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Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory[☆]

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Abstract

In this paper, we introduce the concept of a Q-function defined on a quasi-metric space which generalizes the notion of a τ-function and a w-distance. We establish Ekeland-type variational principles in the setting of quasi-metric spaces with a Q-function. We also present an equilibrium version of the Ekeland-type variational principle in the setting of quasi-metric spaces with a Q-function. We prove some equivalences of our variational principles with Caristi–Kirk type fixed point theorems for multivalued maps, the Takahashi minimization theorem and some other related results. As applications of our results, we derive existence results for solutions of equilibrium problems and fixed point theorems for multivalued maps. We also extend the Nadler's fixed point theorem for multivalued maps to a Q-function and in the setting of complete quasi-metric spaces. As a consequence, we prove the Banach contraction theorem for a Q-function and in the setting of complete quasi-metric spaces. The results of this paper extend and generalize many results appearing recently in the literature.

Keywords: Ekeland-type variational principle; Equilibrium problems; Q-functions; Fixed point theorems; Caristi–Kirk type fixed point theorem; Nadler's fixed point theorem; Banach contraction theorem; Quasi-metric spaces

1. Introduction

In 1972, Ekeland [15] (see also, [16,17]) discovered a pioneer result, now known as Ekeland's variational principle (in short, EVP), that provides an approximate minimizer of a bounded below and lower semicontinuous function in a given neighborhood of a point. This localization property is very useful and explains the importance of this result. EVP is one of the most important results obtained in nonlinear analysis and it has appeared as one of the most useful tools to solve problems in optimization, optimal control theory, game theory, nonlinear equations, dynamical systems, etc; See for example [2,6,16,17,21,24,37] and references therein. Many famous results, namely the Krasnosel'skii–Zabrjeko

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and Caristi–Kirk fixed point theorems, the Petal theorem and the Daneš drop theorem, were discovered around the same time but independently of each other. Soon after, it was found that all these results are equivalent to EVP; See for example [33] and references therein. Since the discovery of EVP, there have also appeared many extensions or equivalent formulations of EVP. In 1996, Kada et al. [27] introduced the concept of a w-distance defined on a metric space and extended EVP, the minimization theorem and the Kirk–Caristi fixed point theorem for a w-distance. Suzuki [35] introduced a more general concept than w-distance, called τ-distance, and established EVP for τ-distance. He also extended most of the results of [27] for τ-distance. It seems that the concept of τ-distance is little more complicated; that's why, Lin and Du [29] introduced the concept of a τ-function which is an extension of a w-distance but independent of τ-distance. They established a generalized EVP for lower semicontinuous from above functions and with a τ-function. They also derived the minimization theorem, nonconvex equilibrium theorem, and common fixed point theorem for a family of multivalued maps and the flower petal theorem.

In this paper, we introduce the concept of a Q-function defined on a quasi-metric space which generalizes the notion of a τ-function and a w-distance. We establish Ekeland-type variational principles, first in the setting of quasi-metric spaces with a Q-function but without any lower semicontinuity assumption on the underlying function and then in the setting of complete quasi-metric spaces with a Q-function. The equilibrium version of the Ekeland-type variational principle in the setting of quasi-metric spaces with a Q-function is also presented. We prove some equivalences of our variational principles with Caristi–Kirk type fixed point theorems for multivalued maps, Takahashi's minimization theorem and some other related results. As applications of our results, we derive existence results for solutions of equilibrium problems and fixed point theorems for multivalued maps. We also extend Nadler's fixed point theorem for multivalued maps to a Q-function and in the setting of complete quasi-metric spaces. As a consequence, we show that the well known Banach contraction theorem also holds good for a Q-function and in the setting of complete quasi-metric spaces. The results of this paper extend and generalize many results appearing recently in the literature.

2. Preliminaries

Throughout the paper, unless otherwise specified, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. Let us recall the following well known definition of a quasi-metric space.

Let X be a nonempty set. A real valued function $d : X \times X \to \mathbb{R}_+$ is said to be a *quasi-metric* on X if the following conditions are satisfied:

(M1) $d(x, y) \geq 0$ for all $x, y \in X$;
(M2) $d(x, y) = 0$ if and only if $x = y$;
(M3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A nonempty set X together with a quasi-metric d is called a *quasi-metric space* and it is denoted by (X, d). Therefore, the concept of a quasi-metric space generalizes the concept of a metric space by lifting the symmetry condition. For the quasi-metric space (X, d), the concepts of Cauchy sequences, convergent sequences and completeness can be defined in the same manner as in the setting of metric spaces. Throughout the paper, unless otherwise specified, X is assumed to be a quasi-metric space with the quasi-metric d.

Now, we introduce the concept of a Q-function on a quasi-metric space X.

Definition 2.1. A function $q : X \times X \to \mathbb{R}_+$ is called a *Q-function* on X if the following conditions are satisfied:

(Q1) For all $x, y, z \in X$, $q(x, z) \leq q(x, y) + q(y, z)$.
(Q2) If $x \in X$ and $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that it converges to a point y (with respect to the quasi-metric) and $q(x, y_n) \leq M$ for some $M = M(x) > 0$, then $q(x, y) \leq M$.
(Q3) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Remark 2.1. If (X, d) is a metric space and in addition to (Q1)–(Q3), the following condition is also satisfied:

(Q4) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} \sup\{q(x_n, x_m) : m > n\} = 0$, and if there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} q(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$,

then the Q-function is called a *τ-function*, introduced by Lin and Du [29]. It has been shown in [29] that every w-distance, introduced and studied by Kada et al. [27], is a τ-function. In fact, if we consider (X, d) as a metric space and replace (Q2) by the following condition:

(Q5) For any $x \in X$, the function $q(x, \cdot) : X \to \mathbb{R}_+$ is lower semicontinuous, then the Q-function is called a *w-distance* on X. Several examples and properties of a w-distance for metric spaces are given in [27,37]. It is easy to see that if $q(x, \cdot)$ is lower semicontinuous, then (Q2) holds. Hence, it is obvious that every w-function is a τ-function and every τ-function is a Q-distance but the converse assertions do not hold.

Example 2.1. (a) Let $X = \mathbb{R}$. Suppose $d : X \times X \to \mathbb{R}_+$ is defined as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |y| & \text{otherwise} \end{cases}$$

and $q : X \times X \to \mathbb{R}_+$ is defined as

$$q(x, y) = |y| \quad \text{for all } x, y \in X.$$

Then it is easy to see that d is a quasi-metric on X and q is a Q-function on X. But q is neither a τ-function nor a w-distance.
(b) Let $X = [0, 1]$. Suppose $d : X \times X \to \mathbb{R}_+$ is defined as

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 2(x - y) & \text{otherwise} \end{cases}$$

and $q : X \times X \to \mathbb{R}_+$ is defined as

$$q(x, y) = |x - y| \quad \text{for all } x, y \in X.$$

Then q is a Q-function on X. However q is neither a τ-function nor a w-distance, because (X, d) is not a metric space.

