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On the constrained equilibrium problems with finite families of players[☆]

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Abstract

In this paper, we consider the equilibrium problem with finite number of families of players such that each family may not have the same number of players and finite number of families of constrained correspondences on the strategy sets. We also consider the case with two finite families of constrained correspondences on the strategies sets. We demonstrate an example of our equilibrium problem. We derive a fixed point theorem for a family of multimaps and a coincidence theorem for two families of multimaps. By using these results, we establish the existence of a solution of our equilibrium problems. The results of this paper generalize some known results in the literature.

Keywords: Constrained equilibrium problems; Debreu Social equilibrium problem; Nash equilibrium problem; Fixed point; Coincidence theorem

1. Introduction

Let I be any finite index set and for each $k \in I$, let J_k be a finite index set. We consider the multiobjective game in the strategy form $\triangle = (X_{k_j}, A_{k_j}, F_{k_j})_{k \in I, j \in J_k}$ with finite families of players such that each family may not have the same (finite) number of players and each player in any family have multivalued gain (or payoff) function.

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For each $k \in I$ and $j \in J_k$, let X_{k_j} denote the strategy set of jth player in kth family, $Y_k = \prod_{j \in J_k} X_{k_j}$, $Y^k = \prod_{l \in I, l \neq k} Y_l$, $A_{k_j} : Y^k \multimap X_{k_j}$ the constrained correspondence which restricts the strategy of the jth player in the kth family to the subset $A_{k_j}(Y^k)$ of X_{k_j} when all the players in other families have chosen their strategies x_{ij}, $i \in I$, $i \neq k$ and $j \in J_i$. Let $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be the cost function of the jth player in the kth family, where ℓ_{k_j} is a natural number and $\mathbb{R}^{\ell_{k_j}}$ is a ℓ_{k_j}-dimensional Euclidean space. In this game, the cost function F_{k_j} is a multivalued function of its strategy and strategies of players in other families. In this problem, we are interested in finding a strategies combination $\bar{y} = (\bar{y}_k)_{k \in I} \in \prod_{k \in I} Y_k = Y$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$, with $\bar{x}_{k_j} \in A_{k_j}(\bar{y}^k)$ and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\mathrm{int}\, \mathbb{R}_+^{\ell_{k_j}}, \tag{1.1}$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in A_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$.

If we let $z_{k_j} = (u^1_{k_j}, \ldots, u^{\ell_{k_j}}_{k_j})$, $\bar{z}_{k_j} = (\bar{u}^1_{k_j}, \ldots, \bar{u}^{\ell_{k_j}}_{k_j})$, then inequality (1.1) implies that there exists $1 \leqslant s \leqslant \ell_{k_j}$ such that $u^s_{k_j} \geqslant \bar{u}^s_{k_j}$, that is, we can choose a collection of objects $\bar{z}_{k_j}$ from $F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that for any collection z_{k_j} chosen from $F_{k_j}(x_{k_j}, \bar{y}^k)$ there is $1 \leqslant s \leqslant \ell_{k_j}$ such that $u^s_{k_j} \geqslant \bar{u}^s_{k_j}$.

For each $k \in I$, let $J_k = \{k\}$ be a singleton set and F_k be a single real valued function. Then above problem is reduced to the Debreu social equilibrium problem [5], which is an extension of the Nash equilibrium problem [14], see also [2]. In 1952, Debreu established the existence of a solution of social equilibrium problem. Since then, many generalizations and applications of these two problems have been appeared in the literature, see [2,16–21] and references therein. We note that these problems have only one family of players and the cost functions are single-valued. There is no constrained correspondences on the strategy sets in the Nash equilibrium problem, while Debreu social equilibrium problem has only one system of constrained correspondence on the strategy sets. If for each $k \in I$, J_k is not a singleton set, our problem is different from Debreu social equilibrium problem. Recently, Lin [8] considered the constrained two families of players competitive equilibrium problems and proved the existence of a solution of this problem. In many problems, we always have finite families of players such that each family have finite players and the cost functions are multimaps; See the following example.

