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MANN-TYPE STEEPEST-DESCENT AND MODIFIED HYBRID STEEPEST-DESCENT METHODS FOR VARIATIONAL INEQUALITIES IN BANACH SPACES

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□ *In this paper, we propose three different kinds of iteration schemes to compute the approximate solutions of variational inequalities in the setting of Banach spaces. First, we suggest Mann-type steepest-descent iterative algorithm, which is based on two well-known methods: Mann iterative method and steepest-descent method. Second, we introduce modified hybrid steepest-descent iterative algorithm. Third, we propose modified hybrid steepest-descent iterative algorithm by using the resolvent operator. For the first two cases, we prove the convergence of sequences generated by the proposed algorithms to a solution of a variational inequality in the setting of Banach spaces. For the third case, we prove the convergence of the iterative sequence generated by the proposed algorithm to a zero of an operator, which is also a solution of a variational inequality.*

Keywords Convergence analysis; Mann-type steepest-descent method; Modified hybrid steepest-descent method; Nonexpansive maps; Resolvent operators; Variational inequalities.

AMS Subject Classification 49J40; 47J20; 47H10; 47H06; 65K10.

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H, and $F : H \to H$ be a nonlinear operator. The variational inequality problem is to find a point $u^* \in C$ such that

$$\langle F(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C. \tag{VI(F, C)}$$

It is well known that the theory of variational inequalities has been playing an important role in the study of many diverse disciplines, for instance, partial differential equations, optimal control, optimization, mathematical programming, mechanics, finance, etc., see, for example, [6, 8, 10, 17, 21, 22, 30, 41, 42] and references therein. It is also known that a great deal of effort has gone into the problem, that is, how to find a solution of VI(F, C) if any; see, for example, [6, 7, 12, 14, 40, 41, 43–45] and references therein.

It is known that if F is a strongly monotone and Lipschitzian mapping on C, then VI(F, C) has a unique solution. It is also known that the VI(F, C) is equivalent to the fixed-point equation

$$u^* = P_C\big(u^* - \mu F(u^*)\big), \tag{1.1}$$

where P_C is the nearest point projection from H onto C, and $\mu > 0$ is an arbitrary fixed constant. If F is a strongly monotone and Lipschitzian mapping on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of (1.1) is a contraction. Hence, Banach contraction principle guarantees that the Picard iterates converge strongly to a unique solution of VI(F, C). Such a method is called the projection method. We remark that the fixed-point formulation (1.1) involves the projection P_C, which may not be easy to compute due to the complexity of the convex set C.

In order to reduce the complexity probably caused by the projection P_C, Yamada [41] (see also [10]) introduced a hybrid steepest-descent method for solving VI(F, C). His idea is the following: Let C be the fixed point set of a nonexpansive mapping $T : H \to H$, that is, $C = \{x \in H : Tx = x\}$. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Let F be strongly monotone and Lipschitzian on C with constants η and κ, respectively. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ of real numbers in $(0, 1)$ satisfying the following conditions:

(L1) $\lim_n \lambda_n = 0$;

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(L2) $\sum_n \lambda_n = \infty$;

(L3) $\lim_n \frac{(\lambda_n - \lambda_{n+1})}{\lambda_{n+1}^2} = 0$.

Starting with an arbitrary initial guess $u_0 \in H$, generate a sequence $\{u_n\}$ by the following algorithm

$$u_{n+1} := Tu_n - \lambda_{n+1}\mu F(Tu_n), \quad \forall n \geq 0. \tag{1.2}$$

Yamada [41, Theorem 3.3 (p. 486)] proved that the sequence $\{u_n\}$ converges strongly to a unique solution of $\mathrm{VI}(F, C)$. An example of a sequence $\{\lambda_n\}$ satisfying conditions (L1)–(L3) is given by

$$\lambda_n = 1/n^\sigma, \quad \text{where } 0 < \sigma \leq 1.$$

We remark that the condition (L3) was used first by Lions [24]. Recently, Xu and Kim [40] continued the convergence study of hybrid steepest-descent algorithm (1.2). Their major contribution is that the strong convergence of (1.2) holds with condition (L3) being replaced by the following condition:

(L3)′ $\lim_n \frac{\lambda_n}{\lambda_{n+1}} = 1$ or equivalently $\lim_n \frac{(\lambda_n - \lambda_{n+1})}{\lambda_{n+1}} = 0$.

It is clear that condition (L3)′ is strictly weaker than condition (L3), coupled with conditions (L1) and (L2). Moreover, (L3)′ includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$, whereas (L3) does not.

Further, inspired by Yamada's algorithm (1.2), Zeng et al. [45] introduced and studied another modified hybrid steepest-descent algorithm with variable parameters for computing approximate solutions of $\mathrm{VI}(F, C)$. Under some suitable assumptions, they proved that the sequence generated by their algorithm converges strongly to a unique solution of $\mathrm{VI}(F, C)$.

Let X be a real Banach space and let $J : X \to 2^{X^*}$ denote the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in X,$$

where X^* denotes the dual space of X, and $\langle ., .\rangle$ denotes the generalized duality pairing. For the rest of the paper, we denote by J the single-valued duality mapping and by $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$ the fixed-point set of a mapping $T : X \to X$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \rightharpoonup^* x$) denotes strong (respectively, weak, weak*) convergence of the sequence $\{x_n\}$ to x.

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Let X be a real Banach space and $F : X \to X$ be a mapping. Assume that C is the fixed point set of a nonexpansive mapping $T : X \to X$, that is,

$$C = \mathrm{Fix}(T) = \{x \in X : Tx = x\}. \tag{1.3}$$

In this paper, we consider the following variational inequality problem in the setting of Banach spaces: Find $u^* \in C$ such that

$$\langle F(u^*), J(u^* - v)\rangle \leq 0, \quad \forall v \in C. \qquad \mathrm{VI}^*(F, C)$$

Here we remind the reader of the following fact: The problem $\mathrm{VI}^*(F, C)$ for an inverse-strongly accretive operator F over a nonempty closed convex subset C of a smooth Banach space X has already been presented in Aoyama et al. [2, 3]. An operator $F : X \to X$ is said to be inverse-strongly accretive if there exists $\alpha > 0$ such that $\langle F(x) - F(y), J(x - y)\rangle \geq \alpha\|F(x) - F(y)\|^2$ for all $x, y \in X$. If F is strongly accretive and strictly pseudocontractive, then F is inverse-strongly accretive. In particular, if we take $F = I - S$, where $S : X \to X$ is a contraction, then $\mathrm{VI}^*(F, C)$ actually reduces to $\mathrm{VI}^*(I - S, \mathrm{Fix}(T))$. In a word, we can provide some examples for the proposed problem $\mathrm{VI}^*(F, \mathrm{Fix}(T))$. For example, we can discuss $\mathrm{VI}^*(I - S, \mathrm{Fix}(T))$, where $S : X \to X$ is a contraction, evolution equations, the problem for solving a zero point of an accretive operator, fixed point problems for the resolvent, and so on.

On the other hand, suppose that E is a Hilbert space and A is the gradient of a convex function Θ. Then the problem $\mathrm{VI}^*(F, C)$ is to find $x \in C$ such that $\Theta(x) = \min_{y \in C} \Theta(y)$. Although for $\mathrm{VI}^*(F, C)$ in a Banach space, the constrained optimization problem is not discussed, that is, the problem in a Banach space is not called the variational inequality problem, but for the sake of convenience we say still that $\mathrm{VI}^*(F, C)$ is the variational inequality problem in a Banach space based on the consideration of the case when E is a Hilbert space. In addition, for the study of the variational inequality problem for monotone operators in a Banach space, see [1, 15, 16] for more details.

In the past decade, several papers have appeared in the literature on the theory of variational inequalities in the setting of Banach spaces; see, for example, [7, 8, 18, 19, 37–41, 43–45] and references therein. In this paper, we extend the steepest-descent method for solving $\mathrm{VI}^*(F, C)$. We propose three different kinds of iteration schemes to compute the approximate solutions of $\mathrm{VI}^*(F, C)$. In the next section, we recall some known definitions and results that will be used in the rest of the paper. In Section 3, we suggest Mann-type steepest-descent iterative algorithm, which is based on two well-known methods: Mann iterative method [25] and steepest-descent method. We prove the convergence of sequences generated by the suggested algorithm to a unique solution of $\mathrm{VI}^*(F, C)$.

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Section 4 deals with the modified hybrid steepest-descent method and the convergence of sequences generated by the proposed algorithms to a unique solution of $\mathrm{VI}^*(F, C)$. In the last section, we propose modified hybrid steepest-descent algorithm by using the resolvent operator and prove the convergence of the iterative sequence generated by the proposed algorithm to a zero of an operator; such a zero is also a solution of $\mathrm{VI}^*(F, C)$.

2. PRELIMINARIES

Let X be a real Banach space; X^* denotes the dual space of X and $\langle ., . \rangle$ denotes the generalized duality pairing. Recall that the norm of a Banach space X is called Gâteaux differentiable (and X is called smooth) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \quad \text{exists}$$

for all x, y in the sphere $U = \{x \in X : \|x\| = 1\}$. Moreover, if for each $y \in U$ this limit is uniformly attained for $x \in U$, then the norm of X is uniformly Gâteaux differentiable. The norm of X is called Fréchet differentiable, if for all $x \in U$, this limit is attained uniformly for $y \in U$. The norm of X is called uniformly Fréchet differentiable (and X is called uniformly smooth) if this limit is attained uniformly for all $(x, y) \in U \times U$. A Banach space X is called strictly convex if $\|x + y\|/2 < 1$, for all $x, y \in U$ with $x \neq y$. It is well known that when X is reflexive, then X is strictly convex if and only if its dual X^* is smooth. Furthermore, if X is smooth, then the duality mapping J is single-valued and norm-to-weak* continuous. If X is uniformly smooth, then the duality mapping J is single-valued and norm-to-norm uniformly continuous on bounded sets of X. If the norm of X is uniformly Gâteaux differentiable, then the duality mapping J is single-valued and norm-to-weak* uniformly continuous on bounded sets of X. A discussion on these and related concepts may be found in [8]. Meantime, there also holds the following result (see Theorem 4.5.3 in [8]): If K is a nonempty closed convex subset of a strictly convex Banach space X and $T : K \to K$ is a nonexpansive mapping, then the set $F(T)$ of fixed points of T is a closed convex subset of K.

If Banach space X admits sequentially continuous duality mapping J from weak topology to weak* topology, then by Lemma 1 in [13], we know that the duality mapping J is single-valued and hence X is also smooth. In this case, duality mapping J is also called weak sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, we have $J(x_n) \rightharpoonup^* J(x)$; see, for example, [13, 26].

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Recall that a mapping F with domain $D(F)$ and range $R(F)$ in X is called δ-strongly accretive if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \geq \delta\|x-y\|^2 \quad \text{for some } \delta \in (0,1).$$

F is called λ-strictly pseudocontractive [5] if for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Fx-Fy)\|^2 \tag{2.1}$$

for some $\lambda \in (0,1)$. It is easy to see that (2.1) can be rewritten as (see [42])

$$\langle (I-F)x - (I-F)y, j(x-y)\rangle \geq \lambda\|(I-F)x-(I-F)y\|^2. \tag{2.2}$$

A multivalued mapping A with domain $D(A)$ and range $R(A)$ in X is called accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0,$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping.

The following proposition will be used frequently throughout the paper. For the sake of completeness, we include its proof.

Proposition 2.1. *Let X be a real smooth Banach space and $F : X \to X$ be a mapping.*

(i) *If F is λ-strictly pseudocontractive, then F is Lipschitzian with constant $\left(1+\frac{1}{\lambda}\right)$.*
(ii) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\lambda}}$.*
(iii) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$.*

Proof. (i) From (2.2), we derive

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\leq \langle (I-F)x-(I-F)y, J(x-y)\rangle \\ &\leq \|(I-F)x-(I-F)y\|\|x-y\|,\end{aligned}$$

which implies that

$$\|(I-F)x-(I-F)y\| \leq \frac{1}{\lambda}\|x-y\|.$$

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Hence

$$\|Fx - Fy\| \le \|(I-F)x - (I-F)y\| + \|x-y\| \le \left(1 + \frac{1}{\lambda}\right)\|x-y\|,$$

and so F is Lipschitzian with constant $\left(1 + \frac{1}{\lambda}\right)$.