Remark 2.2. Let (X, d) be a quasi-metric space and q be a Q-function on X. If $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and subadditive function such that $\eta(0) = 0$, then $\eta \circ q$ is a Q-function on X.

We present some properties of a Q-function which are similar to the properties of a w-distance, see for example [27].

Lemma 2.1. *Let $q : X \times X \to \mathbb{R}_+$ be a Q-function on X and, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}_+$ such that they converge to 0, and let $x, y, z \in X$. Then the following conditions hold:*

(i) *If $q(x_n, y) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$;*
(ii) *If $q(x_n, y_n) \leq \alpha_n$ and $q(x_n, y) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ converges to y;*
(iii) *If $q(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence;*
(iv) *If $q(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence;*
(v) *If $q_1, q_2, \ldots, q_n$ are Q-functions on X, then $q(x, y) = \max\{q_1(x, y), q_2(x, y), \ldots, q_n(x, y)\}$ is also a Q-function on X.*

The proof of this lemma lies on the lines of the proof of Lemma 1 in [27] and therefore, we omit it.

Definition 2.2. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower monotone* if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to some point $x \in X$ and satisfying $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathbb{N}$, we have $f(x) \leq f(x_n)$ for each $n \in \mathbb{N}$.

Remark 2.3. Note that the lower monotonicity is slightly weaker than lower semicontinuity. In other words, every lower monotone function is lower semicontinuous but the converse is not true in general.

Definition 2.3 (*[14]*). A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* (in short, lsca) if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to some point $x \in X$ and satisfying $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathbb{N}$, we have $f(x) \leq \lim_{n\to\infty} f(x_n)$.

Remark 2.4. From the definitions of lower monotone and lsca of a function, it is clear that every lower monotone function is lsca but the converse may not be true in general. But, if f is bounded below then both concepts are equivalent.

Indeed, let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be a sequence such that it converges to some point x and $f(x_{n+1}) \le f(x_n)$ for all $n \in \mathbb{N}$. We claim that $f(x) \le f(x_n)$ for all $n \in \mathbb{N}$, if f is lsca and bounded below.

Since f is bounded below and $f(x_{n+1}) \le f(x_n)$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} f(x_n)$ exists. Let $r = \lim_{n\to\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n)$, then $f(x_n) \ge r$ for all $n \in \mathbb{N}$. The lsca of f implies that $f(x) \le \lim_{n\to\infty} f(x_n) = r$ and thus $f(x) \le r \le f(x_n)$ for all $n \in \mathbb{N}$.

Definition 2.4. Let X be an ordered space with an ordering $\preccurlyeq$ on X.

(i) The ordering $\preccurlyeq$ on X is called a *quasi-order* on X if it is a reflexive and transitive relation.
(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called *decreasing* (with respect to $\preccurlyeq$) if $x_{n+1} \preccurlyeq x_n$ for all $x \in \mathbb{N}$.
(iii) The quasi-order $\preccurlyeq$ on X is said to be *lower closed* if for every $x \in X$, the section $S(x) = \{y \in X : y \preccurlyeq x\}$ is lower closed, that is, if $\{x_n\}_{n\in\mathbb{N}} \subseteq S(x)$ is decreasing with respect to $\preccurlyeq$ and convergent to $\tilde{x} \in X$ with respect to the quasi-metric on X, then $\tilde{x} \in S(x)$.

Definition 2.5. Let (X, d) be a quasi-metric with a quasi-order $\preccurlyeq$ on X. For any $x \in X$, the set $S(x) = \{y \in X : y \preccurlyeq x\}$ is said to be $\preccurlyeq$*-complete* if every decreasing (with respect to $\preccurlyeq$) Cauchy sequence in $S(x)$ converges in it.

3. Ekeland-type variational principles

In this section, we present two generalizations of Ekeland-type variational principle for a Q-function, one in the setting of noncomplete quasi-metric spaces and the other in the setting of complete quasi-metric spaces.

Theorem 3.1. *Let (X, d) be a quasi-metric space (not necessarily complete), $q : X \times X \to \mathbb{R}_+$ a Q-function on X, $\varphi : (-\infty, \infty] \to (0, \infty)$ a nondecreasing function and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper and bounded below function. Define a quasi-order $\preccurlyeq$ on X as*

$$y \preccurlyeq x \quad \textit{iff} \quad x = y \quad \textit{or} \quad q(x, y) \le \varphi(f(x))(f(x) - f(y)). \tag{3.1}$$

Suppose that there exists $\hat{x} \in X$ such that $\inf_{x\in X} f(x) < f(\hat{x})$ and $S(\hat{x}) = \{y \in X : y \preccurlyeq \hat{x}\}$ is $\preccurlyeq$-complete. Then there exists $\bar{x} \in X$ such that

(a) $q(\hat{x}, \bar{x}) \le \varphi\left(f(\hat{x})\right)\left(f(\hat{x}) - f(\bar{x})\right)$
(b) $q(\bar{x}, x) > \varphi(f(\bar{x}))(f(\bar{x}) - f(x))$ *for all $x \in X$, $x \ne \bar{x}$.*

Proof. The reflexivity of $\preccurlyeq$ is obvious. To prove the transitivity of $\preccurlyeq$, we let $x, y, z \in X$ such that $z \preccurlyeq y$ and $y \preccurlyeq x$, that is,

$$z = y \quad \text{or} \quad q(y, z) \le \varphi(f(y))(f(y) - f(z)) \tag{3.2}$$

and

$$y = x \quad \text{or} \quad q(x, y) \le \varphi(f(x))(f(x) - f(y)). \tag{3.3}$$

For $z = y$ or $x = y$, the transitivity of $\preccurlyeq$ is obvious. So, we let $x \ne y \ne z$. Since $q(u, v) \ge 0$ for all $u, v \in X$ and $\varphi(w) > 0$ for all $w \in (-\infty, \infty]$, from (3.2) and (3.3) we have $f(z) \le f(y)$ and $f(y) \le f(x)$. Since φ is nondecreasing, we have $\varphi(f(y)) \le \varphi(f(x))$. By using (Q1) and adding (3.2) and (3.3), we obtain

$$\begin{aligned} q(x, z) &\le q(x, y) + q(y, z) \\ &\le \varphi(f(x))(f(x) - f(y)) + \varphi(f(y))(f(y) - f(z)) \\ &\le \varphi(f(x))(f(x) - f(y)) + \varphi(f(x))(f(y) - f(z)) \\ &= \varphi(f(x))(f(x) - f(z)). \end{aligned}$$

Hence, $z \preccurlyeq x$. Therefore, $\preccurlyeq$ is a quasi-order on X.