We now demonstrate an example of this kind of equilibrium problem. Let $I = \{1, 2, \ldots, m\}$ denote the index set of the companies. For each $k \in I$, let $J_k = \{1, 2, \ldots, n_k\}$ denote the index set of factories in the kth company, F_{k_j} denote the cost function of the jth factory in the kth company. We assume that the products between the factories in the same company are different, while some collections of products are the same and some collections of products are different between different factories in different companies. Therefore, the strategy of jth factory in the kth company depends on the strategies of all factories in different companies. The cost function F_{k_j} of the jth factory in the kth company depends on its strategy and the strategies of factories in other companies. With this strategies combination, each factory can choose a collection of products which minimize the loss of each factory.

In Section 2, we present some known definitions and results which will be used in the sequel, while in Section 3, we derive a fixed point theorem for a family of multimaps and a coincidence theorem for two families of multimaps, which are used in Sections 4 and 5. In Section 4, we establish existence of a solution of our equilibrium problem with finite number of families of players and finite number of families of constraints on strategy sets. We note that the Nash equilibrium theorem [14] is a special case of our result. The results of this section also generalize a result of Lin [8] which is derived by using a fixed point theorem of Park [15] and a coincidence theorem. The last section deals with the equilibrium problem with finite number of families of players and two finite number of families of constraints on strategy sets. In Section 4, we consider the problem that the strategy of each family is influenced by the strategies of other families, while in Section 5, we consider the problem that for each $k \in I$, the strategy of kth family is influenced by the strategies of all the other families and the strategy of kth family will influence all the other families. The result of this paper generalize some known results in the literature.

2. Preliminaries

Let X and Y be nonempty sets. A multimap $T : X \multimap Y$ is a function from X to the power set of Y. Let $A \subset X$, $B \subset Y$, $x \in X$ and $y \in Y$, we define $T(A) = \bigcup\{T(x) \colon x \in A\}$; $x \in T^{-1}(y)$ if and only if $y \in T(x)$ and $T^{-1}(B) = \{x \in X \colon T(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y and $A \subset X$, we denote by $\operatorname{int} A$ and $\bar{A}$ the interior and the closure of A in X, respectively. Let $T : X \multimap Y$, then $\bar{T} : X \multimap Y$ is defined by $\bar{T} = \overline{T(x)}$ for all $x \in X$. T is said to be (i) *upper semicontinuous* (u.s.c.) if for every $x \in X$ and every open set U in Y with $T(x) \subset U$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subset U$ for all $x' \in U(x)$; (ii) *lower semicontinuous* (l.s.c.) if for every $x \in X$ and for every open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $V(x)$ of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in V(x)$; (iii) *continuous* if it is both u.s.c. and l.s.c.; (iv) *closed* if its graph $G_r(T) = \{(x, y) \colon x \in X, \ y \in T(x)\}$ is closed in $X \times Y$; and (v) *compact* if there is a compact subset $K \subset Y$ such that $T(X) \subset K$.

A topological space is said to be *acyclic* if all of its reduced Ĉech homology groups vanish. A multimap $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with acyclic compact values. Let $V(X, Y)$ be the family of acyclic multimaps $T : X \multimap Y$. Throughout this paper, all topological spaces are assumed to be Hausdorff.

Definition 2.1 (Luc [11]). Let Y be a topological vector space with a pointed closed convex solid cone C, i.e., $\operatorname{int} C \neq \emptyset$, then a function $\xi : Y \to \mathbb{R}$ is said to be *monotonically increasing* (respectively, *strict monotonically increasing*) *with respect to* C if $\xi(a) \geqslant \xi(b)$ for all $a - b \in C$ (respectively, $\xi(a) > \xi(b)$ for all $a - b \in \operatorname{int} C$).

Definition 2.2 (Luc and Vargas [12]). Let K be a nonempty convex subset of a topological vector space E and let Z be a topological vector space with a convex cone C. A multimap $G : K \multimap Z$ is said to be *C-quasiconvex* (respectively, *C-quasiconcave*) if for any $z \in Z$, the set

$$\{x \in K: \text{ there is a } y \in G(x) \text{ such that } z - y \in C\}$$

$$(\text{respectively, } \{x \in K: \text{ there is a } y \in G(x) \text{ such that } z - y \in -C\})$$

is convex.