(ii) From (2.1) and (2.2), we obtain

$$\begin{aligned}\lambda\|(I-F)x - (I-F)y\|^2 &\le \|x-y\|^2 - \langle Fx - Fy, J(x-y)\rangle \\ &\le (1-\delta)\|x-y\|^2.\end{aligned}$$

Because $\delta + \lambda > 1 \Leftrightarrow \sqrt{\frac{1-\delta}{\lambda}} \in (0,1)$, we have

$$\|(I-F)x - (I-F)y\| \le \left(\sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\|$$

and hence $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\lambda}}$.

(iii) Because $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\lambda}}$, for each fixed number $\tau \in (0,1)$ we have

$$\begin{aligned}\|x - y - \tau(F(x) - F(y))\| &= \|(1-\tau)(x-y) + \tau[(I-F)x - (I-F)y]\| \\ &\le (1-\tau)\|x-y\| + \tau\|(I-F)x - (I-F)y\| \\ &\le (1-\tau)\|x-y\| + \tau\left(\sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\| \\ &= \left(1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|.\end{aligned}$$

This shows that $I - \tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)$. □

Lemma 2.2 [8, 36]. *Let X be a real smooth Banach space. Then*

$$\|x\|^2 + 2\langle y, J(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall x, y \in X,$$

where $J : X \to X^*$ is the normalized duality mapping.

Lemma 2.3 (Lemma 1.1 in [35]). *Let K be a nonempty closed convex subset of a real Banach space X and $T : K \to K$ be a continuous pseudocontractive map. Then*

(i) *(Theorem 6 in [27]) $A = (2I - T)^{-1}$ is a nonexpansive self-mapping on K;*
(ii) *if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty}\|x_n - Ax_n\| = 0$.*

Let X be a real smooth Banach space and $T : X \to X$ be a nonexpansive mapping. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily and then consider the following mapping $\Gamma_t : X \to X$ defined as

$$\Gamma_t x = tx + (1-t)Tx - t\mu_t Fx, \quad \forall x \in X. \tag{2.3}$$

Then, $\Gamma_t : X \to X$ is a contractive mapping. Indeed, for all $x, y \in X$, using Proposition 2.1(iii) we have

$$\begin{aligned}
\|\Gamma_t x - \Gamma_t y\| &= \|(tx + (1-t)Tx - t\mu_t Fx) - (ty + (1-t)Ty - t\mu_t Fy)\| \\
&= \|(t(I-\mu_t F)x + (1-t)Tx) - (t(I-\mu_t F)y + (1-t)Ty)\| \\
&\le t\|(I-\mu_t F)x - (I-\mu_t F)y\| + (1-t)\|Tx - Ty\| \\
&\le t\left[1 - \mu_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|x-y\| + (1-t)\|x-y\| \\
&= \left[1 - t\mu_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|x-y\|
\end{aligned}$$

and hence $\Gamma_t : X \to X$ is contractive due to $t\mu_t \in (0,1)$. By Banach contraction principle, there exists a unique fixed point x_t of Γ_t in X, that is,

$$x_t = tx_t + (1-t)Tx_t - t\mu_t F(x_t). \tag{2.4}$$

If $T : X \to X$ is a continuous pseudocontractive mapping, then $\Gamma_t : X \to X$, given by (2.3), is a continuous strongly pseudocontractive mapping. Indeed, for all $x, y \in X$, using Proposition 2.1(iii), we have

$$\begin{aligned}
&\langle \Gamma_t x - \Gamma_t y, J(x-y)\rangle \\
&\quad = \langle (tx + (1-t)Tx - t\mu_t Fx) - (ty + (1-t)Ty - t\mu_t Fy), J(x-y)\rangle \\
&\quad = \langle (t(I-\mu_t F)x + (1-t)Tx) - (t(I-\mu_t F)y + (1-t)Ty), J(x-y)\rangle \\
&\quad = t\langle (I-\mu_t F)x - (I-\mu_t F)y, J(x-y)\rangle + (1-t)\langle Tx - Ty, J(x-y)\rangle \\
&\quad \le t\|(I-\mu_t F)x - (I-\mu_t F)y\|\|J(x-y)\| + (1-t)\|x-y\|^2 \\
&\quad \le t\left(1 - \mu_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|^2 + (1-t)\|x-y\|^2 \\
&\quad = \left(1 - t\mu_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|^2
\end{aligned}$$

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and hence $\Gamma_t : X \to X$ is a continuous strongly pseudocontractive mapping due to $t\mu_t \in (0,1)$. By Corollary 2 in [9], there exists a unique fixed point x_t of Γ_t in X, that is,

$$x_t = tx_t + (1-t)Tx_t - t\mu_t F(x_t).$$

Lemma 2.4. *Let X be a real smooth Banach space and $T : X \to X$ be a nonexpansive (or continuous pseudocontractive) mapping with $F(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0,1)$, choose a number $\mu_t \in (0,1)$ arbitrarily and let $\{x_t\}$ be defined by* (2.4). *Suppose that $u \in X$ is a fixed point of T, that is, $u \in C = \mathrm{Fix}(T)$, then*

(i) $\langle F(x_t), J(x_t - u)\rangle \leq 0$;
(ii) $\{x_t\}$ *is bounded.*

Proof. (i) As u is a fixed point of T, we have

$$\begin{aligned}
&\langle x_t - (tx_t + (1-t)u - t\mu_t F(x_t)), J(x_t - u)\rangle \\
&\quad = \langle (tx_t + (1-t)Tx_t - t\mu_t F(x_t)) \\
&\qquad - (tx_t + (1-t)u - t\mu_t F(x_t)), J(x_t - u)\rangle \\
&\quad = (1-t)\langle Tx_t - u, J(x_t - u)\rangle \\
&\quad \leq (1-t)\|x_t - u\|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\langle x_t - (tx_t + (1-t)u - t\mu_t F(x_t)), J(x_t - u)\rangle \\
&\quad = \langle (1-t)(x_t - u) + t\mu_t \langle F(x_t), J(x_t - u)\rangle \\
&\quad \geq (1-t)\|x_t - u\|^2 + t\mu_t \langle F(x_t), J(x_t - u)\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
t\mu_t \langle F(x_t), J(x_t - u)\rangle &\leq \langle x_t - (tx_t + (1-t)u - t\mu_t F(x_t)), J(x_t - u)\rangle \\
&\quad - (1-t)\|x_t - u\|^2 \leq 0,
\end{aligned}$$

that is,

$$\langle F(x_t), J(x_t - u)\rangle \leq 0.$$

(ii) Because F is δ-strongly accretive, we have

$$\begin{aligned}
\langle F(x_t), J(x_t - u)\rangle &= \langle F(x_t) - F(u), J(x_t - u)\rangle + \langle F(u), J(x_t - u)\rangle \\
&\geq \delta\|x_t - u\|^2 + \langle F(u), J(x_t - u)\rangle.
\end{aligned}$$

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Using (i), we get

$$\delta\|x_t - u\|^2 + \langle F(u), J(x_t - u)\rangle \leq 0.$$

Therefore,

$$\delta\|x_t - u\|^2 \leq -\langle F(u), J(x_t - u)\rangle \leq \|F(u)\|\|x_t - u\|, \tag{2.5}$$

which implies that

$$\|x_t - u\| \leq \delta^{-1}\|F(u)\|.$$

This shows that $\{x_t : t \in (0,1)\}$ is bounded. □

A Banach space X is said to satisfy Opial's condition [11] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X \text{ with } x \neq y.$$

By Theorem 1 in [13], we know that if X admits a weak sequentially continuous duality mapping, then X satisfies Opial's condition; see, for example, [13] for further detail.

The following lemma will be needed in the sequel.

Lemma 2.5 (Lemma 2 in [18]; Demiclosedness Principle). *Let K be a nonempty closed convex subset of a reflexive Banach space X that satisfies Opial's condition and suppose that $T : K \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is,*

$$\left.\begin{aligned} x_n &\rightharpoonup x \\ x_n - Tx_n &\to 0 \end{aligned}\right\} \Rightarrow x = Tx.$$

The following lemma is also needed in the sequel.

Lemma 2.6. [11, 12]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \quad \forall n \geq 0,$$

where $\{b_n\}, \{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset [0,1], \sum_{n=0}^{\infty} b_n = \infty$,
(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

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An accretive operator A is m-accretive if $R(I + rA) = X$ for each $r > 0$. In the rest of the section, we assume that A is m-accretive and has a zero (that is, the inclusion $0 \in A(z)$ is solvable). The set of zeros of A is given by

$$A^{-1}(0) = \{z \in D(A) : 0 \in A(z)\}.$$

For each $r > 0$, we denote by J_r the resolvent of A, that is, $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : X \to X$ is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}(0)$ for all $r > 0$. Indeed, observe that

$$\begin{aligned} z \in A^{-1}(0) &\Leftrightarrow 0 \in Az \\ &\Leftrightarrow z \in (I + rA)z \\ &\Leftrightarrow z = (I + rA)^{-1}z \\ &\Leftrightarrow z = J_r z \\ &\Leftrightarrow z \in \mathrm{Fix}(J_r). \end{aligned}$$

We also denote by A_r the Yosida approximation of A, that is, $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from X to $K := \overline{D(A)}$, which will be assumed convex.

Lemma 2.7 (The Resolvent Identity [39]). *For each* $\lambda, \mu > 0$ *and each* $x \in X$,

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Let (E, d) be a metric space. Let $x \in E$ and $D \subset E$, then $d(x, D) = \inf_{v \in D} d(x, v)$. Recall that the set $D \subset E$ is a Chebyshev set, if for all $x \in E$, there exists a unique element $y \in D$ such that $d(x, y) = d(x, D)$. It is well known that every nonempty closed convex subset of a strictly convex and reflexive Banach space X is a Chebyshev set; see Corollary 5.1.19 in [28].

Let $\tilde{\mu}$ be a mean on positive integers $\mathbb{N}$, that is, a continuous linear functional $\tilde{\mu}$ on l^∞ satisfying $\|\tilde{\mu}\| = 1 = \tilde{\mu}(1)$. Then we know that $\tilde{\mu}$ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \le \tilde{\mu}(a) \le \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. According to time and circumstances, we use $\tilde{\mu}_n(a_n)$ instead of $\tilde{\mu}(a)$. A mean $\tilde{\mu}$ on $\mathbb{N}$ is called a Banach limit if

$$\tilde{\mu}_n(a_n) = \tilde{\mu}_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Using Hahn–Banach theorem, we can prove the existence of a Banach limit. We know that if $\tilde{\mu}$ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \tilde{\mu}_n(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Let $\{x_n\}$ be a bounded sequence in X. Then we can define a real-valued continuous convex function $g : X \to \mathbb{R}$ by

$$g(x) = \tilde{\mu}_n \|x_n - x\|^2, \quad \forall x \in X.$$

The following lemma, given in [19], is actually a variant of Lemma 1.2 in [33].

Lemma 2.8. *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence in X, $\tilde{\mu}$ a Banach limit on $\mathbb{N}$, and $z \in C$. Then*

$$\tilde{\mu}_n \|x_n - z\|^2 = \min_{y\in C} \tilde{\mu}_n \|x_n - y\|^2$$

if and only if

$$\tilde{\mu}_n \langle x - z, J(x_n - z)\rangle \le 0, \quad \forall x \in C,$$

where J is the duality mapping of X.