We now construct inductively a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $S(\hat{x})$. To each x_n we let

$$\begin{aligned} S(x_n) &= \left\{y \in S(\hat{x}) : y = x_n \text{ or } q(x_n, y) \le \varphi(f(x_n))(f(x_n) - f(y))\right\} \\ &= \left\{y \in S(\hat{x}) : y \preccurlyeq x_n\right\}. \end{aligned}$$

Let $\hat{x} = x_0$ and assume that x_{n-1} is already known for $n \in \mathbb{N}$. Then choose $x_n \in S(x_{n-1})$ such that

$$f(x_n) \le \inf_{x \in S(x_{n-1})} f(x) + \frac{1}{n}. \tag{3.4}$$

Since $x_n \in S(x_{n-1})$, we have $x_n \preccurlyeq x_{n-1}$ and so $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. Also,

$$q(x_{n-1}, x_n) \le \varphi(f(x_{n-1}))(f(x_{n-1}) - f(x_n))$$

which implies that $f(x_n) \le f(x_{n-1})$ for all $n \in \mathbb{N}$. Therefore, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Since f is bounded below, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to some real number $r \in \mathbb{R}$ such that $r \le f(x_n)$ for all $n \in \mathbb{N}$, that is, $\lim_{n\to\infty} f(x_n) = r$. We claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $S(\hat{x})$.

In fact, if $n < m$, then we have

$$\begin{aligned}
q(x_n, x_m) &\le q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
&\le \varphi(f(x_n))(f(x_n) - f(x_{n+1})) + \varphi(f(x_{n+1}))(f(x_{n+1}) - f(x_{n+2})) \\
&\quad + \cdots + \varphi(f(x_{m-1}))(f(x_{m-1}) - f(x_m)) \\
&\le \varphi(f(x_n))(f(x_n) - f(x_{n+1})) + \varphi(f(x_n))(f(x_{n+1}) - f(x_{n+2})) \\
&\quad + \cdots + \varphi(f(x_n))(f(x_{m-1}) - f(x_m)) \\
&\le \varphi(f(x_n))(f(x_n) - f(x_m)) \\
&\le \varphi(f(x_n))(f(x_n) - r) = \alpha_n,
\end{aligned}$$

where $\alpha_n = \varphi(f(x_n))(f(x_n) - r)$. Since $r \le f(x_n)$ for all $n \in \mathbb{N}$, we always have $f(x_n) - r \ge 0$ and $\lim_{n\to\infty} f(x_n) = r$ implying that $\{\alpha_n\}$ is a sequence in $[0, \infty)$ converging to zero. Then by Lemma 2.1(iii), $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $S(\hat{x})$. Since $S(\hat{x})$ is $\preccurlyeq$-complete, $\{x_n\}_{n\in\mathbb{N}}$ converges to some point $\bar{x} \in S(\hat{x})$. Since $\preccurlyeq$ is transitive, we have $S(x_n) \subseteq S(x_{n-1})$ for all $n \in \mathbb{N}$. Since $\bar{x} \in S(\hat{x}) = S(x_0)$, we have $\bar{x} \in S(x_{n-1})$ for all $n \in \mathbb{N}$. In particular, $\bar{x} \in S(\hat{x})$ implies that

$$q(\hat{x}, \bar{x}) \le \varphi\big(f(\hat{x})\big)\big(f(\hat{x}) - f(\bar{x})\big).$$

Thus (a) holds.

Now we prove that $\{\bar{x}\} = S(\bar{x})$. Let $x \in S(\bar{x})$ but $x \ne \bar{x}$. Then $x \preccurlyeq \bar{x}$ which implies that $f(x) \le f(\bar{x})$. On the other hand, the transitivity of $\preccurlyeq$ implies $x \preccurlyeq \bar{x} \preccurlyeq x_{n-1}$ for all $n \in \mathbb{N}$. The rules for the choice of x_n yield

$$f(\bar{x}) \le f(x_n) \le f(x) + \frac{1}{n}.$$

Since $\lim_{n\to\infty} f(x_n) = r$, we have $f(\bar{x}) \le r \le f(x)$, and hence $f(\bar{x}) \le r \le f(x) \le f(\bar{x})$ implies $f(\bar{x}) = r = f(x)$. Since for all $n \in \mathbb{N}$, $x \preccurlyeq x_n$, we have

$$\begin{aligned}
q(x_n, x) &\le \varphi(f(x_n))(f(x_n) - f(x)) \\
&= \varphi(f(x_n))(f(x_n) - r) \\
&= \alpha_n.
\end{aligned} \tag{3.5}$$

Also, for all $n \in \mathbb{N}$, $\bar{x} \preccurlyeq x_n$, we have

$$\begin{aligned}
q(x_n, \bar{x}) &\le \varphi(f(x_n))(f(x_n) - f(\bar{x})) \\
&= \varphi(f(x_n))(f(x_n) - r) \\
&= \alpha_n.
\end{aligned} \tag{3.6}$$

As above $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$ and converging to zero. By using (3.5) and (3.6) and applying Lemma 2.1(i), we obtain $x = \bar{x}$. Hence $\{\bar{x}\} = S(\bar{x})$.

It follows that $x \notin S(\bar{x})$ whenever $x \ne \bar{x}$ and hence

$$q(\bar{x}, x) > \varphi(f(\bar{x}))(f(\bar{x}) - f(x)) \quad \text{for all } x \in X \text{ and } x \ne \bar{x},$$

that is, (b) holds. □

Remark 3.1. In Theorem 3.1, we did not assume any kind of lower semicontinuity. Instead of it, we assumed that the set $S(\hat{x})$ is $\preccurlyeq$-complete.

The following result is a simplified form of Theorem 3.1.

Theorem 3.2. *Let (X, d) be a complete quasi-metric space, $q : X \times X \to \mathbb{R}_+$ a Q-function on X, $\varphi : (-\infty, \infty] \to (0, \infty)$ a nondecreasing function and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper, lsca and bounded below function. Assume that there exists $\hat{x} \in X$ such that $\inf_{x\in X} f(x) < f(\hat{x})$, then there exists $\bar{x} \in X$ such that*

(a) $q(\hat{x}, \bar{x}) \le \varphi\left(f(\hat{x})\right)\left(f(\hat{x}) - f(\bar{x})\right)$
(b) $q(\bar{x}, x) > \varphi\left(f(\bar{x})\right)\left(f(\bar{x}) - f(x)\right)$ *for all $x \in X$, $x \ne \bar{x}$.*

Proof. Define an ordering $\preccurlyeq$ on X as

$$y \preccurlyeq x \quad \text{iff} \quad x = y \text{ or } q(x, y) \le \varphi\left(f(x)\right)\left(f(x) - f(y)\right). \tag{3.7}$$

As we have seen in the proof of Theorem 3.1, $\preccurlyeq$ is a quasi-order on X.