G is said to be C-convex if for any $x_1, x_2 \in K$ and $0 \leqslant \lambda \leqslant 1$ $\lambda G(x_1) + (1-\lambda)G(x_2) \subset G(\lambda x_1 + (1-\lambda)x_2) + C$.

We now mention some lemmas which will be used in the sequel.

Lemma 2.1 (Ben-El-Mechaiekh et al. [4] and Lin and Park [9]). *Let X be a compact topological space, Y a convex space, $G, T : X \multimap Y$ two multimaps such that for any $x \in X$, $G(x) \neq \emptyset$ and $\operatorname{co} G(x) \subset T(x)$, and $X = \bigcup_{y \in Y} \operatorname{int} G^{-1}(y)$, where $\operatorname{co} G(x)$ denotes the convex hull of $G(x)$. Then T has a continuous selection $f : X \to Y$, that is, there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$.*

Lemma 2.2 (Aubin and Cellina [3]). *Let X and Y be two topological spaces and $T : X \multimap Y$ be a multimap.*

(a) *If X is compact and T is u.s.c. with nonempty compact values, then $T(X)$ is compact.*
(b) *If Y is compact and T is closed, then T is u.s.c.*
(c) *If T is u.s.c. with nonempty closed values, then T is closed.*

Lemma 2.3 (Lin and Yu [10]). *Let X and Y be topological spaces, $F : X \times Y \multimap \mathbb{R}$, $S : X \multimap Y$, $m(x) = \sup F(x, S(x))$ and $M(x) = \{y \in S(x): m(x) \in F(x, y)\}$.*

(a) *If both F and S are l.s.c., then m is l.s.c..*
(b) *If both F and S are u.s.c. with nonempty compact values, then m is u.s.c..*
(c) *If both F and S are continuous multimaps with nonempty compact values, then m is a continuous function and M is an u.s.c. and closed multimap.*

Lemma 2.4 (Tan et al. [16]). *Let X be a topological space and $T : X \multimap \mathbb{R}$ be a continuous multimap with nonempty compact values. Then the function $m : X \multimap \mathbb{R}$ defined by $m(x) = \max T(x)$ is continuous.*

Lemma 2.5 (Park [15]). *Let X be a nonempty compact and convex subset of a locally convex topological vector space E and let $T : X \multimap X$ be an u.s.c. with nonempty closed acyclic values. Then T has at least one fixed point, i.e., there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

Lemma 2.6 (Lee et al. [7]). *Let K be a convex subset of a topological vector space E and let D be a closed convex cone of a topological vector space Z such that $\operatorname{int} D \neq \emptyset$. Let $T : K \multimap Z$ be a multimap and for each fixed $e \in \operatorname{int} D$ and any fixed*

$a \in Z$, let

$$p(z) = \min\{t \in \mathbb{R}\colon z \in a + te - D\}.$$

If T is D-quasiconvex, then $pT : X \multimap \mathbb{R}$ is $\mathbb{R}_+$-quasiconvex.

Lemma 2.7 (Fan [6]). *Let X be a topological vector space and $\{Y_\alpha\}_{\alpha \in \Lambda}$ be a family of compact spaces and $Y = \prod_{\alpha \in \Lambda} Y_\alpha$. If for each $\alpha \in \Lambda$, $f_\alpha : X \multimap Y_\alpha$ is an u.s.c. multimap with nonempty closed values. Then the multimap $f : X \multimap Y$ defined by $f(x) = \prod_{\alpha \in \Lambda} f_\alpha(x)$ is u.s.c.*

3. Fixed point and coincidence theorems

Let Λ be any index set and for each $\alpha \in \Lambda$, let E_α be a topological vector space. Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a family of nonempty convex subsets with each X_α in E_α, $E = \prod_{\alpha \in \Lambda} E_\alpha$, $X = \prod_{\alpha \in \Lambda} X_\alpha$, $X^\alpha = \prod_{\beta \in \Lambda, \beta \neq \alpha} X_\beta$ and we write $X = X^\alpha \times X_\alpha$. For each $x \in X$, $x_\alpha \in X_\alpha$ denotes the αth coordinate and $x^\alpha \in X^\alpha$ the projection of x onto X^α and we also write $x = (x^\alpha, x_\alpha)$.