3. MANN-TYPE STEEPEST-DESCENT METHOD

Let X be a real reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Motivated and inspired by two well-known methods, Mann iteration method and steepest-descent method, we introduce and study the following Mann-type steepest-descent algorithm for computing approximate solutions of $\mathrm{VI}^*(F, C)$.

Algorithm 3.1. Suppose that $\lambda_n, \mu_n \in (0, 1)$ for all $n \ge 0$. Starting with an arbitrary initial guess $x_0 \in X$, generate a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = \lambda_n x_n + (1 - \lambda_n) T x_n, \\ x_{n+1} = y_n - \lambda_n \mu_n F(x_n), \quad \forall n \ge 0. \end{cases} \tag{3.1}$$

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We remark that Mann type steepest-descent method is based on Mann iteration method and steepest-descent method. Indeed, in Algorithm 3.1, one iteration step "$y_n = \lambda_n x_n + (1-\lambda_n)Tx_n$" is taken from Mann iteration method, and another iteration step "$x_{n+1} = y_n - \lambda_n \mu_n F(x_n)$" is taken from steepest-descent method.

We establish the strong convergence of the sequence generated by (2.4) for nonexpansive mappings and in the setting of reflexive Banach spaces that admit weak sequentially continuous duality mappings.

Proposition 3.2. *Let X be a real reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0,1)$, choose a number $\mu_t \in (0,1)$ arbitrarily and let $\{x_t\}$ be defined by* (2.4). *Then as $t \to 0^+$, x_t converges strongly to a unique solution u^* of* $\mathrm{VI}^*(F, C)$.

Proof. Let $u \in C = \mathrm{Fix}(T)$. By Lemma 2.4 (ii), $\{x_t : t \in (0,1)\}$ is bounded, and so are the sets $\{Tx_t : t \in (0,1)\}$ and $\{F(x_t) : t \in (0,1)\}$. Because $x_t = tx_t + (1-t)Tx_t - t\mu_t F(x_t)$, we have

$$\begin{aligned}
\|x_t - Tx_t\| &= \|tx_t + (1-t)Tx_t - t\mu_t F(x_t) - Tx_t\| \\
&= \|t(x_t - Tx_t) - t\mu_t F(x_t)\| \\
&\leq t\|x_t - Tx_t\| + t\mu_t \|F(x_t)\| \\
&\leq t\|x_t - Tx_t\| + t\|F(x_t)\| \to 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

This implies that

$$\lim_{t \to 0^+} \|x_t - Tx_t\| = 0.$$

Note that the set $\{x_t : t \in (0,1)\}$ is bounded. Because X is reflexive, there exists a weakly convergent subsequence $\{x_{t_n}\} \subset \{x_t\}$, where $\{t_n\}$ is a sequence in $(0,1)$ that converges to 0 as $n \to \infty$. Now we suppose that $x_n := x_{t_n}$ and $x_n \rightharpoonup u^*$. Then using Lemma 2.5, we obtain $u^* = Tu^*$. In (2.5), taking $u = u^*$ yields that

$$\|x_n - u^*\|^2 \leq -\delta^{-1}\langle F(u^*), J(x_n - u^*)\rangle.$$

Because J is weak sequentially continuous, we have

$$x_n \to u^* \quad \text{as } n \to \infty,$$

that is, $x_{t_n} \to u^*$ as $n \to \infty$.

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Next we claim that $\{x_t\}$ converges strongly to u^*. Indeed, it suffices to show that for each sequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to v^*$ as $s_k \to 0^+$, one has $v^* = u^*$. Repeating the above argument, we also have $v^* \in \mathrm{Fix}(T)$ (using $\lim_{t\to 0^+} \|x_t - Tx_t\| = 0$ and Lemma 2.5). Now let us show that $v^* = u^*$ and u^* is a unique solution of $\mathrm{VI}^*(F, C)$.

Because the sets $\{x_t\}$ and $\{F(x_t)\}$ are bounded and the duality mapping J is single-valued and weak sequentially continuous from X to X^*, for any $u \in C = \mathrm{Fix}(T)$, according to $x_{s_k} \to v^*$ $(s_k \to 0)$ we derive from Proposition 2.1 (i) that

$$\|F(x_{s_k}) - F(v^*)\| \to 0 \quad \text{as } s_k \to 0,$$

and

$$\begin{aligned}
&|\langle F(x_{s_k}), J(x_{s_k} - u)\rangle - \langle F(v^*), J(v^* - u)\rangle| \\
&\quad = |\langle F(x_{s_k}) - F(v^*), J(x_{s_k} - u)\rangle + \langle F(v^*), J(x_{s_k} - u) - J(v^* - u)\rangle| \\
&\quad \le \|F(x_{s_k}) - F(v^*)\| \|x_{s_k} - u\| \\
&\qquad + |\langle F(v^*), J(x_{s_k} - u) - J(v^* - u)\rangle| \to 0 \quad \text{as } s_k \to 0.
\end{aligned}$$

Therefore, by Lemma 2.4 (i), for any $u \in C = \mathrm{Fix}(T)$, we get

$$\langle F(v^*), J(v^* - u)\rangle = \lim_{s_k \to 0} \langle F(x_{s_k}), J(x_{s_k} - u)\rangle \le 0. \tag{3.2}$$

Similarly, we also can show that

$$\langle F(u^*), J(u^* - u)\rangle = \lim_{t_n \to 0} \langle F(x_{t_n}), J(x_{t_n} - u)\rangle \le 0. \tag{3.3}$$

Taking $u = u^*$ in (3.2) and $u = v^*$ in (3.3), respectively, we obtain

$$\langle F(v^*), J(v^* - u^*)\rangle \le 0,$$

and

$$\langle F(u^*), J(u^* - v^*)\rangle \le 0.$$

Adding the last two inequalities and using the δ-strong accretivity of F, we derive

$$\delta \|u^* - v^*\|^2 \le \langle F(u^*) - F(v^*), J(u^* - v^*)\rangle \le 0.$$

This implies that $v^* = u^*$ and u^* is a unique solution of $\mathrm{VI}^*(F, C)$. □

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Remark 3.3. It is worth pointing out that in Proposition 3.2, the assumption that X is a real reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^* is reasonable. Indeed, by a gauge we mean a continuous strictly increasing function φ defined on $[0,\infty)$ such that $\varphi(0)=0$ and $\lim_{r\to\infty}\varphi(r)=\infty$. We associate with a gauge φ a (generally multivalued) duality map $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}.$$

Clearly, the normalized duality map J corresponds with the gauge $\varphi(t)=t$. Browder [4] initiated the study of certain classes of nonlinear operators by means of a duality map J_φ. Set for $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r)\,dr.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x. In this case, it is easy to see from Lim and Xu [23] that the following inequality holds

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x+y)\rangle$$

for all $x, y \in E$, where $j_\varphi(x+y) \in J_\varphi(x+y)$. Recall that X is said to admit a weak sequentially continuous duality map (see [23, p. 1350]) if there exists a gauge φ such that the duality map J_φ is single-valued and sequentially continuous from X with the weak topology to X^* with the weak* topology. Every l^p $(1 < p < \infty)$ space has a weak sequentially continuous duality map J_φ with the gauge $\varphi(t) = t^{p-1}$. In Proposition 3.2, for the sake of simplicity, we may assume without loss of generality that $J := J_\varphi$ and J is normalized. Therefore, in Proposition 3.2, the assumption that X is a real reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^* is certainly reasonable.

In addition, we note that Song and Chen [34] considered the following variational inequality problem (for short, $VI^*(I-f, C)$): Find $u^* \in C$ such that

$$\langle (I-f)u^*, J(u^*-v)\rangle \leq 0, \quad \forall v \in C,$$

where $C = \mathrm{Fix}(T)$, and f is a fixed contractive self-mapping on X. They established a result (Theorem 2.2 in [34]) for $\mathrm{VI}^*(I-F, C)$ similar to Proposition 3.2. But the assumptions in Proposition 3.2 are different from those in Theorem 2.2 in [34].

To prove the strong convergence of the sequence generated by (3.1), we need the following lemma.

Lemma 3.4 [37]. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;
(ii) $\limsup_{n\to\infty}\beta_n \leq 0$;
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Now we prove the strong convergence of the sequence generated by Algorithm 3.1 to a solution of $\mathrm{VI}^*(F, C)$.

Theorem 3.5. *Let X be a real reflexive Banach space that admits a weak sequentially continuous duality mapping J from X to X^*. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{x_n\}$ be defined by* (3.1), *where $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 0$ *and* $\sum_{n=0}^{\infty} \lambda_n\mu_n = \infty$;
(ii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(iii) *either* $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = 1$.

Then $\{x_n\}$ converges strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$.

Proof. First, we claim that $\lim_{n\to\infty} \lambda_n = 0$. Indeed, because $\{\mu_n\}$ is a sequence in $(0,1)$, we have

$$\lambda_n = \mu_n \frac{\lambda_n}{\mu_n} \leq \frac{\lambda_n}{\mu_n} \to 0 \quad \text{as } n \to \infty,$$

that is, $\lim_{n\to\infty} \lambda_n = 0$.

Second, we claim that $\{x_n\}$ is bounded. Indeed, take $u \in C = \mathrm{Fix}(T)$. Then, using Proposition 2.1(iii), we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|y_n - \lambda_n\mu_n F(x_n) - u\| \\ &= \|\lambda_n x_n + (1-\lambda_n)Tx_n - \lambda_n\mu_n F(x_n) - u\|\end{aligned}$$

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$$
\begin{aligned}
&= \|\lambda_n[(I-\mu_n F)x_n - u] + (1-\lambda_n)(Tx_n - u)\| \\
&\le (1-\lambda_n)\|Tx_n - u\| + \lambda_n\|(I-\mu_n F)x_n - u\| \\
&\le (1-\lambda_n)\|x_n - u\| + \lambda_n[\|(I-\mu_n F)x_n - (I-\mu_n F)u\| \\
&\quad + \|(I-\mu_n F)u - u\|] \\
&\le (1-\lambda_n)\|x_n - u\| + \lambda_n\left[1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|x_n - u\| \\
&\quad + \lambda_n\mu_n\|F(u)\| \\
&= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u\| + \lambda_n\mu_n\|F(u)\| \\
&= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u\| \\
&\quad + \lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\| \\
&\le \max\left\{\|x_n - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\}.
\end{aligned}
$$

By induction,

$$
\|x_n - u\| \le \max\left\{\|x_0 - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\}, \quad \forall n \ge 0.
$$

Hence $\{x_n\}$ is bounded and so are the sequences $\{Tx_n\}$ and $\{F(x_n)\}$. Now we claim that

$$
\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.4}
$$

Indeed, we have (for some approximate constant $M > 0$)

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\lambda_n x_n + (1-\lambda_n)Tx_n - \lambda_n\mu_n F(x_n) \\
&\quad - (\lambda_{n-1}x_{n-1} + (1-\lambda_{n-1})Tx_{n-1} - \lambda_{n-1}\mu_{n-1}F(x_{n-1}))\| \\
&= \|\lambda_n(I-\mu_n F)x_n + (1-\lambda_n)Tx_n \\
&\quad - (\lambda_{n-1}(I-\mu_{n-1}F)x_{n-1} + (1-\lambda_{n-1})Tx_{n-1})\| \\
&\le \|(1-\lambda_n)(Tx_n - Tx_{n-1}) \\
&\quad + (\lambda_n - \lambda_{n-1})[(I-\mu_{n-1}F)x_{n-1} - Tx_{n-1}]\| \\
&\quad + \lambda_n\|(I-\mu_n F)x_n - (I-\mu_{n-1}F)x_{n-1}\| \\
&\le (1-\lambda_n)\|Tx_n - Tx_{n-1}\|
\end{aligned}
$$