We prove that $\preccurlyeq$ is lower closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to x \in X$ (with respect to the quasi-metric d) and $x_{n+1} \preccurlyeq x_n$ for all $n \in \mathbb{N}$, that is,

$$q(x_n, x_{n+1}) \le \varphi\left(f(x_n)\right)\left(f(x_n) - f(x_{n+1})\right). \tag{3.8}$$

As in the proof of Theorem 3.1, $f(x_n) \le f(x_{n+1})$ for all $n \in \mathbb{N}$. Therefore, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Since f is bounded below, $\lim_{n\to\infty} f(x_n)$ exists. Let $r = \lim_{n\to\infty} f(x_n)$, then $r \le f(x_n)$ for all $n \in \mathbb{N}$. The lsca of f implies that $f(x) \le \lim_{n\to\infty} f(x_n)$ and thus $f(x) \le r \le f(x_n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be any arbitrary fixed and for all $m \in \mathbb{N}$ with $m > n$, as in the proof of Theorem 3.1, we have

$$\begin{aligned} q(x_n, x_m) &\le \varphi\left(f(x_n)\right)\left(f(x_n) - f(x_m)\right) \\ &\le \varphi\left(f(x_n)\right)\left(f(x_n) - f(x)\right) \end{aligned}$$

because $-f(x) \ge -f(x_m)$ for all $m \in \mathbb{N}$. Since n is arbitrary fixed, $\varphi\left(f(x_n)\right)$ and $f(x_n) - f(x)$ are fixed nonnegative real numbers because $f(x) \le f(x_n)$ for all $n \in \mathbb{N}$. We let $M = \varphi\left(f(x_n)\right)\left(f(x_n) - f(x)\right)$. Then by using (Q2), we obtain

$$q(x_n, x_m) \le M \Rightarrow q(x_n, x) \le M \qquad \text{for arbitrary fixed } n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$, we have

$$q(x_n, x) \le M = \varphi\left(f(x_n)\right)\left(f(x_n) - f(x)\right).$$

Thus, $x \preccurlyeq x_n$ and hence $\preccurlyeq$ is lower closed for any $z \in X$. By the definition of lower closedness, $S(z) = \{y \in X : y \preccurlyeq z\}$ is lower closed.

We now construct inductively a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. To each x_n we let

$$\begin{aligned} S(x_n) &= \{y \in X : y = x_n \text{ or } q(x_n, y) \le \varphi\left(f(x_n)\right)\left(f(x_n) - f(y)\right)\} \\ &= \{y \in X : y \preccurlyeq x_n\}. \end{aligned}$$

Then for all $n \in \mathbb{N}$, $S(x_n)$ is a lower closed subset of a complete quasi-metric space and hence $\preccurlyeq$-complete. The result follows from Theorem 3.1. □

Remark 3.2. (i) Theorems 3.1 and 3.2 extend and generalize Theorem 2.1 in [14], Theorem 1.1 in [16], Theorem 1 in [17], Theorem 3 in [27], Theorem 2.1 in [29] and Theorem 3 in [32]; See also references therein.
(ii) In [26], Hamel established an Ekeland-type variational principle (similar to Theorem 3.2) in the setting of uniform spaces generated by a family of quasi-metrics. He proved his results for sequentially lower monotone functions and for a quasi-metric. Above, Theorems 3.1 and 3.2 are proved for lsca functions, which are more general than lower monotone functions, and for a Q-function. As it is seen in the examples below that the concept of a Q-function and a quasi-metric are not comparable, the results of this paper are different from those considered in [26].

Example 3.1. Let $(X, \|\cdot\|)$ be a normed space. Then a function $q : X \times X \to \mathbb{R}_+$ defined as $q(x, y) = \|y\|$ for all $x, y \in X$, is a Q-function. But it is not a quasi-metric on X.

Example 3.2. Let $X = \mathbb{R}$. Define a function $d : X \times X \to \mathbb{R}_+$ as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |x| & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric on X but it is not a Q-function. We remark that every metric d is a Q-function.

Corollary 3.1. *Let X, q, f and φ be the same as in Theorem 3.2. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and subadditive function such that $\eta(0) = 0$. Assume that there exists $\hat{x} \in X$ such that $\inf_{x \in X} f(x) < f(\hat{x})$, then there exists $\bar{x} \in X$ such that*

(a) $\eta\left(q(\hat{x}, \bar{x})\right) \le \varphi\left(f(\hat{x})\right)\left(f(\hat{x}) - f(\bar{x})\right)$
(b) $\eta\left(q(\bar{x}, x)\right) > \varphi\left(f(\bar{x})\right)\left(f(\bar{x}) - f(x)\right)$ *for all* $x \in X$, $x \neq \bar{x}$.

Proof. From Remark 2.2, $\eta \circ q$ is a Q-function on X. By applying Theorem 3.2 with a Q-function $\eta \circ q$, we obtain the conclusion. □

Remark 3.3. Very recently, Bosch et al. [8] established a result similar to Corollary 3.1 but for a Minkowski gauge and in the setting of locally complete spaces. In addition to our assumptions on η, they also assumed that it is continuous. Therefore, the main result in [8] and Corollary 3.1 are not comparable.

4. Some equivalences

Throughout the paper, 2^X denotes the set of all subsets of X. We now present a Caristi–Kirk type fixed point theorem, Takahashi's minimization theorem and an equilibrium version of Ekeland-type variational principle for a Q-function in the setting of complete quasi-metric spaces. We also prove the equivalences among these results and Theorem 3.2.

Theorem 4.1. *Let (X, d) be a complete quasi-metric space, $q : X \times X \to \mathbb{R}_+$ a Q-function on X, $\varphi : (-\infty, \infty] \to (0, \infty)$ a nondecreasing function and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper, lsca and bounded below function. Then the following statements are equivalent reformulations of Theorem 3.2.*

(i) *(Caristi–Kirk Type Fixed Point Theorem). Let $T : X \to 2^X$ be a multivalued map with nonempty values. If the condition*

$$\text{for all } y \in T(x) : q(x, y) \le \varphi(f(x))(f(x) - f(y)) \tag{4.1}$$

is satisfied, then T has an invariant point in X, that is, there exists $\bar{x} \in X$ such that $\{\bar{x}\} = T(\bar{x})$.
If the condition

$$\text{there exists } y \in T(x) : q(x, y) \le \varphi(f(x))(f(x) - f(y)) \tag{4.2}$$

is satisfied, then T has a fixed point in X, that is, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

(ii) *(Takahashi's Minimization Theorem). Assume that for each $\hat{x} \in X$ with $\inf_{z \in X} f(z) < f(\hat{x})$, there exists $x \in X$ such that*

$$x \neq \hat{x} \quad \text{and} \quad q(\hat{x}, x) \le \varphi\left(f(\hat{x})\right)\left(f(\hat{x}) - f(x)\right). \tag{4.3}$$

Then there exists $\bar{x} \in X$ such that $f(\bar{x}) = \inf_{y \in X} f(y)$.