Very recently, Ansari et al. [1] established the following fixed point theorem for a family of multimaps.

Theorem 3.1 (Ansari et al. [1]). *For each $\alpha \in \Lambda$, let $\psi_\alpha : X^\alpha \multimap X_\alpha$ be a multimap. Assume that the following conditions hold*:

(a) *For each $\alpha \in \Lambda$ and for all $x^\alpha \in X^\alpha$, $\psi_\alpha(x^\alpha)$ is a nonempty convex set*;
(b) *For each $\alpha \in \Lambda$, $X^\alpha = \bigcup\{\mathrm{int}_{X^\alpha}\, \psi_\alpha^{-1}(x_\alpha)\colon x_\alpha \in X_\alpha\}$*;
(c) *There exist a nonempty compact subset K^α of X^α and a nonempty compact convex subset D_α of X_α such that for each $x^\alpha \in X^\alpha \backslash K^\alpha$ there exists $y_\alpha \in D_\alpha$ such that $x^\alpha \in \mathrm{int}_{X^\alpha}\, \psi^{-1}(y_\alpha)$.*

Then there exists $\bar{x} = (\bar{x}_\alpha)_{\alpha \in \Lambda} \in X$ such that $\bar{x}_\alpha \in \psi_\alpha(\bar{x}^\alpha)$ for all $\alpha \in \Lambda$.

Now we establish a coincidence theorem for two families of multimaps which will be used in Section 5 to prove the existence of a solution of equilibrium problems with finite families of players and two finite number of families of constraints on strategy sets.

Theorem 3.2. *For each $\alpha \in \Lambda$, let X_α be a nonempty compact convex subset of a locally convex topological vector space E_α and let $\psi_\alpha : X^\alpha \multimap X_\alpha$ and $\phi_\alpha : X_\alpha \multimap X^\alpha$ be multimaps. For each $\alpha \in \Lambda$, assume that the following conditions hold*:

(a) *For all $x^\alpha \in X^\alpha$, $\psi_\alpha(x^\alpha)$ is a nonempty convex subset of X_α*;
(b) *$X^\alpha = \bigcup\{\mathrm{int}_{X^\alpha}\, \psi_\alpha^{-1}(x_\alpha)\colon x_\alpha \in X_\alpha\}$*;
(c) *For all $x_\alpha \in X_\alpha$, $\phi_\alpha(x_\alpha)$ is a nonempty convex subset of X^α*;
(d) *$X_\alpha = \bigcup\{\mathrm{int}_{X_\alpha}\, \phi_\alpha^{-1}(x^\alpha)\colon x^\alpha \in X^\alpha\}$.*

Then there exist $\bar{x} = (\bar{x}_\alpha)_{\alpha\in\Lambda} \in X$ *and* $\bar{u} = (\bar{u}_\alpha)_{\alpha\in\Lambda} \in X$ *such that* $\bar{x}_\alpha \in \psi_\alpha(\bar{u}^\alpha)$ *and* $\bar{u}^\alpha \in \phi_\alpha(\bar{x}_\alpha)$ *for all* $\alpha \in \Lambda$.