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$$
\begin{aligned}
&+ |\lambda_n - \lambda_{n-1}| \|(I - \mu_{n-1}F)x_{n-1} - Tx_{n-1}\| \\
&+ \lambda_n [\|(I - \mu_n F)x_n - (I - \mu_n F)x_{n-1}\| + \|(\mu_{n-1} - \mu_n)F(x_{n-1})\|] \\
\le\ & (1 - \lambda_n)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M \\
&+ \lambda_n \left[\left(1 - \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) \right) \|x_n - x_{n-1}\| + |\mu_{n-1} - \mu_n| M \right] \\
=\ & \left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right) \right) \|x_n - x_{n-1}\| + \lambda_n |\mu_{n-1} - \mu_n| M \\
&+ |\lambda_n - \lambda_{n-1}| M.
\end{aligned}
$$

Put

$$
\begin{aligned}
\alpha_n &= \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right), \\
\beta_n &= \left(1 - \sqrt{\frac{1-\delta}{\lambda}} \right)^{-1} \left| \frac{\mu_{n-1}}{\mu_n} - 1 \right|,
\end{aligned}
$$

and $\gamma_n = |\lambda_n - \lambda_{n-1}|M$. Then it follows that

$$
\|x_{n+1} - x_n\| \le (1 - \alpha_n)\|x_n - x_{n-1}\| + \alpha_n \beta_n + \gamma_n. \tag{3.5}
$$

It is readily seen from conditions (i)–(iii) that one of the following conditions (a) and (b) holds:

$$
\text{(a)} \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \beta_n = 0, \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < \infty;
$$

$$
\text{(b)} \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \alpha_n \beta_n < \infty, \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < \infty.
$$

Thus, applying Lemma 3.4 to (3.5), we conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

We now show that

$$
\|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.6}
$$

Indeed, observe that

$$
\begin{aligned}
\|x_{n+1} - Tx_n\| &= \|\lambda_n x_n + (1 - \lambda_n)Tx_n - \lambda_n \mu_n F(x_n) - Tx_n\| \\
&\le \lambda_n \|x_n - Tx_n\| + \lambda_n \|F(x_n)\|.
\end{aligned}
$$

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So it follows from (3.4) that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \lambda_n\|x_n - Tx_n\| + \lambda_n\|F(x_n)\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Let $u^* = \lim_{t\to 0^+} x_t$, where $\{x_t\}$ is defined by (2.4). By Proposition 3.2, there exists a unique solution u^* of $\mathrm{VI}^*(F, C)$, that is,

$$\langle F(u^*), J(u^* - v)\rangle \le 0, \quad \forall v \in C. \tag{3.7}$$

Next, let us show that

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle \le 0. \tag{3.8}$$

Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle = \lim_{k\to\infty} \langle F(u^*), J(u^* - x_{n_k})\rangle.$$

Because X is reflexive and because $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup x^*$. In terms of Lemma 2.5, we deduce from $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$ that $x^* \in C = \mathrm{Fix}(T)$. Because the duality mapping J is weak sequentially continuous from X to X^*, from (3.7) we obtain

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle = \langle F(u^*), J(u^* - x^*)\rangle \le 0.$$

Finally, let us show that $x_n \to u^*$ as $n \to \infty$. Indeed, observe that

$$x_{n+1} - (\lambda_n x_n + (1 - \lambda_n)u^* - \lambda_n\mu_n F(x_n)) = x_{n+1} - u^* - \lambda_n((I - \mu_n F)x_n - u^*).$$

Using Lemma 2.2 and Proposition 2.1(iii), we have

$$\begin{aligned}\|x_{n+1} - u^*\|^2 &= \|x_{n+1} - (\lambda_n x_n + (1 - \lambda_n)u^* - \lambda_n\mu_n F(x_n)) \\ &\qquad + \lambda_n((I - \mu_n F)x_n - u^*)\|^2 \\ &\le \|x_{n+1} - (\lambda_n x_n + (1 - \lambda_n)u^* - \lambda_n\mu_n F(x_n))\|^2 \\ &\qquad + 2\lambda_n\langle (I - \mu_n F)x_n - u^*, J(x_{n+1} - u^*)\rangle \\ &= (1 - \lambda_n)^2\|Tx_n - u^*\|^2 \\ &\qquad + 2\lambda_n\langle (I - \mu_n F)x_n - (I - \mu_n F)u^*, J(x_{n+1} - u^*)\rangle \\ &\qquad + 2\lambda_n\langle (I - \mu_n F)u^* - u^*, J(x_{n+1} - u^*)\rangle\end{aligned}$$

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$$
\begin{aligned}
&\le \left(1-\lambda_n\right)^2\|x_n-u^*\|^2 \\
&\quad + 2\lambda_n\|(I-\mu_nF)x_n-(I-\mu_nF)u^*\|\|x_{n+1}-u^*\| \\
&\quad + 2\lambda_n\mu_n\langle F(u^*),J(u^*-x_{n+1})\rangle \\
&\le (1-\lambda_n)^2\|x_n-u^*\|^2 \\
&\quad + 2\lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-u^*\|\|x_{n+1}-u^*\| \\
&\quad + 2\lambda_n\mu_n\langle F(u^*),J(u^*-x_{n+1})\rangle \\
&\le \left(1-\lambda_n\right)^2\|x_n-u^*\|^2 \\
&\quad + \lambda_n\left[\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2\|x_n-u^*\|^2+\|x_{n+1}-u^*\|^2\right] \\
&\quad + 2\lambda_n\mu_n\langle F(u^*),J(u^*-x_{n+1})\rangle \\
&\le \left(1-\lambda_n\right)^2\|x_n-u^*\|^2+\lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-u^*\|^2 \\
&\quad + \lambda_n\|x_{n+1}-u^*\|^2+2\lambda_n\mu_n\langle F(u^*),J(u^*-x_{n+1})\rangle.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\|x_{n+1}-u^*\|^2 &\le \left(1-\lambda_n+\frac{\lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)}{1-\lambda_n}\right)\|x_n-u^*\|^2 \\
&\quad + \frac{2\lambda_n\mu_n}{1-\lambda_n}\langle F(u^*),J(u^*-x_{n+1})\rangle \\
&= \left(1-\frac{\lambda_n\mu_n}{1-\lambda_n}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)+\frac{\lambda_n^2}{1-\lambda_n}\right)\|x_n-u^*\|^2 \\
&\quad + \frac{2\lambda_n\mu_n}{1-\lambda_n}\langle F(u^*),J(u^*-x_{n+1})\rangle \\
&= \left(1-\frac{\lambda_n\mu_n}{1-\lambda_n}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-u^*\|^2+\frac{\lambda_n\mu_n}{1-\lambda_n}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right) \\
&\quad \cdot\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\left[\frac{\lambda_n}{\mu_n}\|x_n-u^*\|^2+2\langle F(u^*),J(u^*-x_{n+1})\rangle\right].
\end{aligned}
$$

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Put

$$\alpha_n = \frac{\lambda_n \mu_n}{1-\lambda_n}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right),$$

$$\beta_n = \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\left[\frac{\lambda_n}{\mu_n}\|x_n - u^*\|^2 + 2\langle F(u^*), J(u^* - x_{n+1})\rangle\right],$$

and $\gamma_n = 0$. Then it follows that

$$\|x_{n+1} - u^*\|^2 \le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n\beta_n + \gamma_n. \tag{3.9}$$

Because $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$, we have $\sum_{n=0}^{\infty}\frac{\lambda_n\mu_n}{1-\lambda_n} = \infty$ and hence $\sum_{n=0}^{\infty}\alpha_n = \infty$. Note that $\lim_{n\to\infty}\lambda_n/\mu_n = 0$ and $\limsup_{n\to\infty}\langle F(u^*), J(u^* - x_{n+1})\rangle \le 0$ due to (3.8). Thus, according to the boundedness of $\{x_n - u^*\}$, we have $\limsup_{n\to\infty}\beta_n \le 0$. Consequently, applying Lemma 3.4 to (3.9), we conclude that $\lim_{n\to\infty}\|x_n - u^*\| = 0$. □

Remark 3.6. Theorem 2.4 in [34] is closely related to Theorem 3.5 but with different assumptions. However, it involves the variational inequality problem $\mathrm{VI}^*(I-F, C)$.

4. MODIFIED HYBRID STEEPEST-DESCENT METHOD

Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Motivated and inspired by Algorithm 1.1 in [45], we suggest and analyze the following modified hybrid steepest-descent algorithms for finding the approximate solutions of $\mathrm{VI}^*(F, C)$.

Algorithm 4.1. Suppose $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are three sequences of positive numbers in $(0, 1]$. Starting with an arbitrary initial guess $x_0 \in X$, generate a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = Tx_{n-1} - \lambda_n\mu_n F(Tx_{n-1}), \\ x_n = \alpha_n y_n + (1-\alpha_n)Tx_n. \end{cases} \tag{4.1}$$

Algorithm 4.2. Suppose $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are three sequences of positive numbers in $(0, 1]$. Starting with an arbitrary initial guess $x_0 \in X$, generate a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ x_{n+1} = Ty_n - \lambda_n\mu_n F(Ty_n). \end{cases} \tag{4.2}$$

We first establish the convergence of the sequence defined by (2.4) for continuous pseudocontractive mappings and in the setting of reflexive and strictly convex Banach spaces with uniformly Gâteaux differentiable norms.

Proposition 4.3. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $T : X \to X$ is a continuous pseudocontractive mapping and $C = Fix(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0, 1)$, choose a number $\mu_t \in (0, 1)$ arbitrarily and let $\{x_t\}$ be defined by (2.4). Then as $t \to 0^+$, x_t converges strongly to a unique solution u^* of VI*(F, C).*

Proof. First, we prove that the uniqueness of a solution of $\mathrm{VI}^*(F, C)$. Indeed, suppose that u^* and $v^* \in C = \mathrm{Fix}(T)$ are two solutions of $\mathrm{VI}^*(F, C)$. Then, we have

$$\langle F(u^*), J(u^* - v^*)\rangle \leq 0$$

and

$$\langle F(v^*), J(v^* - u^*)\rangle \leq 0.$$

Adding up the last two inequalities, we have

$$\delta\|u^* - v^*\|^2 \leq \langle F(u^*) - F(v^*), J(u^* - v^*)\rangle \leq 0.$$

Hence $u^* = v^*$. Below we use $u^* \in C = \mathrm{Fix}(T)$ to denote a unique solution of $\mathrm{VI}^*(F, C)$.

According to Lemma 2.4 (ii), $\{x_t : t \in (0, 1)\}$ is bounded. Because F is a Lipschitzian mapping with constant $1 + 1/\lambda$, we have $\|F(x_t) - F(u)\| \leq (1 + 1/\lambda)\|x_t - u\|$. So, the set $\{F(x_t) : t \in (0, 1)\}$ is bounded. Noticing that $x_t = tx_t + (1 - t)Tx_t - t\mu_t F(x_t)$, we have

$$Tx_t = x_t + \frac{t\mu_t}{1 - t}F(x_t),$$

and hence

$$\|Tx_t\| \leq \|x_t\| + \frac{t\mu_t}{1 - t}\|F(x_t)\| \leq \|x_t\| + \frac{t}{1 - t}\|F(x_t)\|.$$

Because $\lim_{t\to 0^+} \frac{t}{(1-t)} = 0$, there exists $t_0 \in (0, 1)$ such that $\{t/(1 - t) : t \in (0, t_0)\}$ is bounded. So, $\{Tx_t : t \in (0, t_0)\}$ is bounded. This implies that

$$\|x_t - Tx_t\| = \frac{t\mu_t}{1 - t}\|F(x_t)\| \leq \frac{t}{1 - t}\|F(x_t)\| \to 0, \quad \text{as } t \to 0^+.$$

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From Lemma 2.3, the mapping $A = (2I - T)^{-1} : X \to X$ is nonexpansive, $F(A) = F(T)$, and $\lim_{t\to 0^+} \|x_t - Ax_t\| = 0$, where I denotes the identity operator of X.