(iii) *(Equilibrium Version of Ekeland-Type Variational Principle). Let $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying the following conditions:*
(E1) *For all $x, y, z \in X$, $F(x, z) \le F(x, y) + F(y, z)$;*
(E2) *For each fixed $x \in X$, the function $F(x, \cdot) : X \mapsto \mathbb{R} \cup \{+\infty\}$ is proper and lsca;*
(E3) *There exists $\hat{x} \in X$ such that $\inf_{x \in X} F(\hat{x}, x) > -\infty$.*
Then, there exists $\bar{x} \in X$ such that
(aa) $\varphi\left(f(\hat{x})\right) F(\hat{x}, \bar{x}) + q(\hat{x}, \bar{x}) \le 0$
(bb) $\varphi\left(f(\bar{x})\right) F(\bar{x}, x) + q(\bar{x}, x) > 0$ *for all* $x \in X$, $x \neq \bar{x}$.

Proof. Theorem 3.2⇒ (i): From Theorem 3.2(b), there exists $\bar{x} \in X$ such that

$$q(\bar{x}, x) > \varphi(f(\bar{x}))(f(\bar{x}) - f(x)) \quad \text{for all } x \in X,\ x \neq \bar{x}. \tag{4.4}$$

We claim that $\{\bar{x}\} = T(\bar{x})$ (respectively, $\bar{x} \in T(\bar{x})$). Otherwise all $y \in T(\bar{x}) \subseteq X$ are such that $y \neq \bar{x}$. Then from (4.1) (respectively, from (4.2)) and (4.4), we have

$$q(\bar{x}, y) \leq \varphi(f(\bar{x}))(f(\bar{x}) - f(y)) \quad \text{and} \quad q(\bar{x}, y) > \varphi(f(\bar{x}))(f(\bar{x}) - f(y))$$

which can not hold simultaneously.
(i)⇒ (ii): Define a multivalued map $T : X \to 2^X$ as

$$T(x) = \{y \in X : q(x, y) \leq \varphi(f(x))(f(x) - f(y))\} \quad \text{for all } x \in X.$$

Then T satisfies (4.1) and by (i), there exists $\bar{x} \in X$ such that $\{\bar{x}\} = T(\bar{x})$. By assumption, for all $\hat{x} \in X$ there exists $x \in X$ such that $x \neq \hat{x}$, we have $x \in T(\hat{x})$ and so $T(\hat{x}) \setminus \{\hat{x}\} \neq \emptyset$ whenever $\inf_{z \in X} f(z) < f(\hat{x})$. Hence we must have $f(\bar{x}) = \inf_{x \in X} f(x)$.
(ii) ⇒ (iii): Define a function $f : X \to \mathbb{R} \cup \{+\infty\}$ by $f(x) = F(\hat{x}, x)$ for all $x \in X$, where $\hat{x}$ is the same as given in condition (E3). Then by condition (E3), we have $\inf_{x \in X} f(x) > -\infty$ and hence f is bounded below and so along with condition (E2), f is proper, lsca and bounded below. Assume that (bb) does not hold. Then for all $x \in X$, there exists $y \in X$ such that

$$y \neq x \quad \text{and} \quad \varphi(f(x))F(x, y) + q(x, y) \leq 0. \tag{4.5}$$

By condition (E1), we have

$$F(\hat{x}, y) - F(\hat{x}, x) \leq F(x, y)$$

and thus (4.5) becomes

$$\varphi(f(x))\big(F(\hat{x}, y) - F(\hat{x}, x)\big) + q(x, y) \leq \varphi(f(x))F(x, y) + q(x, y) \leq 0. \tag{4.6}$$

That is, for all $x \in X$, there exists $y \in X$ such that

$$y \neq x \quad \text{and} \quad \varphi(f(x))(f(y) - f(x)) + q(x, y) \leq 0$$

or

$$y \neq x \quad \text{and} \quad q(x, y) \leq \varphi(f(x))(f(x) - f(y)).$$

Then by (ii), there exists $\bar{x} \in X$ such that $f(\bar{x}) \leq f(z)$ for all $z \in X$. Substitute x by $\bar{x}$ in (4.6), we obtain

$$\text{there exists } y \in X \text{ such that} \quad y \neq \bar{x} \quad \text{and} \quad \varphi(f(\bar{x}))\big(F(\hat{x}, y) - F(\hat{x}, \bar{x})\big) + q(\bar{x}, y) \leq 0,$$

that is,

$$\varphi(f(\bar{x}))(f(y) - f(\bar{x})) + q(\bar{x}, y) \leq 0 \quad \text{or} \quad q(\bar{x}, y) \leq \varphi(f(\bar{x}))(f(\bar{x}) - f(y)). \tag{4.7}$$

Since $y \neq \bar{x}$, by Lemma 2.1(i) neither $q(\bar{x}, y) = 0$ nor $q(\bar{x}, \bar{x}) = 0$ and so we have $q(\bar{x}, y) > 0$. From (4.7) we obtain

$$0 < \varphi(f(\bar{x}))(f(\bar{x}) - f(y)) \Rightarrow f(y) < f(\bar{x})$$

a contradiction.
(iii) ⇒ Theorem 3.2: Define a function $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ as $F(x, y) = f(y) - f(x)$ for all $x, y \in X$ with $\hat{x} \in \text{dom}(f)$. Then by the hypothesis, F satisfies all the conditions of (iii). Then (iii) implies the existence of $\bar{x} \in X$ such that (a) and (b) hold. □

Remark 4.1. (i) Hamel [26] proved similar results to Theorem 4.1 for sequentially lower monotone functions and in the setting of uniform spaces generated by a family of quasi-metrics. Theorem 4.1 is proved for lsca functions which are more general than lower monotone functions, and for a Q-function. As we have seen above that Q functions and quasi-metrics are not comparable, the results of this paper are different from those considered in [26].
(ii) Theorem 4.1(i) generalizes Theorem 2.2 in [3], Theorem $(2.1)'$ in [10] and a result in [11].

(iii) Theorem 4.1(ii) extends and generalizes Theorem 1 in [27,36].
(iv) Theorem 4.1(iii) generalizes Theorem 2.1 in [29].

Corollary 4.1. *Let X, q, f and φ be the same as in Theorem 4.1. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and subadditive function such that $\eta(0) = 0$ and let $T : X \to 2^X$ be a multivalued map with nonempty values. If for all $x \in X$, there is a $y \in T(x)$ satisfying*

$$\eta(q(x, y)) \leq \varphi(f(x))(f(x) - f(y)),$$

then T has a fixed point in X.