Proof. From conditions (a), (b) and Lemma 2.1, for each $\alpha\in\Lambda$, $\psi_\alpha : X^\alpha \multimap X_\alpha$ has a continuous selection $f_\alpha : X^\alpha \to X_\alpha$. Similarly, it follows from conditions (c), (d) and Lemma 2.1 that for each $\alpha\in\Lambda$, $\phi_\alpha : X_\alpha \multimap X^\alpha$ has a continuous selection $g_\alpha : X_\alpha \to X^\alpha$. Let $h : X \to X$ be defined by $h(x) = \prod_{\alpha\in\Lambda} f_\alpha(g_\alpha(x_\alpha))$ for $x = (x_\alpha)_{\alpha\in\Lambda} \in X$. Then h is a continuous function on X. Since X is a compact convex subset of a locally convex topological vector space $E = \prod_{\alpha\in\Lambda} E_\alpha$, by Tychonoff fixed point theorem, there exists $\bar{x} = (\bar{x}_\alpha)_{\alpha\in\Lambda} \in X$ such that $\bar{x} = h(\bar{x}) = \prod_{\alpha\in\Lambda} f_\alpha(g_\alpha(\bar{x}_\alpha))$, that is, for each $\alpha\in\Lambda$, $\bar{x}_\alpha = f_\alpha(g_\alpha(\bar{x}_\alpha))$. For each $\alpha\in\Lambda$, let $\bar{u}^\alpha = g_\alpha(\bar{x}_\alpha)$, then $\bar{u}^\alpha \in X^\alpha$. Hence $\bar{x}_\alpha = f_\alpha(\bar{u}^\alpha) \in \psi_\alpha(\bar{u}^\alpha)$ and $\bar{u}^\alpha = g_\alpha(\bar{x}_\alpha) \in \phi_\alpha(\bar{x}_\alpha)$ for all $\alpha\in\Lambda$. □

Let I be any finite index set and for each $k \in I$, let J_k be a finite index set. For each $k\in I$ and $j \in J_k$, let X_{k_j} be a nonempty compact convex subset of a topological vector space E_{k_j}. We write $Y_k = \prod_{j\in J_k} X_{k_j}$, $Y = \prod_{k\in I} Y_k$, $Y^k = \prod_{l\in I, l\neq k} Y_l$ and $Y = Y^k \times Y_k$.

By using Theorem 3.1, we derive the following result which will be used in the sequel.

Theorem 3.3. *For each* $k\in I$ *and* $j\in J_k$, *let* $\psi_{k_j} : Y^k \multimap X_{k_j}$ *be a multimap. For each* $k \in I$, *assume that the following conditions hold*:

(a) *For each* $j \in J_k$ *and for all* $y^k \in Y^k$, $\psi_{k_j}(y^k)$ *is a convex set*;
(b) $Y^k = \bigcup \mathrm{int}_{Y^k}\{\bigcap_{j\in J_k} \psi_{k_j}^{-1}(x_{k_j}) \colon (x_{k_j})_{j\in J_k} \in Y_k\}$.

Then there exists $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$ *such that* $\bar{x}_{k_j} \in \psi_{k_j}(\bar{y}^k)$ *for all* $k \in I$ *and* $j \in J_k$, *where* $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$ *and* $\bar{y}^k = (\bar{y}_l)_{l\in I, l\neq k}$.

Proof. For each $k\in I$, let $g_k : Y^k \multimap Y_k$ be defined by $g_k(y^k) = \prod_{j\in J_k} \psi_{k_j}(y^k)$. Then $g_k : Y^k \multimap Y_k$ is a multimap with convex values.

Note that for each $k\in I$, $u_k = (v_{k_j})_{j\in J_k} \in Y_k$,

$$\begin{aligned} y^k \in g_k^{-1}(u_k) &\Leftrightarrow u_k \in g_k(y^k) = \prod_{j\in J_k} \psi_{k_j}(y^k) \\ &\Leftrightarrow v_{k_j} \in \psi_{k_j}(y^k) \quad \text{for all } j \in J_k \\ &\Leftrightarrow y^k \in \psi_{k_j}^{-1}(v_{k_j}) \quad \text{for all } j \in J_k \\ &\Leftrightarrow y^k \in \bigcap_{j\in J_k} \psi_{k_j}^{-1}(v_{k_j}). \end{aligned}$$

