 0$, then we define

$$p(x,y) = \frac{f(y) - f(x)}{\nabla f(x)^\top \eta(y,x)}.$$

We have to show that $p(x,y) > 0$. Suppose that $f(y) > f(x)$. Then by η-pseudoconvexity of $-f$, we have $\nabla f(x)^\top \eta(y,x) > 0$. Hence $p(x,y) > 0$. Similarly, if $f(y) < f(x)$, then we have $\nabla f(x)^\top \eta(y,x) < 0$ by η-pseudoconvexity of f. Therefore $p(x,y) > 0$.

To prove the converse, we first show that f is η-pseudoconvex, i.e., for any $x, y \in K$,

$$\nabla f(x)^\top \eta(y,x) \geq 0 \text{ implies } f(y) \geq f(x).$$

If $\nabla f(x)^\top \eta(y,x) \geq 0$, then we have

$$f(y) - f(x) = p(x,y)\nabla f(x)^\top \eta(y,x) \geq 0.$$

Thus $f(y) \geq f(x)$. Likewise, we can prove that $-f$ is η-pseudoconvex. Hence f is η-pseudolinear. ∎

REMARK 1. Proposition 1 generalizes an early result by Kortanek and Evans (1967), see also [4]. Proposition 2 generalizes a result by Chew and Choo (1984).

PROPOSITION 3. *Let $f : D \to \mathbb{R}^n$ be an η-pseudolinear function defined on an open set D of $\mathbb{R}^n$ and let $F : \mathbb{R} \to \mathbb{R}$ be differentiable with $F'(t) > 0$ on $F'(t) < 0$ for all $t \in \mathbb{R}$. Then the composite function $F \circ f$ is also η-pseudolinear.*

Proof. Let $g(x) = F(f(x))$ for all $x \in D$. It suffices to prove the result for $F'(t) > 0$ since the negative of an η-pseudolinear function is η-pseudolinear. We have

$$\nabla g(x)^\top \eta(y,x) = F'(f(x))\nabla f(x)^\top \eta(y,x).$$

Then $\nabla g(x)^\top \eta(y,x) \geq 0$ (≤ 0) implies $\nabla f(x)^\top \eta(y,x) \geq 0$ (≤ 0) since F is strictly increasing. This yields $f(y) \geq f(x)$ ($f(y) \leq f(x)$), due to η-pseudolinearity of f. Thus $g(y) \geq g(x)$ ($g(y) \leq g(x)$) since F is strictly increasing. Hence g is η-pseudolinear. ∎

The following example shows that Proposition 3 no longer holds if $F'(t) = 0$ for some t.

EXAMPLE 3. Let f, η and D be defined as in Example 1 and let $F(t) = t^3$ defined on $\mathbb{R}$. Obviously, $F'(0) = 0$.

For $x = (0,0)$, $y = (0, -\frac{\pi}{3})$, we have $\nabla g(0,0) = 0$ and therefore, $\nabla g(0,0)^\top \eta(y,x) = 0$. But $g(y) = F(f(y)) = (\sin(-\frac{\pi}{3}))^3 = -\frac{3\sqrt{3}}{8} < 0 = g(x)$. Thus g is not η-pseudoconvex, so not η-pseudolinear.

3. Characterizations of Solution Sets

We consider the following problem:

$$\min \ f(x) \qquad \text{subject to} \quad x \in K \tag{P}$$

where $f : D \to \mathbb{R}$, D is an open subset of $\mathbb{R}^n$, and K is an η-convex set of D.

We assume throughout this section that the solution set

$$\bar{S} := \arg \min_{x \in K} f(x)$$

is nonempty.

PROPOSITION 4. *If f is a pre-invex function on K, then the solution set $\bar{S}$ of problem (P) is an η-convex set.*

Proof. Let $x_1, x_2 \in \bar{S}$. Then $f(x_1) \leq f(y)$ and $f(x_2) \leq f(y)$ for all $y \in K$. Since f is pre-invex, we have

$$\begin{aligned} f(x_1 + t\eta(x_2, x_1)) &\leq tf(x_2) + (1-t)f(x_1), \quad \text{for all } t \in [0,1] \\ &\leq tf(y) + (1-t)f(y) \\ &= f(y). \end{aligned}$$

Hence $x_1 + t\eta(x_2, x_1) \in \bar{S}$, and so, $\bar{S}$ is an η-convex set. ■

REMARK 2. From the proof of Proposition 1, it is easy to show that the solution set $\bar{S}$ of problem (P) is η-convex if $f : D \to \mathbb{R}$ is η-pseudolinear where $\eta : K \times K \to \mathbb{R}^n$ satisfies condition C.

Now we state a first-order characterization of the solution set of an η-pseudolinear program in terms of any of its solutions.