Proof. From Remark 2.2, $\eta \circ q$ is a Q-function on X and by using Theorem 4.1(i), we obtain the desired conclusion. □

Remark 4.2. (i) Corollary 4.1 generalizes Theorem 4.2 in [20] in the following ways:

(a) X is a complete quasi-metric space in Corollary 4.1 while it is complete metric space in Theorem 4.2 in [20].
(b) f is bounded and lsca in Corollary 4.1 while is bounded below and lower semicontinuous in Theorem 4.2 in [20].
(c) In Corollary 4.1, η is not necessarily continuous.

(ii) Corollary 4.1 also generalizes and extends Theorem 3.17 in [28] in several ways.

By using the same arguments as in the proof of Theorem 2.2 in [29], a common fixed point theorem can be easily derived for a family of multivalued maps similar to Theorem 2.2 in [29]. Since the proof is straightforward, we do not mention it here.

As a particular case of Theorem 4.1(iii), we derive the following result by taking $\varphi(f(x)) = \frac{1}{\varepsilon}$ for all $x \in X$ and for any given $\varepsilon > 0$.

Corollary 4.2 (*Equilibrium Version of Ekeland-Type Variational Principle*). *Let (X, d) be a complete quasi-metric space and $q : X \times X \to \mathbb{R}_+$ be a Q-function on X. Let $F : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:*

(E1) *For all $x, y, z \in X$, $F(x, z) \leq F(x, y) + F(y, z)$;*
(E2) *For each fixed $x \in X$, the function $F(x, \cdot) : X \mapsto \mathbb{R}$ is lsca and bounded below.*

Then, for any $\varepsilon > 0$ and for any $\hat{x} \in X$, there exists $\bar{x} \in X$ such that

(aa) $F(\hat{x}, \bar{x}) + \varepsilon q(\hat{x}, \bar{x}) \leq 0$
(bb) $F(\bar{x}, x) + \varepsilon q(\bar{x}, x) > 0$ *for all $x \in X$, $x \neq \bar{x}$.*

Remark 4.3. Corollary 4.2 can be seem as an extension of Theorem 2.1 of Bianchi et al. [6] to the setting of complete quasi-metric spaces and for a Q-function.

5. Equilibrium problem and existence results

The equilibrium problem is a unified model of several problems, for example, optimization problems, saddle point problems, Nash equilibrium problems, variational inequality problems, nonlinear complementarity problems, fixed point problems, etc. In the last decade, it has emerged as a new direction of research in nonlinear analysis, optimization, optimal control, game theory, mathematical economics, etc. Most of the results on the existence of solutions of equilibrium problems are studied in the setting of topological vector spaces by using some kind of fixed point (Fan–Browder type fixed point) theorem or KKM type theorem. In [7,31], Blum, Oettli and Théra first gave the existence of a solution of an equilibrium problem in the setting of complete metric spaces. They have also showed that their existence result for a solution of the equilibrium problem is equivalent to an Ekeland-type variational principle for bifunctions, a Caristi–Kirk fixed point theorem for multivalued maps [11] and a maximal element theorem.

Let K be a nonempty subset of a metric space X and let $F : K \times K \to \mathbb{R}$ be a real valued function. The *equilibrium problem* (in short, EP) is to find $\bar{x} \in K$ such that

$$F(\bar{x}, x) \geq 0 \quad \text{for all } x \in K.$$

For further details on equilibrium problems, we refer to [1,4–7,9,12,13,18,19,22,23,25] and references therein.

Definition 5.1. Let K be a nonempty subset of a metric space X, $F : K \times K \to \mathbb{R}$ a real valued function and q a Q-function on X. Let $\varepsilon > 0$ be given. A point $\bar{x}$ is called an ε-*solution* of EP if

$$F(\bar{x}, y) + \varepsilon q(\bar{x}, y) \geq 0 \quad \text{for all } y \in K. \tag{5.1}$$

It is called *strictly* ε-*solution* of EP if the inequality in (5.1) is strict for all $x \neq y$.

We note that Corollary 4.2(bb) gives the existence of a strict ε-solution of EP for any $\varepsilon > 0$.

Now we prove the existence of a solution of equilibrium problem without any convexity assumption.

Theorem 5.1. *Let K be a nonempty compact subset of a complete metric space X and q be a Q-function on X. Let $F : K \times K \to \mathbb{R}$ be a real valued function satisfying the following conditions:*

(E1) *For all $x, y, z \in K$, $F(x, z) \leq F(x, y) + F(y, z)$;*
(E2) *For each fixed $x \in K$, the function $F(x, \cdot) : K \mapsto \mathbb{R}$ is lsca and bounded below;*
(E3) *For each fixed $y \in K$, the function $F(\cdot, y) : K \mapsto \mathbb{R}$ is upper semicontinuous.*

Then there exists a solution $\bar{x} \in K$ of EP.

Proof. By Corollary 4.2, for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that

$$F(x_n, y) + \frac{1}{n} q(x_n, y) \geq 0 \quad \text{for all } y \in K,$$

that is, for each $n \in \mathbb{N}$, $x_n \in K$ is an ε-solution of EP for $\varepsilon = \frac{1}{n}$. Since K is compact, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $n \to \infty$. Then by upper semicontinuity of $F(\cdot, y)$ on K, we have

$$F(\bar{x}, y) \geq \limsup_{k \to \infty} \left(F(x_{n_k}, y) + \frac{1}{n_k} q(x_{n_k}, y) \right) \geq 0 \quad \text{for all } y \in K$$

and thus $\bar{x}$ is a solution of EP. □

On the lines of Theorem 4.1 in [6] we can easily derive the existence results for a solution of EP when K is not necessarily compact.

Remark 5.1. (i) Theorem 5.1 generalizes Proposition 3.2 in [6] in the following ways:

(a) In Theorem 5.1, we did not assume that $F(x, x) = 0$ for all $x \in X$.
(b) In Theorem 5.1, $F(x, \cdot)$ is lsca, while it is lower semicontinuous in [6].

(ii) We notice that the product of n complete quasi-metric spaces (X_i, d_i) is a complete quasi-metric space (X, d), where $X = \prod_{i=1}^{n} X_i$, $d(x, y) = \max_{1 \leq i \leq n}\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$, $x = (x_1, x_2, \ldots, x_n) \in X$ and $y = (y_1, y_2, \ldots, y_n) \in X$. By Lemma 2.1(v), $q(x, y) = \max_{1 \leq i \leq n}\{q_1(x_1, y_1), \ldots, q_n(x_n, y_n)\}$ is a Q-function on X, where q_i is a Q-function on X_i for all $i = 1, 2, \ldots, n$. Therefore, Theorem 2.2 in [6] can be easily extended to complete quasi-metric spaces and Q-functions as in Theorem 5.1.