Therefore, $g_k^{-1}(u_k) = \bigcap_{j\in J_k} \psi_{k_j}^{-1}(v_{k_j})$. By condition (b), for each $k \in I$,

$$Y^k = \bigcup\{\mathrm{int}_{Y^k}\, g_k^{-1}(u_k) \colon u_k \in Y_k\}.$$

Therefore, for each $y^k \in Y^k$, there exists $u_k \in Y_k$ such that

$$y^k \in \operatorname{int}_{Y^k} g_k^{-1}(u_k) \subseteq g_k^{-1}(u_k)$$

and so $u_k \in g_k(y^k)$. Hence for any $k \in I$, $g_k(y^k)$ is nonempty for each $y^k \in Y^k$. Then from Theorem 3.1, there exists $\bar{y}=(\bar{y}_k)_{k\in I} \in Y$ such that $\bar{y}_k \in g_k(\bar{y}^k)=\prod_{j\in J_k} \psi_{k_j}(\bar{y}^k)$. Let $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$, then $\bar{x}_{k_j} \in \psi_{k_j}(\bar{y}^k)$ for all $k \in I$ and $j \in J_k$. □

4. Equilibrium problems with finite families of players and finite families of constraints on strategy sets

In this section, we establish the existence of a solution of our equilibrium problem with finite number of families of players and finite number of families of constraints on strategy sets by using Theorems 3.1 and 3.3.

Let I be any index set and for each $k \in I$, let J_k be a finite index set. For each $k \in I$ and $j \in J_k$, let X_{k_j} be a nonempty compact convex subset of a topological vector space E_{k_j}. We write $Y_k = \prod_{j\in J_k} X_{k_j}$, $Y = \prod_{k\in I} Y_k$, $Y^k = \prod_{l\in I, l\neq k} Y_l$ and $Y = Y^k \times Y_k$. For each $k \in I$ and $j \in J_k$, let $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \setminus \{0\}$ be a fixed vector and $W_k = \prod_{j\in I_k} W_{k_j}$ and let $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be the payoff multimap and $A_{k_j} : Y^k \multimap X_{k_j}$ be the constraints. For each $k \in I$, let $A_k : Y^k \multimap Y_k$ be defined by $A_k(y^k) = \prod_{j\in J_k} A_{k_j}(y^k)$, where $y_k = (x_{k_j})_{j\in J_k} \in Y_k$ and $y^k = (y_l)_{l\in I, l\neq k} \in Y^k$, also let $S^{W_k} : Y_k \times Y^k \multimap \mathbb{R}$ be defined by

$$S^{W_k}(y_k, y^k) = \sum_{j\in J_k} W_{k_j} \cdot F_{k_j}(x_{k_j}, y^k)$$

$$= \left\{ u\colon\ u = \sum_{j\in J_k} W_{k_j} \cdot z_{k_j},\ \text{ for } z_{k_j} \in F_{k_j}(x_{k_j}, y^k) \right\}.$$

For each $k \in I$, let $M^{W_k} : Y^k \multimap Y_k$ be defined by

$$M^{W_k}(y^k) = \{ y_k \in A_k(y^k)\colon\ \inf S^{W_k}(y_k, y^k) = \inf S^{W_k}(A_k(y^k), y^k) \}$$

and

$$M(y) = \prod_{k\in I} M^{W_k}(y^k).$$

Throughout the paper, we shall use the above-mentioned notations, unless otherwise specified.

Theorem 4.1. *For each $k \in I$ and $j \in J_k$, let $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be a nonempty multimap, $A_{k_j} : Y^k \multimap X_{k_j}$ be a multimap with nonempty convex values such that for* *all* $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ *is open in* Y^k *and* $\bar{A}_{k_j}$ *is an u.s.c. multimap. Assume that*

(i) *for each* $k \in I$, $S^{W_k}: Y_k \times Y^k \multimap \mathbb{R}$ *is a continuous multimap with nonempty compact values,*
(ii) *for each* $y^k \in Y^k$, *the multimap* $u_k \multimap S^{W_k}(u_k, y^k)$ *is* $\mathbb{R}_+$*-quasiconvex.*

Then there exists $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, *where* $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$, *with* $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ *and there exists* $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ *such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ *and for each* $k \in I$ *and* $j \in J_k$.