We claim that the set $\{x_t : t \in (0, t_0)\}$ is relatively compact. Indeed, suppose that $x_n := x_{t_n}$ and

$$g(x) = \mu_n \|x_n - x\|^2, \quad \forall x \in X,$$

where $\{t_n\}$ is a sequence in $(0, t_0)$ that converges to 0 as $n \to \infty$ and $\tilde{\mu}_n$ is a Banach limit. Define the set

$$K = \left\{ x \in X : g(x) = \inf_{y \in X} g(y) \right\}.$$

Because X is a reflexive Banach space, K is a nonempty bounded closed convex subset of X according to Theorem 1.3.11 in [36]. Also, because $\lim_{n\to\infty} \|x_n - Ax_n\| = 0$, we deduce that for all $x \in K$,

$$\begin{aligned} g(Ax) = \tilde{\mu}_n \|x_n - Ax\|^2 &\le \tilde{\mu}_n \big(\|x_n - Ax_n\| + \|Ax_n - Ax\| \big)^2 \\ &\le \tilde{\mu}_n \|x_n - x\|^2 = g(x). \end{aligned}$$

Hence, $Ax \in K$. This shows that K is invariant under A. Because $\mathrm{Fix}(A) = \mathrm{Fix}(T)$, we can pick $u \in F(A) = F(T)$ arbitrarily. Because every nonempty closed convex subset of a strictly convex and reflexive Banach space X is a Chebyshev set according to Corollary 5.1.19 in [28], there exists a unique $\hat{u} \in K$ such that

$$\|u - \hat{u}\| = \inf_{x \in K} \|u - x\|.$$

Because $u = Tu = Au$ and $A\hat{u} \in K$, we have

$$\|u - A\hat{u}\| = \|Au - A\hat{u}\| \le \|u - \hat{u}\|.$$

Hence $\hat{u} = A\hat{u}$. Using Lemma 2.8, we have

$$\tilde{\mu}_n \langle x - \hat{u}, j(x_n - \hat{u}) \rangle \le 0, \quad \forall x \in X$$

Taking $x = \hat{u} - F(\hat{u})$ in the last inequality, we conclude from Lemma 2.4(i) that

$$\tilde{\mu}_n \|x_n - \hat{u}\|^2 \le -\delta^{-1} \tilde{\mu}_n \langle F(\hat{u}), J(x_n - \hat{u}) \rangle \le 0.$$

This implies that

$$\tilde{\mu}_n \|x_n - \hat{u}\|^2 = 0.$$

Hence, there exists a subsequence of $\{x_n\}$ that converges strongly to $\hat{u} \in C = F(T)$. Without loss of generality, we may assume that $\{x_n\}$ converges strongly to $\hat{u}$.

Next, we claim that $\hat{u}$ is a solution of $\mathrm{VI}^*(F, C)$. Indeed, take $u \in C = \mathrm{Fix}(T)$ arbitrarily. By Lemma 2.4 (i), we have

$$\langle F(x_t), J(x_t - u)\rangle \leq 0. \tag{4.3}$$

Note that the duality mapping J is single-valued and norm-to-weak* uniformly continuous on bounded sets of a Banach space X with uniformly Gâteaux differentiable norm. Hence we deduce from $x_n \to \hat{u}$ that as $n \to \infty$,

$$\|F(x_n) - F(\hat{u})\| \to 0$$

and

$$\begin{aligned}
&|\langle F(x_n), J(x_n - u)\rangle - \langle F(\hat{u}), J(\hat{u} - u)\rangle| \\
&\quad = |\langle F(x_n) - F(\hat{u}), J(x_n - u)\rangle + \langle F(\hat{u}), J(x_n - u) - J(\hat{u} - u)\rangle| \\
&\quad \leq \|F(x_n) - F(\hat{u})\|\|x_n - u\| + |\langle F(\hat{u}), J(x_n - u) - J(\hat{u} - u)\rangle| \to 0.
\end{aligned}$$

Therefore, it follows immediately from (4.3) that

$$\langle F(\hat{u}), J(\hat{u} - u)\rangle = \lim_{n\to\infty} \langle F(x_n), J(x_n - u)\rangle \leq 0.$$

Consequently, $\hat{u} \in C = \mathrm{Fix}(T)$ is a solution of $\mathrm{VI}^*(F, C)$ and hence $\hat{u} = u^*$ by uniqueness. All in all, we have proved that the set $\{x_t\}$ is relatively compact and every cluster point of $\{x_t\}$ (as $t \to 0^+$) equals u^*. Therefore, $x_t \to u^*$ as $t \to 0^+$. □

Remark 4.4. Theorem 2.1 in [34] is closely related to Proposition 4.3 but with different assumptions. However, it involves the following variational inequality problem (for short, $VI^*(I - F, C)$) with F a fixed Lipschitzian strongly pseudocontractive self-mapping in X.

Proposition 4.5. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm. Suppose $T : X \to X$ is a continuous pseudocontractive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $u^* = \lim_{t\to 0^+} z_t$ exists, where $\{z_t\}$ is defined by* (2.4). *Then*

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle \leq 0.$$

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Proof. Using equality

$$z_t - x_n = (1-t)(Tz_t - x_n) + t((I - \mu_t F)z_t - x_n)$$

and inequality

$$\langle Tx - Ty, J(x-y)\rangle \le \|x-y\|^2,$$

we obtain that

$$\begin{aligned}
\|z_t - x_n\|^2 &= (1-t)\langle Tz_t - x_n, J(z_t - x_n)\rangle + t\langle (I-\mu_t F)z_t - x_n, J(z_t - x_n)\rangle \\
&= (1-t)(\langle Tz_t - Tx_n, J(z_t - x_n)\rangle + \langle Tx_n - x_n, J(z_t - x_n)\rangle) \\
&\quad + t\langle (I-\mu_t F)z_t - z_t, J(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\
&\le (1-t)(\|z_t - x_n\|^2 + \|Tx_n - x_n\|\|J(z_t - x_n)\|) \\
&\quad + t\langle (I-\mu_t F)z_t - z_t, J(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\
&\le \|z_t - x_n\|^2 + \|Tx_n - x_n\|\|z_t - x_n\| + t\langle (I-\mu_t F)z_t - z_t, J(z_t - x_n)\rangle \\
&= \|z_t - x_n\|^2 + \|Tx_n - x_n\|\|z_t - x_n\| - t\mu_t\langle F(z_t), J(z_t - x_n)\rangle,
\end{aligned}$$

and hence

$$\langle F(z_t), J(z_t - x_n)\rangle \le \frac{\|x_n - Tx_n\|}{t\mu_t}\|z_t - x_n\|. \tag{4.4}$$

Because both $\{z_t\}$ and $\{x_n\}$ are bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, taking the superior limit in (4.4), we obtain that

$$\limsup_{n\to\infty}\langle F(z_t), J(z_t - x_n)\rangle \le 0. \tag{4.5}$$

On the other hand, because $\{z_t - x_n\}$ is bounded and the duality mapping J is single-valued and norm-to-weak* uniformly continuous on bounded sets of a Banach space X with uniformly Gâteaux differentiable norm, so we conclude that (using the condition $z_t \to u^*$ $(t \to 0^+)$)

$$\|F(z_t) - F(u^*)\| \le \left(1 + \frac{1}{\lambda}\right)\|z_t - u^*\| \to 0,$$

and

$$\begin{aligned}
&|\langle F(z_t), J(z_t - x_n)\rangle - \langle F(u^*), J(u^* - x_n)\rangle| \\
&\quad = |\langle F(z_t) - F(u^*), J(z_t - x_n)\rangle + \langle F(u^*), J(z_t - x_n) - J(u^* - x_n)\rangle| \\
&\quad \le \|F(z_t) - F(u^*)\|\|z_t - x_n\| \\
&\qquad + |\langle F(u^*), J(z_t - x_n) - J(u^* - x_n)\rangle| \to 0 \quad \text{as } t \to 0^+.
\end{aligned}$$

Thus, for an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for all n

$$|\langle F(z_t), J(z_t - x_n)\rangle - \langle F(u^*), J(u^* - x_n)\rangle| < \varepsilon, \quad \forall t \in (0, \delta).$$

and hence

$$\langle F(u^*), J(u^* - x_n)\rangle < \langle F(z_t), J(z_t - x_n)\rangle + \varepsilon, \quad \forall t \in (0, \delta).$$

Using (4.5), we have

$$\limsup_{n\to\infty}\langle F(u^*), J(u^* - x_n)\rangle \le \limsup_{n\to\infty}\langle F(z_t), J(z_t - x_n)\rangle + \varepsilon \le \varepsilon.$$

By the arbitrariness of ε, we have $\limsup_{n\to\infty}\langle F(u^*),\ J(u^* - x_n)\rangle \le 0$. The proof is complete. □

Remark 4.6. Theorem 2.3 in [34] is closely related to Proposition 4.5 but with different assumptions. Because the assumptions on F are very different, the proofs for these two results are also different.

Now we show that the sequences $\{x_n\}$ generated by (4.1) and (4.2) converge strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$ under suitable restrictions on the variable parameters $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$.

Theorem 4.7. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $T : X \to X$ is a nonexpansive mapping and $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{x_n\}$ be defined by* (4.1)*, where $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are three sequences of positive numbers in* $(0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$.

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, taking a fixed $u \in C = \mathrm{Fix}(T)$, we have

$$\begin{aligned}\|x_n - u\| &= \|\alpha_n y_n + (1 - \alpha_n)Tx_n - u\| \\ &\le \alpha_n\|y_n - u\| + (1 - \alpha_n)\|Tx_n - u\| \\ &\le \alpha_n\|y_n - u\| + (1 - \alpha_n)\|x_n - u\|.\end{aligned}$$

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So, $\|x_n - u\| \le \|y_n - u\|$ for all $n \ge 0$. Thus, by Proposition 2.1(iii), we have

$$\begin{aligned}
\|x_n - u\| &\le \|y_n - u\| \\
&= \|Tx_{n-1} - \lambda_n \mu_n F(Tx_{n-1}) - u\| \\
&= \|\lambda_n (I - \mu_n F) Tx_{n-1} + (1 - \lambda_n) Tx_{n-1} - u\| \\
&\le \lambda_n \|(I - \mu_n F) Tx_{n-1} - u\| + (1 - \lambda_n)\|Tx_{n-1} - u\| \\
&\le \lambda_n \|(I - \mu_n F) Tx_{n-1} - (I - \mu_n F)u\| + \lambda_n \|(I - \mu_n F)u - u\| \\
&\quad + (1 - \lambda_n)\|Tx_{n-1} - u\| \\
&\le \lambda_n \left(1 - \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Tx_{n-1} - u\| + \lambda_n \mu_n \|F(u)\| \\
&\quad + (1 - \lambda_n)\|Tx_{n-1} - u\| \\
&= \left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Tx_{n-1} - u\| + \lambda_n \mu_n \|F(u)\| \\
&\le \left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n-1} - u\| + \lambda_n \mu_n \|F(u)\| \\
&= \left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n-1} - u\| \\
&\quad + \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1} \|F(u)\| \\
&\le \max\left\{\|x_{n-1} - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1} \|F(u)\|\right\}.
\end{aligned}$$

By induction,

$$\|x_n - u\| \le \max\left\{\|x_0 - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1} \|F(u)\|\right\}, \quad \forall n \ge 0.$$

Hence $\{x_n\}$ is bounded. Because T is nonexpansive and F is Lipschitzian, $\{Tx_n\}$ and $\{F(Tx_n)\}$ are bounded and so is $\{y_n\}$ due to (4.1). From condition (i), we have

$$\|x_n - Tx_n\| = \alpha_n \|y_n - Tx_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.6}$$

Put $z_t = tz_t + (1 - t)Tz_t - t\mu_t F(z_t)$. Then it follows from Proposition 4.3 that as $t \to 0^+$, z_t converges strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$. It follows from (4.6) and Proposition 4.5 that