Definition 5.2. Let (X, d) be a complete quasi-metric space and q be a Q-function on X. We say that $x_0 \in X$ satisfies *Condition* (A) if and only if every sequence $\{x_n\} \subseteq X$ satisfying $F(x_0, x_n) \leq 1/n$ for all $n \in \mathbb{N}$ and $F(x_n, x) + \frac{1}{n} q(x_n, x) \geq 0$ for all $x \in X$ and $n \in \mathbb{N}$, has a convergent subsequence.

This definition is introduced and considered by Oettli and Théra [31] in the setting of a complete metric space.

The following result provides the existence of a solution of EP under the condition (A) but without the compactness assumption.

Theorem 5.2. *Let (X, d) be a complete quasi-metric space, q a Q-function on X and $F : X \times X \to \mathbb{R}$ satisfy condition* (E1)–(E2) *of Corollary 4.2 and upper semicontinuous in the first argument. If some $x_0 \in X$ satisfies Condition* (A)*, then there exists a solution $\bar{x} \in X$ of EP.*

Proof. Take $\varepsilon = \frac{1}{n}$ in Corollary 4.2. Then for each $n \in \mathbb{N}$ and for every $x_0 \in X$, there exists $x_n \in X$ such that

$$F(x_0, x_n) + \frac{1}{n} q(x_0, x_n) \leq 0 \tag{5.2}$$

and

$$F(x_n, x) + \frac{1}{n} q(x_n, x) > 0 \quad \text{for all } x \in X. \tag{5.3}$$

Since $q(x_0, x_n) \geq 0$, (5.2) implies that $F(x_0, x_n) \leq 0$ for all $n \in \mathbb{N}$. From Condition (A), there exists a subsequence of $\{x_n\}$ converges to some point $\bar{x} \in X$. By using the upper semicontinuity of $F(\cdot, x)$ and (5.3), we obtain that $\bar{x}$ is a solution of EP. □

Remark 5.2. (i) Theorem 5.2 extends Theorem 6(a) in [31] for a Q-function and in the setting of complete quasi-metric spaces.

(ii) In Theorems 5.1 and 5.2, we have not assumed that $F(x, x) = 0$ for all $x \in X$. This assumption, a sort of convexity condition on the underlying function F and convexity structure on the underlying set K, is required in almost all the results on the existence of a solution of EP appearing in the literature; See, for example, [1,4–7,9,12,13,18,19,22,23,25] and references therein. But in Theorems 5.1 and 5.2, neither is a kind of convexity condition required on the function F nor on the convexity structure on the set K. Therefore, the results of this section are new in the literature.

The following theorem provides the equivalence between the equilibrium version of Ekeland-type variational principle, the equilibrium problem, Caristi–Kirk type fixed point theorems and Oettli and Théra type theorems.

Theorem 5.3. *Let (X, d) be a complete quasi-metric space and $q : X \times X \to \mathbb{R}_+$ be a Q-function on X. Let $F : X \times X \to \mathbb{R}$ be a function satisfying the conditions* (E1) *and* (E2) *of Corollary* 4.2. *Then the following statements are equivalent:*

(i) *(Equilibrium Form of Ekeland-Type Variational Principle). For every $\hat{x} \in X$, there exists $\bar{x} \in X$ such that*

$$\bar{x} \in \hat{S} := \{x \in X : F(\hat{x}, x) + q(\hat{x}, x) \leq 0, x \neq \hat{x}\}$$

and

$$F(\bar{x}, x) + q(\bar{x}, x) > 0 \quad \textit{for all } x \in X \textit{ and } x \neq \bar{x}.$$

(ii) *(Existence of Solutions of EP). Assume that*

$$\begin{cases} \textit{for every } \tilde{x} \in \hat{S}, \ \textit{there exists } x \in X \\ \textit{such that} \quad x \neq \tilde{x} \quad \textit{and} \quad F(\tilde{x}, x) + q(\tilde{x}, x) \leq 0. \end{cases}$$

Then there exists $\bar{x} \in \hat{S}$ such that $F(\bar{x}, x) \geq 0$ for all $x \in X$.

(iii) *(Caristi–Kirk Type Fixed Point Theorem). Let $T : X \to 2^X$ be a multivalued mapping such that*

$$\begin{cases} \textit{for every } \tilde{x} \in \hat{S}, \ \textit{there exists } x \in T(\tilde{x}) \textit{ satisfying} \\ F(\tilde{x}, x) + q(\tilde{x}, x) \leq 0. \end{cases}$$

Then there exists $\bar{x} \in \hat{S}$ such that $\bar{x} \in T(\bar{x})$.

(iv) *(Oettli and Théra Type Theorem). Let D be a subset of X such that*

$$\begin{cases} \textit{for every } \tilde{x} \in \hat{S} \setminus D, \textit{there exists } x \in X \\ \textit{such that} \quad x \neq \tilde{x} \quad \textit{and} \quad F(\tilde{x}, x) + q(\hat{x}, x) \leq 0. \end{cases}$$

Then there exists $\bar{x} \in \hat{S} \cap D$.

The proof of this theorem lies on the lines of the proof of Theorem 5 in [31] and therefore we omit it.

6. A generalization of Nadler's fixed point theorem

In this section, we present a generalization of Nadler's fixed point theorem [30] to complete quasi-metric spaces with a Q-function. As a particular case, we show that the well known Banach contraction theorem holds good in the setting of a complete quasi-metric space with a Q-function.

Definition 6.1. Let (X, d) be a quasi-metric space. A multivalued map $T : X \to 2^X$ is said to be *q-contractive* if there exist a Q-function q on X and $r \in [0, 1)$ such that for all $x, y \in X$ and $u \in T(x)$ there is a $v \in T(y)$ satisfying

$$q(u, v) \leq rq(x, y).$$

The real number r is called *q-contractivity constant*.

In particular, a single valued map $f : X \to X$ is said to be *q-contractive* if there exist a Q-function q on X and $r \in [0, 1)$ such that

$$q(f(x), f(y)) \leq rq(x, y) \quad \text{for all } x, y \in X.$$

Remark 6.1. If X is a metric space and q is a w-distance on X, then the definition of a q-contractivity reduces to the definition of w-contractivity introduced and studied by Takahashi [37]. If X is a normed space, then it is also considered and studied in [34].

Now, we present a generalization of Nadler's fixed point theorem to a q-contractive multivalued map and in the setting of complete quasi-metric spaces.

