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A NOTE ON GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES

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In this paper, we consider the generalized vector variational-like inequalities and prove the existence of their solutions in the setting of Hausdorff topological vector spaces and H-spaces.

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INTRODUCTION

The vector variational inequality (in short, VVI) in a finite dimensional Euclidean space has been introduced in Ref. 1. Since then, it has been intensively studied in abstract spaces by Chen *et al.* (Refs. 2–4), Lai and Yao (Ref. 5), Siddiqi *et al.* (Ref. 6), Yang (Refs. 7–8) and Yu and Yao (Ref. 9). It has been extended and generalized in many different directions. Lee *et al.* (Refs. 10–11) and Chen and Craven (Ref. 12) have studied VVI for point-tó-set maps, which is called generalized vector variational inequality (GVVI), and obtained some existence theorems. They have also shown some relations between weak vector minimization, a GVVI, and the optimization of a utility function over a set of efficient points. Recently, vector variational-like inequality (VVLI)

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has been introduced and studied in Ref. 13. Inspired and motivated by the applications of GVVI and VVLI, we have introduced and studied generalized vector variational-like inequalities (GVVLI), which include VVLI and GVVI as special cases, in the setting of reflexive Banach spcaes (see Ref. 14). In this paper, we study the existence of solution of GVVLI in the setting of Hausdorff topological vector spaces and H-spaces (Ref. 15). Several special cases were also discussed.

Let X and Y be two Hausdorff topological vector spaces and K be a nonempty convex subset of X. Let $T: K \rightrightarrows L(X, Y)$ be a point-to-set map, where $L(X, Y)$ is the space of all continuous linear maps from X into Y. Let $\eta: K \times K \to X$ be a given map and $C: K \rightrightarrows Y$ be a point-to-set map such that $\forall x \in K$, $C(x)$ is a closed pointed convex cone with $\operatorname{int} C(x) \neq \emptyset$, where $\operatorname{int} C(x)$ denotes the interior of the set $C(x)$.

We consider the following generalized vector variational-like inequality problem (GVVLIP), that is to find $x_0 \in K$ such that

$$\langle s, \eta(x, x_0)\rangle \notin -\operatorname{int} C(x_0), \quad \forall s \in T(x_0) \text{ and } x \in K,$$

where $\langle s, y\rangle$ denotes the evaluation of the linear map s at y.

When $\eta(x, x_0) = x - x_0$, (GVVLIP) reduces to the following generalized vector variational inequality problem (GVVIP) considered by Lee *et al.* (Refs. 10–11):

(GVVIP): Find $x_0 \in K$ such that

$$\langle s, x - x_0\rangle \notin -\operatorname{int} C(x_0), \quad \forall s \in T(x_0) \text{ and } x \in K.$$

When $\eta(x, x_0) = x - x_0$, $Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+$, (GVVLIP) reduces to the following generalized variational inequality problem (GVIP):

(GVIP): Find $x_0 \in K$ such that

$$\inf_{s \in T(x_0)} \langle s, x - x_0\rangle \geqslant 0, \quad \forall x \in K.$$

When T is a map from X into $L(X, Y)$, (GVVLIP) becomes the following vector variational-like inequality problem (VVLIP) (see Ref. 13):

(VVLIP): Find $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0) \rangle \notin -\operatorname{int} C(x_0), \quad \forall x \in K.$$

When T is a map from X into $L(X, Y)$ and $\eta(x, x_0) = x - x_0$, (GVVLIP) is equivalent to the following vector variational inequality problem (VVIP) (see Refs. 2–4):

(VVIP): Find $x_0 \in K$ such that

$$\langle T(x_0), x - x_0 \rangle \notin -\operatorname{int} C(x_0), \quad \forall x \in K.$$

2. EXISTENCE THEORY IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

Through out in this section, we will consider X and Y as Hausdorff topological vector spaces. The bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous. For the proof of main result of this section, we need the following concepts and result:

DEFINITION 2.1 A point-to-set map $F: X \rightrightarrows Y$ is said to be *lower semicontinuous* at $x \in X$ if $x_n \to x$, $x_n \in X$ $\forall n$ and $y \in F(x)$ imply that there exists $y_n \in F(x_n)$ $\forall n$ such that $y_n \to y$. F is lower semicontinuous on X if it is lower semicontinuous at each of its points.