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle \le 0. \tag{4.7}$$

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Finally, we claim that $x_n \to x^*$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned}
\|x_n - u^*\|^2 &= (1-\alpha_n)\langle Tx_n - u^*, J(x_n - u^*)\rangle + \alpha_n\langle y_n - u^*, J(x_n - u^*)\rangle \\
&= (1-\alpha_n)\langle Tx_n - u^*, J(x_n - u^*)\rangle + \alpha_n\langle (1-\lambda_n)Tx_{n-1} \\
&\quad + \lambda_n(I-\mu_n F)Tx_{n-1} - u^*, J(x_n - u^*)\rangle \\
&\le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n(1-\lambda_n)\langle Tx_{n-1} - u^*, J(x_n - u^*)\rangle \\
&\quad + \alpha_n\lambda_n\langle (I-\mu_n F)Tx_{n-1} - u^*, J(x_n - u^*)\rangle \\
&\le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n(1-\lambda_n)\|Tx_{n-1} - u^*\|\|J(x_n - u^*)\| \\
&\quad + \alpha_n\lambda_n(\langle (I-\mu_n F)Tx_{n-1} - (I-\mu_n F)u^*, J(x_n - u^*)\rangle \\
&\quad + \langle (I-\mu_n F)u^* - u^*, J(x_n - u^*)\rangle) \\
&\le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n(1-\lambda_n)\|Tx_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \alpha_n\lambda_n(\|(I-\mu_n F)Tx_{n-1} - (I-\mu_n F)u^*\|\|J(x_n - u^*)\| \\
&\quad + \langle (I-\mu_n F)u^* - u^*, J(x_n - u^*)\rangle) \\
&\le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n(1-\lambda_n)\|Tx_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \alpha_n\lambda_n\left(1 - \mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Tx_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \alpha_n\lambda_n\mu_n\langle F(u^*), J(u^* - x_n)\rangle \\
&= (1-\alpha_n)\|x_n - u^*\|^2 \\
&\quad + \alpha_n\left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Tx_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \alpha_n\lambda_n\mu_n\langle F(u^*), J(u^* - x_n)\rangle \\
&\le (1-\alpha_n)\|x_n - u^*\|^2 \\
&\quad + \alpha_n\left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \alpha_n\lambda_n\mu_n\langle F(u^*), J(u^* - x_n)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_n - u^*\|^2 &\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{n-1} - u^*\|\|x_n - u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_n)\rangle \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\frac{\|x_{n-1} - u^*\|^2 + \|x_n - u^*\|^2}{2} \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_n)\rangle.
\end{aligned}$$

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This yields that

$$\begin{aligned}\|x_n - u^*\|^2 &\le \frac{1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\|x_{n-1}-u^*\|^2 \\ &\quad + \frac{2\lambda_n\mu_n}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\langle F(u^*), J(u^*-x_n)\rangle \\ &= \left(1 - \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\right)\|x_{n-1}-u^*\|^2 \\ &\quad + \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\cdot\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*), J(u^*-x_n)\rangle.\end{aligned}$$

Put

$$b_n = \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)},$$

and

$$c_n = \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*), J(u^*-x_n)\rangle.$$

Then the last inequality can be rewritten as

$$\|x_n - u^*\|^2 \le (1-b_n)\|x_{n-1}-u^*\|^2 + b_n c_n. \tag{4.8}$$

Because $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$, we have $\sum_{n=0}^{\infty}\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)} = \infty$ and hence $\sum_{n=0}^{\infty} b_n = \infty$. Note that $\limsup_{n\to\infty}\langle F(u^*), J(u^*-x_n)\rangle \le 0$ due to (4.7). Thus, $\limsup_{n\to\infty} c_n \le 0$. Consequently, applying Lemma 2.6 to (4.8), we conclude that $\lim_{n\to\infty}\|x_n - u^*\| = 0$. □

Theorem 4.8. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose $T : X \to X$ is a nonexpansive mapping and $C = Fix(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{x_n\}$ be defined by* (4.2)*, and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are three sequences of positive numbers in* $(0, 1]$ *satisfying the conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$.

Then, $\{x_n\}$ *converges strongly to a unique solution* u^* *of* $\mathrm{VI}^*(F, C)$.

Proof. First, we claim that $\{x_n\}$ is bounded. Taking a fixed $u \in C = F(T)$, we have

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n x_n + (1-\alpha_n)Ty_n - u\| \\ &\le (1-\alpha_n)\|Ty_n - u\| + \alpha_n\|x_n - u\| \\ &\le (1-\alpha_n)\|y_n - u\| + \alpha_n\|x_n - u\|.\end{aligned}$$

So, $\|y_n - u\| \le \|x_n - u\|$ for all $n \ge 0$. Thus we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|Ty_n - \lambda_n\mu_n F(Ty_n) - u\| \\ &= \|\lambda_n(I-\mu_n F)Ty_n + (1-\lambda_n)Ty_n - u\| \\ &\le \lambda_n\|(I-\mu_n F)Ty_n - u\| + (1-\lambda_n)\|Ty_n - u\| \\ &\le \lambda_n\|(I-\mu_n F)Ty_n - (I-\mu_n F)u\| + \lambda_n\|(I-\mu_n F)u - u\| \\ &\quad + (1-\lambda_n)\|Ty_n - u\| \\ &\le \lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Ty_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &\quad + (1-\lambda_n)\|Ty_n - u\| \\ &= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Ty_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &\le \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|y_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &\le \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u\| \\ &\quad + \lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\cdot\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\| \\ &\le \max\left\{\|x_n - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\}.\end{aligned}$$

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By induction, we derive

$$\|x_n - u\| \le \max\left\{\|x_0 - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1} \|F(u)\|\right\}, \quad \forall n \ge 0.$$

This shows that $\{x_n\}$ is bounded, and so is $\{y_n\}$. Because T is nonexpansive and F is Lipschitzian, we know that $\{Ty_n\}$ and $\{F(Ty_n)\}$ are bounded. From condition (i), we obtain

$$\|y_n - Ty_n\| = \alpha_n \|x_n - Ty_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

Put $z_t = tz_t + (1-t)Tz_t - t\mu_t F(z_t)$. Then it follows from Proposition 4.3 that as $t \to 0^+$, z_t converges strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$.

From (4.9) and Proposition 4.5, it follows that

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - y_n)\rangle \le 0. \tag{4.10}$$

Furthermore, using (4.8) and (4.9), we have

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|Ty_n - \lambda_n\mu_n F(Ty_n) - (\alpha_n x_n + (1-\alpha_n)Ty_n)\| \\ &\le \alpha_n\|x_n - Ty_n\| + \lambda_n\mu_n\|F(Ty_n)\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Because the duality map J is single-valued and norm-to-weak* uniformly continuous on bounded sets in Banach space X with uniformly Gâteaux differentiable norm, we have

$$\lim_{n\to\infty} \langle F(u^*), J(u^* - x_{n+1}) - J(u^* - y_n)\rangle = 0.$$

So, from (4.10) we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_{n+1})\rangle \\ &\quad = \limsup_{n\to\infty} (\langle F(u^*), J(u^* - y_n)\rangle + \langle F(u^*), J(u^* - x_{n+1}) - J(u^* - y_n)\rangle) \\ &\quad \le \limsup_{n\to\infty} \langle F(u^*), J(u^* - y_n)\rangle \\ &\qquad + \limsup_{n\to\infty} \langle F(u^*), J(u^* - x_{n+1}) - J(u^* - y_n)\rangle \\ &\quad = \limsup_{n\to\infty} \langle F(u^*), J(u^* - y_n)\rangle \le 0.\end{aligned} \tag{4.11}$$

Finally, we claim that $x_n \to u^*$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned}\|x_{n+1} - u^*\|^2 &= \langle Ty_n - \lambda_n\mu_n F(Ty_n) - u^*, J(x_{n+1} - u^*)\rangle \\ &= \langle \lambda_n(I - \mu_n F)Ty_n + (1-\lambda_n)Ty_n - u^*, J(x_{n+1} - u^*)\rangle\end{aligned}$$

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$$
\begin{aligned}
&= \lambda_n\langle (I-\mu_n F)Ty_n - u^*, J(x_{n+1}-u^*)\rangle \\
&\quad + (1-\lambda_n)\langle Ty_n - u^*, J(x_{n+1}-u^*)\rangle \\
&= \lambda_n(\langle (I-\mu_n F)Ty_n - (I-\mu_n F)u^*, J(x_{n+1}-u^*)\rangle \\
&\quad + \langle (I-\mu_n F)u^* - u^*, J(x_{n+1}-u^*)\rangle) \\
&\quad + (1-\lambda_n)\langle Ty_n - u^*, J(x_{n+1}-u^*)\rangle \\
&\le \lambda_n(\|(I-\mu_n F)Ty_n - (I-\mu_n F)u^*\|\|x_{n+1}-u^*\| \\
&\quad + \langle (I-\mu_n F)u^* - u^*, J(x_{n+1}-u^*)\rangle) \\
&\quad + (1-\lambda_n)\|Ty_n - u^*\|\|x_{n+1}-u^*\| \\
&\le \lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Ty_n - u^*\|\|x_{n+1}-u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^*-x_{n+1})\rangle \\
&\quad + (1-\lambda_n)\|Ty_n - u^*\|\|x_{n+1}-u^*\| \\
&= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|Ty_n - u^*\|\|x_{n+1}-u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^*-x_{n+1})\rangle \\
&\le \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|y_n - u^*\|\|x_{n+1}-u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^*-x_{n+1})\rangle \\
&\le \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u^*\|\|x_{n+1}-u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^*-x_{n+1})\rangle \\
&\le \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\frac{\|x_n - u^*\|^2 + \|x_{n+1}-u^*\|^2}{2} \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^*-x_{n+1})\rangle,
\end{aligned}
$$

which hence implies that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \le{}& \frac{1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\|x_n-u^*\|^2 \\
&+ \frac{2\lambda_n\mu_n}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\langle F(u^*), J(u^*-x_{n+1})\rangle
\end{aligned}
$$

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$$= \left(1 - \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\right)\|x_n - u^*\|^2$$

$$+ \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)} \cdot \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*), J(u^* - x_{n+1})\rangle.$$

Put

$$b_n = \frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)},$$

and

$$c_n = \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*), J(u^* - x_{n+1})\rangle.$$

Then the last inequality can be rewritten as

$$\|x_{n+1} - x^*\|^2 \le (1-b_n)\|x_n - x^*\|^2 + b_n c_n. \tag{4.12}$$

Because $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$, we have $\sum_{n=0}^{\infty}\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)} = \infty$ and hence $\sum_{n=0}^{\infty} b_n = \infty$. Note that $\limsup_{n\to\infty}\langle F(u^*), J(u^* - x_{n+1})\rangle \le 0$ due to (4.11). Thus, $\limsup_{n\to\infty} c_n \le 0$. Consequently, applying Lemma 2.6 to (4.12), we conclude that $\lim_{n\to\infty}\|x_n - u^*\| = 0$. □

Remark 4.9. Theorem 3.1 in [35] is closely related to Theorems 4.7 and 4.8. However, the iterative scheme in [35, Theorem 3.1] is different from the ones in Theorems 4.7 and 4.8. It involves the variational inequality problem $VI^*(I-F, C)$, where F is a fixed contractive self-mapping in X.

5. MODIFIED HYBRID STEEPEST-DESCENT METHOD WITH RESOLVENT OPERATORS

Let X be a uniformly smooth Banach space and A be an m-accretive operator in X with $C = A^{-1}(0) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. In this section, we propose and study the following modified hybrid steepest-descent

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algorithms with resolvent operators for computing the approximate solutions of $\mathrm{VI}^*(F, C)$.

Algorithm 5.1. Suppose $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ are three sequences in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$. Starting with an arbitrary initial guess $x_0 \in X$, generate a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} = J_{r_n} y_n - \lambda_n \mu_n F(J_{r_n} y_n). \end{cases} \tag{5.1}$$

Algorithm 5.2. Suppose $\{\alpha_n\}$ and $\{\lambda_n\}$ are two sequences in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$. Starting with an arbitrary initial guess $x_0 \in X$, generate a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} = y_n - \lambda_n F(y_n). \end{cases} \tag{5.2}$$

To prove the main results of this section, we need the following results.