Theorem 6.1. *Let (X, d) be a complete quasi-metric space and let $T : X \to 2^X$ be a q-contractive multivalued map such that for all $x \in X$, $T(x)$ is a nonempty closed subset of X. Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $q(\bar{x}, \bar{x}) = 0$.*

Proof. Let q be a Q-function on X and $r \in [0, 1)$ be a q-contractivity constant. Let $x_0 \in X$ and $x_1 \in T(x_0)$ be fixed. Then by the definition of a q-contractive multivalued map, there exists $x_2 \in T(x_1)$ such that

$$q(x_1, x_2) \leq rq(x_0, x_1).$$

Continue in this way, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in T(x_n)$ and

$$q(x_n, x_{n+1}) \leq rq(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}.$$

Now for all $n \in \mathbb{N}$, we have

$$q(x_n, x_{n+1}) \leq rq(x_{n-1}, x_n) \leq r^2 q(x_{n-2}, x_{n-1}) \leq \cdots \leq r^n q(x_0, x_1)$$

and hence, for any $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\leq r^n q(x_0, x_1) + r^{n+1} q(x_0, x_1) + \cdots + r^{m-1} q(x_0, x_1) \\ &= r^n \left(1 + r + r^2 \cdots + r^{m-n-1}\right) q(x_0, x_1) \\ &= \frac{r^n}{1 - r} q(x_0, x_1). \end{aligned}$$

Let $\alpha_n = \frac{r^n}{1-r}$, then $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_n \to 0$ as $n \to \infty$. By Lemma 2.1(iii), $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some point $\bar{x} \in X$. Let $n \in \mathbb{N}$ be any arbitrary fixed number, and for all $m \in \mathbb{N}$ with $m > n$, as above, we have

$$q(x_n, x_m) \leq \frac{r^n}{1 - r} q(x_0, x_1). \tag{6.1}$$

Let $M = \frac{r^n}{1-r} q(x_0, x_1)$, then $M \geq 0$. By using (Q2), we obtain

$$q(x_n, x_m) \leq M \Rightarrow q(x_n, \bar{x}) \leq M \quad \text{for arbitrary fixed } n \in \mathbb{N}$$

since $\{x_m\}$ converges to $\bar{x}$. Since n was arbitrary fixed, we have

$$q(x_n, \bar{x}) \leq \frac{r^n}{1-r} q(x_0, x_1) \quad \text{for all } n \in \mathbb{N}. \tag{6.2}$$

By hypothesis, we also have $z_n \in T(\bar{x})$ such that

$$q(x_n, z_n) \leq r q(x_{n-1}, \bar{x}).$$

Therefore, by using (6.2), for any $n \in \mathbb{N}$, we have

$$q(x_n, z_n) \leq r q(x_{n-1}, \bar{x}) \leq \frac{r^n}{1-r} q(x_0, x_1). \tag{6.3}$$

From (6.2) and (6.3) and applying Lemma 2.1(ii), we obtain that $\{z_n\}$ converges to $\bar{x}$. Since $T(\bar{x})$ is closed, we have $\bar{x} \in T(\bar{x})$.

Now we prove that $q(\bar{x}, \bar{x}) = 0$, where $\bar{x} \in T(\bar{x})$. For such $\bar{x}$, there exists $y_1 \in T(\bar{x})$ such that $q(\bar{x}, y_1) \leq r q(\bar{x}, \bar{x})$. As above, we obtain a sequence $\{y_n\}$ in X such that $y_{n+1} \in T(y_n)$ and

$$q(\bar{x}, y_{n+1}) \leq r q(\bar{x}, y_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$, we have

$$q(\bar{x}, y_n) \leq r q(\bar{x}, y_{n-1}) \leq \cdots \leq r^n q(\bar{x}, \bar{x}). \tag{6.4}$$

Since $q(\bar{x}, \bar{x})$ is a fixed nonnegative real number and $r^n \geq 0$ for all $n \in \mathbb{N}$ and $r^n \to 0$ as $n \to \infty$, by Lemma 2.1(iv), $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to some point $\bar{y} \in X$. Let $M = \sup_{n \in \mathbb{N}} r^n q(\bar{x}, \bar{x})$, then $M > 0$. From (6.4) and by using (Q2), we obtain

$$q(\bar{x}, y_n) \leq M \Rightarrow q(\bar{x}, \bar{y}) \leq M = \sup_{n \in \mathbb{N}} r^n q(\bar{x}, \bar{x})$$

and thus $q(\bar{x}, \bar{y}) \leq 0$. Hence $q(\bar{x}, \bar{y}) = 0$. For any $n \in \mathbb{N}$, we have

$$q(x_n, \bar{y}) \leq q(x_n, \bar{x}) + q(\bar{x}, \bar{y}) \leq \frac{r^n}{1-r} q(x_0, x_1) \quad \text{by using (6.2).} \tag{6.5}$$

From (6.2) and (6.5) and by using Lemma 2.1(i), we obtain $\bar{x} = \bar{y}$. Hence $q(\bar{x}, \bar{x}) = 0$. □

Remark 6.2. (i) Theorem 6.1 generalizes Theorem 2.2.9 in [37] in the following ways:

(a) In Theorem 6.1, X is assumed to be complete quasi-metric space, while it is complete metric space in Theorem 2.2.9 in [37].
(b) Theorem 6.1 holds for Q-functions, while Theorem 2.2.9 in [37] is true for w-distance which is a stronger assumption than Q-functions.

(ii) Theorem 6.1 also generalizes Nadler's fixed point theorem [30] and Theorem 2.1 in [34] in several ways.

As a consequence of Theorem 6.1, we show that the well known Banach contraction theorem also holds good for a q-contractive function and in the setting of complete quasi-metric spaces.

Corollary 6.1. *Let (X, d) be a complete quasi-metric space. If $T : X \to X$ is a q-contractive map, then there exists a unique fixed point $\bar{x} \in X$ of T, that is, $\bar{x} = T(\bar{x})$, and, such $\bar{x}$ satisfies $q(\bar{x}, \bar{x}) = 0$.*

Proof. The existence of the fixed point $\bar{x} \in X$ of T that satisfies $q(\bar{x}, \bar{x}) = 0$ follows from Theorem 6.1. We prove the uniqueness of the fixed point $\bar{x} \in X$ of T. Let $\bar{y}$ be another fixed point of T then $\bar{y} = T(\bar{y})$. Since T is q-contractive map, we have

$$q(\bar{x}, \bar{y}) = q(T(\bar{x}), T(\bar{y})) \leq r q(\bar{x}, \bar{y}).$$

Since $r \in [0, 1)$, we have $q(\bar{x}, \bar{y}) = 0$. By using Lemma 2.1(i) and $q(\bar{x}, \bar{x}) = 0$, we obtain $\bar{x} = \bar{y}$. □

Remark 6.3. (i) Corollary 6.1 generalizes Corollary 2.2.10 in [37] in the following ways:

(a) In Corollary 6.1, X is assumed to be complete quasi-metric space, while it is complete metric space in Corollary 2.2.10 in [37].
(b) Corollary 6.1 holds for Q-functions, while Corollary 2.2.10 in [37] is true for w-distance which is a stronger assumption than Q-functions.

(ii) Corollary 6.1 also generalizes the well-known Banach contraction theorem in several ways.

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