F is said to be upper semicontinuous at $x \in X$ if $x_n \to x$ in X and $y_n \to y$ in Y with $y_n \in F(x_n)$ imply $y \in F(x)$.
F is upper semicontinuous on X if it is upper semicontinuous at each of its points.

LEMMA 2.1 (Ref. 16). *Let X be a Hausdorff topological vector space and K be a nonempty subset of X. Let A be a subset of $X \times X$ having the following properties:*

(i) For any $x \in K$, the set $\{y \in K : (y, x) \in A\}$ is closed;
(ii) For any $y \in K$, the set $\{x \in K : (y, x) \notin A\}$ is convex (or empty);
(iii) $(x, x) \in A$, $\forall x \in K$;
(vi) For a nonempty compact convex subset $D \subset K$ with each $y \in K$, $\exists x \in K$ and $(y, x) \notin A$.

Then, $\exists y^* \in K$ such that $\{y^*\} \times K \subset A$.

Now we prove the existence theorem for the (GVVLIP) by using lower semicontinuity but not upper semicontinuity of the point-to-set map T.

THEOREM 2.1 *Let K be a convex subset of X. Assume that:*

(a) $C: K \rightrightarrows Y$ is a point-to-set map such that $\forall x \in K, C(x)$ is a closed pointed convex cone with $\operatorname{int} C(x) \neq \emptyset$;
(b) the point-to-set map $W: K \rightrightarrows Y$ defined by $W(x) = Y \setminus \{-\operatorname{int} C(x)\}$, $\forall x \in K$, is upper semicontinuous;
(c) $T: K \rightrightarrows L(X, Y)$ is lower semicontinuous;
(d) $\eta: K \times K \to X$ is continuous in the second argument such that $\eta(x, x) = 0$, $\forall x \in K$;
(e) the map $x \mapsto \langle s, \eta(x, y) \rangle$, for each fixed $y \in K$ and $\forall s \in T(y)$, is affine;
(f) for a nonempty compact convex subset $D \subset K$ with each $y \in D \setminus K$, $\exists x \in D$ such that for any $s \in T(y)$,

$$\langle s, \eta(x, y) \rangle \in -\operatorname{int} C(y).$$

Then the (GVVLIP) is solvable.

Proof Let

$$A = \{(y, x) \in K \times K : \langle s, \eta(x, y) \rangle \notin -\operatorname{int} C(y), \ \forall s \in T(y)\}.$$

To prove this theorem, we will show that all the assumptions of Lemma 2.1 are satisfied.

Since $C(x)$ is a closed pointed convex cone with $\operatorname{int} C(x) \neq \emptyset$, $\forall x \in K$ then by assumption (d), we have

$$\langle s, \eta(x, x) \rangle \notin -\operatorname{int} C(x), \ \forall s \in T(x) \text{ and } x \in K. \tag{1}$$

From the definition of A, we have $\forall x \in K, (x, x) \in A \Leftrightarrow \langle s, \eta(x, x) \rangle \notin -\operatorname{int} C(x), \forall s \in T(x)$. Now, let $A_x = \{y \in K: (y, x) \in A\}$, for each fixed $x \in K$, then we show that A_x is closed.

Let $\{y_n\}$ be a net in A_x converges to some $y_* \in K$. By the lower semicontinuity of T, for any $s_* \in T(y_*)$, $\exists s_n \in T(y_n) \, \forall n$ such that the sequence $\{s_n\}$ converges to $s_* \in L(X, Y)$. Since $y_n \in A_x \, \forall n$, we have

$$\langle s_n, \eta(x, y_n)\rangle \notin -\operatorname{int} C(y_n),$$

$$\text{or} \quad \langle s_n, \eta(x, y_n)\rangle \in Y \setminus \{-\operatorname{int} C(y_n)\}. \tag{2}$$

Since $\eta(\cdot,\cdot)$ is continuous in the second argument and $y_n \to y_*$, we have $\eta(x, y_n) \to \eta(x, y_*)$. Since $s_n \to s$ and $\langle\cdot,\cdot\rangle$ is continuous, we have

$$\langle s_n, \eta(x, y_n)\rangle \to \langle s, \eta(x, y_*)\rangle.$$

By assumption (*b*), we have

$$\langle s, \eta(x, y_*)\rangle \in Y \setminus \{-\operatorname{int} C(y_*)\},$$

$$\text{or} \quad \langle s, \eta(x, y_*)\rangle \notin -\operatorname{int} C(y_*).$$

Thus $y_* \in A_x$ and consequently A_x is closed.
Now we show that for each fixed $y \in K$, the set $A_y = \{x \in K : (y, x) \notin A\}$ is convex.