Proposition 5.3. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a nonexpansive mapping with $C = Fix(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0,1)$, choose a number $\mu_t \in (0,1)$ arbitrarily and let $\{x_t\}$ be defined by* (2.4). *Then as $t \to 0^+$, x_t converges strongly to a unique solution u^* of* $\mathrm{VI}^*(F, C)$.

Proof. According to Lemma 2.4 (ii), $\{x_t : t \in (0,1)\}$ is bounded. Because F is a Lipschitzian mapping with constant $1 + 1/\lambda$, we have $\|F(x_t) - F(u)\| \leq (1 + 1/\lambda)\|x_t - u\|$. So, the set $\{F(x_t) : t \in (0,1)\}$ is bounded. Noticing that $x_t = tx_t + (1-t)Tx_t - t\mu_t F(x_t)$, we have

$$Tx_t = x_t + \frac{t\mu_t}{1-t} F(x_t).$$

This implies that

$$\|x_t - Tx_t\| = \frac{t\mu_t}{1-t}\|F(x_t)\| \leq \frac{t}{1-t}\|F(x_t)\| \to 0 \quad \text{as } t \to 0^+.$$

Note that $\{x_t : t \in (0,1)\}$ is bounded. Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and define $g : X \to \mathbb{R}$ by

$$g(x) = \tilde{\mu}_n \|x_n - x\|^2, \quad \forall x \in X,$$

where $\tilde{\mu}$ is a Banach limit on l^∞. Let

$$K = \left\{ x \in X : g(x) = \min_{y \in X} g(y) \right\}.$$

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It is easily seen that K is a nonempty closed convex bounded subset of X. Because (note that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$)

$$g(Tx) = \tilde{\mu}_n \|x_n - Tx\|^2 = \tilde{\mu}_n \|Tx_n - Tx\|^2 \leq \tilde{\mu}_n \|x_n - x\|^2 = g(x),$$

it follows that $T(K) \subset K$, that is, K is invariant under T. Because a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, T has a fixed point, say $\hat{u}$, in K. Because $\hat{u}$ is also a minimizer of g over X, it follows that, for $t \in (0,1)$ and $x \in X$,

$$0 \leq \frac{g(\hat{u} + t(x - \hat{u})) - g(\hat{u})}{t} = \tilde{\mu}_n \frac{\|(x_n - \hat{u}) + t(\hat{u} - x)\|^2 - \|x_n - \hat{u}\|^2}{t}.$$

The uniform smoothness of X implies that the duality map J is norm-to-norm uniformly continuous on bounded sets of X. Letting $t \to 0^+$, we find the two limits above can be interchanged and obtain

$$\tilde{\mu}_n \langle x - \hat{u}, J(x_n - \hat{u})\rangle \leq 0, \quad \forall x \in X.$$

Taking $x = \hat{u} - F(\hat{u})$ in the last inequality, we conclude from Lemma 2.4(i) that

$$\tilde{\mu}_n \|x_n - \hat{u}\|^2 \leq -\delta^{-1} \tilde{\mu}_n \langle F(\hat{u}), J(x_n - \hat{u})\rangle \leq 0.$$

This implies that

$$\tilde{\mu}_n \|x_n - \hat{u}\|^2 = 0.$$

and hence there exists a subsequence that is still denoted $\{x_n\}$ such that $x_n \to \hat{u} \in C = \mathrm{Fix}(T)$.

Now we claim that $\hat{u}$ is a solution of $\mathrm{VI}^*(F, C)$. Indeed, take $u \in C = \mathrm{Fix}(T)$ arbitrarily. By Lemma 2.4(i), we have

$$\langle F(x_t), J(x_t - u)\rangle \leq 0. \tag{5.3}$$

Note that the duality map J is single-valued and norm-to-norm uniformly continuous on bounded sets of X. Hence we deduce from $x_n \to \hat{u}$ that as $n \to \infty$,

$$\|F(x_n) - F(\hat{u})\| \to 0$$

and

$$\begin{aligned}
&|\langle F(x_n), J(x_n - u)\rangle - \langle F(\hat{u}), J(\hat{u} - u)\rangle| \\
&\quad = |\langle F(x_n) - F(\hat{u}), J(x_n - u)\rangle + \langle F(\hat{u}), J(x_n - u) - J(\hat{u} - u)\rangle| \\
&\quad \leq \|F(x_n) - F(\hat{u})\| \|x_n - u\| + \|F(\hat{u})\| \|J(x_n - u) - J(\hat{u} - u)\| \to 0.
\end{aligned}$$

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Therefore, it follows immediately from (5.3) that

$$\langle F(\hat{u}), J(\hat{u} - u)\rangle = \lim_{n\to\infty} \langle F(x_n), J(x_n - u)\rangle \le 0.$$

Consequently, $\hat{u} \in C = \mathrm{Fix}(T)$ is a solution of $\mathrm{VI}^*(F, C)$.

Further, we claim that the uniqueness of a solution of $\mathrm{VI}^*(F, C)$. Indeed, suppose that u^* and $v^* \in C = \mathrm{Fix}(T)$ are two solutions of the $\mathrm{VI}^*(F, C)$. Then, we have

$$\langle F(u^*), J(u^* - v^*)\rangle \le 0$$

and

$$\langle F(v^*), J(v^* - u^*)\rangle \le 0.$$

Adding up the last two inequalities, we have

$$\delta\|u^* - v^*\|^2 \le \langle F(u^*) - F(v^*), J(u^* - v^*)\rangle \le 0.$$

Hence $u^* = v^*$. Below we use $u^* \in C = \mathrm{Fix}(T)$ to denote a unique solution of $\mathrm{VI}^*(F, C)$.

Finally, we claim that $x_t \to u^*$ as $t \to 0^+$. Indeed, assume that there exists another subsequence $\{x_m\}$ of $\{x_t\}$ such that $x_m \to \bar{u} \in C = \mathrm{Fix}(T)$. Repeating the same argument as above, we can deduce that $\bar{u}$ is a solution of $\mathrm{VI}^*(F, C)$. In terms of the uniqueness of solutions of $\mathrm{VI}^*(F, C)$, we have $\bar{u} = u^*$. Therefore, $x_t \to u^*$ as $t \to 0^+$. □

Remark 5.4. Theorem 4.1 in Xu [38] is closely related to Proposition 5.3 but with different assumptions. However, it involves the variational inequality problem $VI^*(I - F, C)$, where F is a fixed nonexpansive self-mapping in X.

Proposition 5.5. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a nonexpansive mapping with $C = \mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $u^* = \lim_{t\to 0^+} x_t$ exists, where $\{x_t\}$ is defined by* (2.4)*, then*

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle \le 0.$$

Proof. Because $x_t - x_n = (1 - t)(Tx_t - x_n) + t((I - \mu_t F)x_t - x_n)$, we have

$$\|x_t - x_n\|^2 = (1 - t)\langle Tx_t - x_n, J(x_t - x_n)\rangle + t\langle (I - \mu_t F)x_t - x_n, J(x_t - x_n)\rangle$$

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$$
\begin{aligned}
&= (1-t)(\langle Tx_t - Tx_n, J(x_t - x_n)\rangle + \langle Tx_n - x_n, J(x_t - x_n)\rangle) \\
&\quad + t\langle (I-\mu_t F)x_t - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\
&\le \|x_t - x_n\|^2 + \|Tx_n - x_n\|\|J(x_t - x_n)\| \\
&\quad + t\langle (I-\mu_t F)x_t - x_t, J(x_t - x_n)\rangle \\
&\le \|x_t - x_n\|^2 + \|Tx_n - x_n\|\|x_t - x_n\| - t\mu_t\langle F(x_t), J(x_t - x_n)\rangle,
\end{aligned}
$$

and hence

$$
\langle F(x_t), J(x_t - x_n)\rangle \le \frac{\|x_n - Tx_n\|}{t\mu_t}\|x_t - x_n\|. \tag{5.4}
$$

Because both $\{x_t\}$ and $\{x_n\}$ are bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, taking the superior limit in (5.4), we obtain that

$$
\limsup_{n\to\infty}\langle F(x_t), J(x_t - x_n)\rangle \le 0. \tag{5.5}
$$

Further, taking the superior limit as $t \to 0^+$ in (5.5) and noticing the fact that the two limits are interchangeable due to the fact the duality map J is norm-to-norm uniformly continuous on bounded sets, we obtain $\limsup_{n\to\infty}\langle F(u^*), J(u^* - x_n)\rangle \le 0$. □

Now we show that the sequence $\{x_n\}$ generated by (5.1) converges strongly to a zero u^* of A, which is a unique solution of $\mathrm{VI}^*(F, C)$ under some suitable assumptions. □

Theorem 5.6. *Let X be a uniformly smooth Banach space and A be an m-accretive operator in X with $C = A^{-1}(0) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Given a point $x_0 \in K = \overline{D(A)}$ and given sequences $\{\lambda_n\}_{n=0}^{\infty}$ in $(0,1)$, $\{\mu_n\}_{n=0}^{\infty}$ in $(0,1)$, $\{\alpha_n\}_{n=0}^{\infty}$ in $[0,1]$, suppose that the sequences $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, and $\{r_n\}_{n=0}^{\infty}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=0}^{\infty}\lambda_n\mu_n = \infty$;
(ii) $r_n \ge \epsilon$ *for all* n *and* $\{\alpha_n\} \subset [a, b]$, *for some* $a, b \in (0,1)$;
(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty}|\mu_{n+1} - \mu_n| < \infty$ *and* $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (5.1), converges strongly to a zero u^* of A, which is a unique solution u^* of $\mathrm{VI}^*(F, C)$.

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Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, taking a fixed $u \in C = A^{-1}(0)$ arbitrarily, we have

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n x_n + (1-\alpha_n)J_{r_n}x_n - u\| \\ &\le \alpha_n\|x_n - u\| + (1-\alpha_n)\|J_{r_n}x_n - u\| \\ &\le \alpha_n\|x_n - u\| + (1-\alpha_n)\|x_n - u\| = \|x_n - u\|.\end{aligned}$$

So, $\|y_n - u\| \le \|x_n - u\|$ for all $n \ge 0$. Thus, by Proposition 2.1(iii), we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|J_{r_n}y_n - \lambda_n\mu_n F(J_{r_n}y_n) - u\| \\ &= \|\lambda_n(I-\mu_n F)J_{r_n}y_n + (1-\lambda_n)J_{r_n}y_n - u\| \\ &\le \lambda_n\|(I-\mu_n F)J_{r_n}y_n - u\| + (1-\lambda_n)\|J_{r_n}y_n - u\| \\ &\le \lambda_n\|(I-\mu_n F)J_{r_n}y_n - (I-\mu_n F)u\| + \lambda_n\|(I-\mu_n F)u - u\| \\ &\quad + (1-\lambda_n)\|J_{r_n}y_n - u\| \\ &\le \lambda_n\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|J_{r_n}y_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &\quad + (1-\lambda_n)\|J_{r_n}y_n - u\| \\ &= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|J_{r_n}y_n - u\| + \lambda_n\mu_n\|F(u)\| \\ &= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|J_{r_n}y_n - u\| \\ &\quad + \lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\| \\ &\le \max\left\{\|J_{r_n}y_n - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\} \\ &\le \max\left\{\|y_n - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\} \\ &\le \max\left\{\|x_n - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\}.\end{aligned}$$

By induction,

$$\|x_n - u\| \le \max\left\{\|x_0 - u\|, \left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|F(u)\|\right\}, \quad \forall n \ge 0.$$

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Thus, $\{x_n\}$ is bounded and so is $\{y_n\}$. Because each J_{r_n} is nonexpansive and F is Lipschitzian, $\{J_{r_n}x_n\}$, $\{J_{r_n}y_n\}$, and $\{F(J_{r_n}y_n)\}$ are bounded. From condition (i), we have

$$\|x_{n+1} - J_{r_n}y_n\| = \lambda_n\mu_n\|F(J_{r_n}y_n)\| \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