THEOREM 1. *Let $f : D \to \mathbb{R}$ be differentiable on an open set D and let f be η-pseudolinear on an η-convex subset $K \subset D$ where η satisfies condition C and $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Let $\bar{x} \in \bar{S}$. Then $\bar{S} = \tilde{S} = \hat{S}$ where*

$$\tilde{S} := \{x \in K : \nabla f(x)^\top \eta(\bar{x}, x) = 0\}, \tag{4}$$

$$\hat{S} := \{x \in K : \nabla f(\bar{x})^\top \eta(\bar{x}, x) = 0\}. \tag{5}$$

Proof. The point $x \in \bar{S}$ if and only if $f(x) = f(\bar{x})$. By Proposition 1, we have $f(x) = f(\bar{x})$ if and only if $\nabla f(x)^\top \eta(\bar{x}, x) = 0$. Also $f(\bar{x}) = f(x)$ if and only if $\nabla f(\bar{x})^\top \eta(x, \bar{x}) = 0$. The latter is equivalent to $\nabla f(\bar{x})^\top \eta(\bar{x}, x) = 0$ since $\eta(\bar{x}, x) = -\eta(x, \bar{x})$. ■

COROLLARY 1. *Let f and η be the same as in Theorem 1. Then $\bar{S} = \tilde{S}_1 = \hat{S}_1$ where*

$$\tilde{S}_1 := \{x \in K : \nabla f(x)^\top \eta(\bar{x}, x) \geq 0\},$$

$$\hat{S}_1 := \{x \in K : \nabla f(\bar{x})^\top \eta(\bar{x}, x) \geq 0\}.$$

Proof. It is clear from Theorem 1 that $\bar{S} \subset \tilde{S}_1$. We prove that $\tilde{S}_1 \subset \bar{S}$. Assume that $x \in \tilde{S}_1$, that is,

$$x \in K \quad \text{such that} \quad \nabla f(x)^\top \eta(\bar{x}, x) \geq 0.$$

In view of Proposition 2, there exists a function p defined on $K \times K$ such that $p(x, \bar{x}) > 0$ and

$$f(\bar{x}) = f(x) + p(x, \bar{x}) \nabla f(x)^\top \eta(\bar{x}, x) \geq f(x).$$

This implies that $x \in \bar{S}$, and hence $\tilde{S}_1 \subset \bar{S}$. Similarly we can prove that $\bar{S} = \hat{S}_1$, using the identity $\eta(x, \bar{x}) = -\eta(\bar{x}, x)$. ■

THEOREM 2. *In problem* (P), *assume that* f *is differentiable on* D *and* η-*pseudolinear on an* η-*convex set* $K \subset D$ *where* η *satisfies condition* C *and* $\eta(x,y) + \eta(y,x) = 0$ *for all* $x, y \in K$. *If* $\bar{x} \in \bar{S}$, *then* $\bar{S} = S^* = S_1^*$ *where*

$$S^* := \{x \in K : \nabla f(\bar{x})^\top \eta(\bar{x}, x) = \nabla f(x)^\top \eta(x, \bar{x})\},$$

$$S_1^* := \{x \in K : \nabla f(\bar{x})^\top \eta(\bar{x}, x) \geq \nabla f(x)^\top \eta(x, \bar{x})\}.$$

Proof. (i) $\bar{S} \subset S^*$. Let $x \in \bar{S}$. It follows from Theorem 1 that

$$\nabla f(x)^\top \eta(\bar{x}, x) = 0 = \nabla f(\bar{x})^\top \eta(\bar{x}, x).$$

Since $\eta(\bar{x}, x) = -\eta(x, \bar{x})$, we have

$$\nabla f(x)^\top \eta(x, \bar{x}) = 0 = \nabla f(\bar{x})^\top \eta(\bar{x}, x).$$

Thus $x \in S^*$, and hence $\bar{S} \subset S^*$.

(ii) $S^* \subset S_1^*$ is obvious.

(iii) $S_1^* \subset \bar{S}$. Assume that $x \in S_1^*$. Then $x \in K$ satisfies

$$\nabla f(\bar{x})^\top \eta(\bar{x}, x) \geq \nabla f(x)^\top \eta(x, \bar{x}). \tag{6}$$

Suppose that $x \notin \bar{S}$. Then $f(x) > f(\bar{x})$. By η-pseudoconvexity of $-f$ we have

$$\nabla f(\bar{x})^\top \eta(x, \bar{x}) > 0.$$

Since $\eta(x, \bar{x}) = -\eta(\bar{x}, x)$, we have $\nabla f(\bar{x})^\top \eta(\bar{x}, x) < 0$. Using (6), we have

$$\nabla f(x)^\top \eta(x, \bar{x}) < 0 \quad \text{or} \quad \nabla f(x)^\top \eta(\bar{x}, x) > 0.$$

In view of Proposition 2, there exists a function p defined on $K \times K$ such that $p(x, \bar{x}) > 0$, and

$$f(\bar{x}) = f(x) + p(x, \bar{x}) \nabla f(x)^\top \eta(\bar{x}, x) > f(x),$$

a contradiction. Hence $x \in \bar{S}$. ∎

REMARK 3. This section generalizes results by Jeyakumar and Yang (1995).

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η-Pseudolinearità

RIASSUNTO

Il lavoro introduce la nozione di η-pseudolinearità. Dopo avere ottenuto alcune caratterizzazioni delle funzioni η-pseudolineari, si derivano caratterizzazioni dell'insieme delle soluzioni di un programma η-pseudolineare. Lo studio generalizza diversi risultati sulle funzioni e sui programmi pseudolineari.

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