Indeed, let $x_1, x_2 \in A_y$ and α, $\beta \geqslant 0$ such that $\alpha + \beta = 1$. As $C(y)$ is a cone, we have $\forall s \in T(y)$

$$\alpha\langle s, \eta(x_1, y)\rangle \in -\operatorname{int} C(y) \tag{3}$$

and

$$\beta\langle s, \eta(x_2, y)\rangle \in -\operatorname{int} C(y). \tag{4}$$

Adding (3) and (4), we get

$$\alpha\langle s, \eta(x_1, y)\rangle + \beta\langle s, \eta(x_2, y)\rangle \in -\operatorname{int} C(y) - \operatorname{int} C(y) \subseteq -\operatorname{int} C(y).$$

By assumption (*e*), we have

$$\langle s, \eta(\alpha x_1 + \beta x_2, y)\rangle \in -\operatorname{int} C(y)$$

$\Rightarrow \alpha x_1 + \beta x_2 \in A_y$ and hence A_y is convex.
Then by Lemma 2.1, $\exists x_0 \in K$ such that $\{x_0\} \times K \subset A$.

$$\Rightarrow x_0 \in K \colon \langle s, \eta(x, x_0)\rangle \notin -\operatorname{int} C(x_0), \ \forall s \in T(x_0) \text{ and } x \in K.$$

3. EXISTENCE THEORY WITHOUT CONVEXITY

In this section, by using the technique of Chen (Ref. 4) and Lee *et al.* (Ref. 11), we establish the existence theorem for solution of a special case of (GVVLI) by replacing the convexity assumption with merely topological properties.

Now we give some definitions concerning the H-space, H-convexity and H-KKM theorem (see Ref. 15).

DEFINITION 3.1 Let X be a topological space and let $\{\Gamma_A\}$ be a given family of nonempty contractible subsets of X, indexed by finite subsets of X.

A pair $(X, \{\Gamma_A\})$ is said to be a H-space, if $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

Let $(X, \{\Gamma_A\})$ be an H-space. A subset $D \subset X$ is called H-convex, if for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.

A subset $D \subset X$ is called *weakly H-convex*, if $\Gamma_A \cap D$ is nonempty and contractible for every finite subset $A \subset D$. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is a *H-space*.

A subset $K \subset X$ is called *H-compact*, if there exists a compact and weakly *H-convex* set $D \subset X$ such that $K \cup A \subset D$ for every finite subset $A \subset X$.

A point-to-set map $F: X \rightrightarrows X$ is called H-KKM *map*, if $\Gamma_A \subset \cup_{x \in A} F(x)$, for every finite subset $A \subset X$.

THEOREM 3.1 (Ref. 15). *Let $(X, \{\Gamma_A\})$ be an H-space, and let $F: X \rightrightarrows X$ be an H-KKM point-to-set map such that:*

(i) for each $x \in X$, $F(x)$ is compactly closed, that is $B \cap F(x)$ is closed in B, for every compact set $B \subset X$;
(ii) there exists a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for each weakly H-convex set D with $K \subset D \subset X$, we have $\cap_{x \in D}(F(x) \cap D) \subset L$.

Then $\cap_{x \in D} F(x) \neq \emptyset$.