Now we claim that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{5.7}$$

In order to prove (5.7), we estimate $\|x_{n+1} - x_n\|$ first. From (5.1), we have

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \\ y_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})J_{r_{n-1}}x_{n-1}. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} = {} & (1-\alpha_n)(J_{r_n}x_n - J_{r_{n-1}}x_{n-1}) + \alpha_n(x_n - x_{n-1}) \\ & + (x_{n-1} - J_{r_{n-1}}x_{n-1})(\alpha_n - \alpha_{n-1}). \end{aligned} \tag{5.8}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| \le {} & (1-\alpha_n)\|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| + \alpha_n\|x_n - x_{n-1}\| \\ & + \|x_{n-1} - J_{r_{n-1}}x_{n-1}\||\alpha_n - \alpha_{n-1}|. \end{aligned} \tag{5.9}$$

Observe that by Lemma 2.7 there exists the resolvent identity

$$J_{r_n}x_n = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right).$$

If $r_{n-1} \le r_n$, we conclude that

$$\begin{aligned} \|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right) - J_{r_{n-1}}x_{n-1}\right\| \\ &\le \left\|\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n - x_{n-1}\right\| \\ &= \left\|\frac{r_{n-1}}{r_n}(x_n - x_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(J_{r_n}x_n - x_{n-1})\right\| \\ &\le \frac{r_{n-1}}{r_n}\|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{r_n}\|J_{r_n}x_n - x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}x_n - x_{n-1}\|. \end{aligned} \tag{5.10}$$

Repeating the same argument as above, we can derive

$$\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \le \|y_n - y_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}y_n - y_{n-1}\|. \tag{5.11}$$

Substituting (5.10) into (5.11), we obtain

$$\begin{aligned}\|y_n - y_{n-1}\| &\le (1-\alpha_n)\left(\|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}x_n - x_{n-1}\|\right)\\ &\quad + \alpha_n\|x_n - x_{n-1}\| + \|x_{n-1} - J_{r_{n-1}}x_{n-1}\||\alpha_n - \alpha_{n-1}|\\ &\le \|x_n - x_{n-1}\| + M_1\big(|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}|\big),\end{aligned} \tag{5.12}$$

where M_1 is a constant such that

$$M_1 > \max\left\{\frac{\|J_{r_n}x_n - x_{n-1}\|}{\epsilon}, \|x_{n-1} - J_{r_{n-1}}x_{n-1}\|\right\}.$$

On the other hand, we have from (5.1)

$$\begin{cases} x_{n+1} = J_{r_n}y_n - \lambda_n\mu_n F(J_{r_n}y_n),\\ x_n = J_{r_{n-1}}y_{n-1} - \lambda_{n-1}\mu_{n-1}F(J_{r_{n-1}}y_{n-1}).\end{cases}$$

Simple calculations show that

$$\begin{aligned}x_{n+1} - x_n &= (I - \lambda_n\mu_n F)J_{r_n}y_n - (I - \lambda_n\mu_n F)J_{r_{n-1}}y_{n-1}\\ &\quad + (\lambda_{n-1}\mu_{n-1} - \lambda_n\mu_n)F(J_{r_{n-1}}y_{n-1}).\end{aligned}$$

It follows from Proposition 2.1(iii) and (5.11) that

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|(I - \lambda_n\mu_n F)J_{r_n}y_n - (I - \lambda_n\mu_n F)J_{r_{n-1}}y_{n-1}\|\\ &\quad + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\|F(J_{r_{n-1}}y_{n-1})\|\\ &\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\|\\ &\quad + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\|F(J_{r_{n-1}}y_{n-1})\|\\ &\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\\ &\quad \times\left(\|y_n - y_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}y_n - y_{n-1}\|\right)\\ &\quad + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\|F(J_{r_{n-1}}y_{n-1})\|\\ &\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|y_n - y_{n-1}\|\\ &\quad + M_2\big(|r_n - r_{n-1}| + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\big),\end{aligned} \tag{5.13}$$

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where M_2 is a constant such that

$$M_2 > \max\left\{\frac{\|J_{r_n}y_n - y_{n-1}\|}{\epsilon}, \|F(J_{r_{n-1}}y_{n-1})\|, M_1\right\}.$$

Substituting (5.12) into (5.13), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|y_n - y_{n-1}\| \\
&\quad + M_2\big(|r_n - r_{n-1}| + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\big) \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\big(\|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}|)\big) + M_2\big(|r_n - r_{n-1}| + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|\big) \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}|) + M_2(|r_n - r_{n-1}| + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|) \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x_{n-1}\| + 2M_2(|r_n - r_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}| + |\lambda_n\mu_n - \lambda_{n-1}\mu_{n-1}|) \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x_{n-1}\| + 2M_2(|r_n - r_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|). \qquad (5.14)
\end{aligned}$$

Similarly we can prove (5.14) if $r_{n-1} \ge r_n$. Using assumptions (i), (iii), we have

$$\sum_{n=0}^{\infty} \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) = \infty,$$

and

$$\sum_{n=1}^{\infty} \big(|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|\big) < \infty.$$

Therefore, Lemma 2.6 is applicable to (5.14) and we obtain

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty.$$

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By condition (ii) and (5.1), we have

$$\begin{aligned}\|y_n - x_n\| &= (1-\alpha_n)\|J_{r_n}x_n - x_n\| \\ &\le (1-a)(\|J_{r_n}x_n - J_{r_n}y_n\| + \|J_{r_n}y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\le (1-a)(\|x_n - y_n\| + \|J_{r_n}y_n - x_{n+1}\| + \|x_{n+1} - x_n\|),\end{aligned}$$

which implies that

$$\|x_n - y_n\| \le \frac{1-a}{a}\big(\|J_{r_n}y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\big).$$

This together with (5.6)–(5.7) imply that

$$\|x_n - y_n\| \to 0 \quad \text{as } n \to \infty.$$

So, we obtain

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - y_n\| + \|y_n - J_{r_n}x_n\| \\ &= \|x_n - y_n\| + \alpha_n\|x_n - J_{r_n}x_n\| \\ &\le \|x_n - y_n\| + b\|x_n - J_{r_n}x_n\|,\end{aligned}$$

which implies that

$$\|x_n - J_{r_n}x_n\| \le \frac{1}{1-b}\|x_n - y_n\|,$$

and hence

$$\|x_n - J_{r_n}x_n\| \to 0 \quad \text{as } n \to \infty.$$

Taking a fixed number r such that $\epsilon > r > 0$, we derive from Lemma 2.7

$$\begin{aligned}\|J_{r_n}x_n - J_r x_n\| &= \left\|J_r\left(\frac{r}{r_n}x_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\ &\le \left(1 - \frac{r}{r_n}\right)\|x_n - J_{r_n}x_n\| \\ &\le \|x_n - J_{r_n}x_n\|. \qquad (5.15)\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\ &\le \|x_n - J_{r_n}x_n\| + \|x_n - J_{r_n}x_n\| \\ &= 2\|J_{r_n}x_n - x_n\|. \qquad (5.16)\end{aligned}$$

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So, we obtain

$$\|x_n - J_r x_n\| \to 0 \quad \text{as } n \to \infty. \tag{5.17}$$

Put $x_t = tx_t + (1-t)J_r x_t - t\mu_t F(x_t)$. Then it follows from Proposition 5.3 that as $t \to 0^+$, x_t converges strongly to a unique solution u^* of $\mathrm{VI}^*(F, C)$ where $C = A^{-1}(0) = \mathrm{Fix}(J_r)$. It follows from (5.17) and Proposition 5.5 that

$$\limsup_{n\to\infty} \langle F(u^*), J(u^* - x_n)\rangle \le 0. \tag{5.18}$$

Finally, we claim that $x_n \to x^*$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned}
\|x_{n+1} - u^*\|^2 &= \langle (I - \lambda_n\mu_n F)J_{r_n}y_n - u^*, J(x_{n+1} - u^*)\rangle \\
&= \langle (I - \lambda_n\mu_n F)J_{r_n}y_n - (I - \lambda_n\mu_n F)u^*, J(x_{n+1} - u^*)\rangle \\
&\quad + \langle (I - \lambda_n\mu_n F)u^* - u^*, J(x_{n+1} - u^*)\rangle \\
&\le \|(I - \lambda_n\mu_n F)J_{r_n}y_n - (I - \lambda_n\mu_n F)u^*\|\|x_{n+1} - u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_{n+1})\rangle \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|J_{r_n}y_n - u^*\|\|x_{n+1} - u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_{n+1})\rangle \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|y_n - u^*\|\|x_{n+1} - u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_{n+1})\rangle \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - u^*\|\|x_{n+1} - u^*\| \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_{n+1})\rangle \\
&\le \left(1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\frac{\|x_n - u^*\|^2 + \|x_{n+1} - u^*\|^2}{2} \\
&\quad + \lambda_n\mu_n\langle F(u^*), J(u^* - x_{n+1})\rangle.
\end{aligned}$$

This yields that

$$\|x_{n+1} - u^*\|^2 \le \frac{1 - \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)}{1 + \lambda_n\mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)}\|x_n - u^*\|^2$$

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$$+\frac{2\lambda_n\mu_n}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\langle F(u^*),J(u^*-x_{n+1})\rangle$$

$$=\left(1-\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\right)\|x_n-u^*\|^2$$

$$+\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}\cdot\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*),J(u^*-x_{n+1})\rangle.$$

Put

$$b_n=\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)},$$

and

$$c_n=\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\langle F(u^*),J(u^*-x_n)\rangle.$$

Then the last inequality can be rewritten as

$$\|x_{n+1}-u^*\|^2\le(1-b_n)\|x_n-u^*\|^2+b_nc_n. \tag{5.19}$$

Because $\sum_{n=0}^{\infty}\lambda_n\mu_n=\infty$, we have $\sum_{n=0}^{\infty}\frac{2\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}{1+\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)}=\infty$ and hence $\sum_{n=0}^{\infty}b_n=\infty$. Note that $\limsup_{n\to\infty}\langle F(u^*),J(u^*-x_{n+1})\rangle\le 0$ due to (5.18). Thus, $\limsup_{n\to\infty}c_n\le 0$. Consequently, applying Lemma 2.5 to (5.19), we conclude that $\lim_{n\to\infty}\|x_n-u^*\|=0$. □

By the careful analysis of the proof of Theorem 5.6, we can obtain the following result. Because its proof is much simpler than that of Theorem 5.6, we omit its proof.

Theorem 5.7. *Let X be a uniformly smooth Banach space and A be an m-accretive operator in X with $C=A^{-1}(0)\neq\emptyset$. Assume that $F:X\to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda>1$. Given a point $x_0\in K=\overline{D(A)}$ and given sequences $\{\lambda_n\}_{n=0}^{\infty}$ in $(0,1)$, $\{\alpha_n\}_{n=0}^{\infty}$ in $[0,1]$, suppose that the sequences $\{\lambda_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, and $\{r_n\}_{n=0}^{\infty}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\lambda_n=0$ *and* $\sum_{n=0}^{\infty}\lambda_n=\infty$;
(ii) $r_n\ge\epsilon$ *for all* n *and* $\{\alpha_n\}\subset[a,b]$, *for some* $a,b\in(0,1)$;
(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ *and* $\sum_{n=0}^{\infty}|r_{n+1}-r_n|<\infty$.

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Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated by* (5.2), *converges strongly to a zero* u^* *of* A, *which is a unique solution* u^* *of* $\mathrm{VI}^*(F, C)$.

Remark 5.8. Theorem 2.1 in Qin and Su [31] is closely related to Theorems 5.6 and 5.7. However, it involves only the problem of finding a zero point of an m-accretive operator in X but does not involve any variational inequality problem. In addition, the iterative schemes in these results are very different due to the second step.

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