Let $(X, \{\Gamma_A\})$ be an H-space and (Y, P) be an order topological vector space with a closed pointed convex cone P such that $\operatorname{int} P \neq \emptyset$. Given a point-to-set map $T: X \rightrightarrows L(X, Y)$ and a map $\eta: X \times X \to X$. Then we consider a special case of (GVVLIP), but in a more general context:

(GVVLIP)′: Find $x_0 \in X$ such that

$$\langle s, \eta(x, x_0) \rangle \notin -\operatorname{int} P, \ \forall s \in T(x_0) \text{ and } x \in X. \tag{5}$$

When $\eta(x, x_0) = x - x_0$, (GVVLIP)′ reduces to the following (GVVIP)′ considered by Lee *et al.* (Ref. 11):
(GVVIP)′ Find $x_0 \in X$ such that

$$\langle s, x - x_0 \rangle \notin -\operatorname{int} P, \ \forall s \in T(x_0) \text{ and } x \in X. \tag{6}$$

Now we prove the existence theorem for the (GVVLIP)′.

THEOREM 3.2 *Assume that*

(a) $T: X \rightrightarrows L(X, Y)$ *is lower semicontinuous;*
(b) $\eta: X \times X \to X$ *is continuous such that* $\eta(x, x) = 0, \ \forall x \in X$:
(c) for each $y \in X$, $B_y = \{x \in X: \exists s \in T(y) \text{ such that } \langle s, \eta(x, y) \rangle \in -\operatorname{int} P\}$ *is H-convex or empty;*
(d) there exists a compact set $L \subset X$ *and an H-compact set* $K \subset X$ *such that for every weakly H-convex set D with* $K \subset D \subset X$, *we have*

$$\{y \in D: \langle s, \eta(x, y) \rangle \notin -\operatorname{int} P, \ \forall s \in T(y) \text{ and } x \in D\} \subset L.$$

Then the (GVVLIP)′ is solvable.

Proof Define a point-to-set map $F: X \rightrightarrows X$ by

$$F(x) = \{y \in X: \langle s, \eta(x, y) \rangle \notin -\operatorname{int} P, \ \forall s \in T(y)\}, \ \forall x \in X.$$

We first prove that F is an H-KKM map. Suppose that F is not an H-KKM map. Then there exists a finite subset $A \subset X$ such that $\Gamma_A \subset \cup_{x \in A} F(x)$. Thus, there exists $z \in \Gamma_A$ such that $z \notin F(x)$, $\forall x \in A$, namely, $\exists s \in T(z)$ such that $\langle s, \eta(x, z) \rangle \in -\operatorname{int} P$, $\forall x \in A$.
But from assumption (*c*), $A \subset B_z$. Since B_z is *H-convex*, $\Gamma_A \subset B_z$. Therefore, $z \in B_z$ and hence $\exists s \in T(z)$ such that $\langle s, \eta(z, z) \rangle \in -\operatorname{int} P$. But from assumption (*b*), $\langle s, \eta(z, z) \rangle = 0, \forall z \in B_z$ and hence $0 \in -\operatorname{int} P$. But P is a closed pointed convex cone, we have $0 \notin -\operatorname{int} P$. Therefore we reach to a contradiction and hence F is an H-KKM map.

From the proof of Theorem 2.1, we have $F(x)$, $\forall x \in X$ is closed. Therefore, the condition (*i*) of Theorem 3.1 holds. It is easy to see that the assumption (*d*) of this theorem is same as condition (*ii*) of Theorem 3.1. Thus by Theorem 3.1, we have

$$\bigcap_{x \in X} F(x) \neq \emptyset,$$

that is, $\exists x_0 \in X$ such that

$$\langle s, \eta(x, x_0) \rangle \notin -\operatorname{int} P, \ \forall s \in T(x_0) \text{ and } x \in X.$$

If $\eta(x, y) = x - y$ then we have the following existence result for (GVVIP)′.

COROLLARY 3.1 (Ref. 11) *Assume that*

(*a*) *$T: X \rightrightarrows L(X, Y)$ is lower semicontinuous;*
(*b*) *for each $y \in X$, $B_y = \{x \in X: \exists s \in T(y)$ such that $\langle s, x - y \rangle \in -\operatorname{int} P\}$ is H-convex or empty;*
(*c*) *there exists a compact set $L \subset X$ and an H-compact set $K \subset X$ such that for every weakly H-convex set D with $K \subset D \subset X$, we have*

$$\{y \in D: \langle s, x - y \rangle \notin -\operatorname{int} P, \ \forall s \in T(y) \text{ and } x \in D\} \subset L.$$

Then the (GVVIP)′ is solvable.

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