Proof. From the proof of Theorem 3.3, for each $k \in I$ and $y_k \in Y_k$, we have $A_k^{-1}(y_k) = \bigcap_{j\in J_k} A_{k_j}^{-1}(x_{k_j})$, where $y_k = (x_{k_j})_{j\in J_k}$. Since for each $k \in I$ and $j \in J_k$, $A_{k_j}^{-1}(x_{k_j})$ is open, $A_k^{-1}(y_k)$ is open and since for each $k \in I$, $\bar{A}_k(\bar{y}^k) = \prod_{j\in J_k} \bar{A}_{k_j}(\bar{y}^k)$, by Lemma 2.7, $\bar{A}_k$ is an u.s.c. multimap with nonempty convex values. For each $k \in I$ and for all $n \in \mathbb{N}$, we define $H_{k,n}: Y^k \multimap Y_k$ by

$$H_{k,n}(y^k) = \left\{ u_k \in A_k(y^k):\ \min S^{W_k}(u_k, y^k) < \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n} \right\}.$$

Since $\bar{A}_k$ is an u.s.c multimap with nonempty closed values, it follows from Lemma 2.2 (c) that $\bar{A}_k$ is closed. Since $\bar{A}_k$ is a multimap with nonempty compact values and S^{W_k} is a continuous multimap with compact values, it follows from Lemma 2.2 (a), $S^{W_k}(\bar{A}_k(y^k), y^k)$ is a compact set. Therefore, there exist $u_k \in \bar{A}_k(y^k)$ and $v_k \in S^{W_k}(u_k, y^k)$ such that

$$v_k = \min S^{W_k}(u_k, y^k) = \min S^{W_k}(\bar{A}_k(y^k), y^k).$$

Since $u_k \in \bar{A}_k(y^k)$, there exists a net $u_{k_\alpha} \in A_k(y^k)$ such that $u_{k_\alpha} \to u_k$. By Lemma 2.4, the function $x \mapsto \min S^{W_k}(x, y^k)$ is continuous. Therefore, for each $n \in \mathbb{N}$

$$\min S^{W_k}(u_{k_\alpha}, y^k) - \min S^{W_k}(u_k, y^k) < \frac{1}{n}$$

for sufficiently large α. This shows that

$$\min S^{W_k}(u_{k_\alpha}, y^k) < \min S^{W_k}(u_k, y^k) + \frac{1}{n} = \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n}$$

and $u_{k_\alpha} \in H_{k,n}(y^k)$ for sufficiently large α. Hence for all $n \in \mathbb{N}$ and $y^k \in Y^k$, $H_{k,n}(y^k)$ is nonempty.

Since for each $y^k \in Y^k$, the multimap $u_k \multimap S^{W_k}(u_k, y^k)$ is $\mathbb{R}_+$-quasiconvex and $A_{k_j}(y^k)$ is convex, $H_{k,n}(y^k)$ is convex.

Indeed, let $u_k, u'_k \in H_{k,n}(y^k)$ and $\lambda \in [0,1]$, then $u_k, u'_k \in A_k(y^k)$ and

$$\min S^{W_k}(u_k, y^k) < \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n}$$

and

$$\min S^{W_k}(u'_k, y^k) < \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n}.$$

Let m_k and m'_k be such that

$$m_k = \min S^{W_k}(u_k, y^k) \in S^{W_k}(u_k, y^k)$$

and

$$m'_k = \min S^{W_k}(u'_k, y^k) \in S^{W_k}(u'_k, y^k),$$

then

$$m_k < a \text{ and } m'_k < a, \quad \text{where } a = \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n}.$$

By the definition of $\mathbb{R}_+$-quasiconvexity of the multimap $u_k \multimap S^{W_k}(u_k, y^k)$, there exists $b \in S^{W_k}(\lambda u_k + (1-\lambda)u'_k, y^k)$ such that

$$\min S^{W_k}(\lambda u_k + (1-\lambda)u'_k, y^k) \leqslant b < \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n} = a.$$

Since for all $k \in I$ and $j \in J_k$, A_{k_j} has convex values and so A_k and therefore $\lambda u_k + (1-\lambda)u'_k \in A_k(y^k)$. This shows that $\lambda u_k + (1-\lambda)u'_k \in H_{k,n}(y^k)$ and hence $H_{k,n}(y^k)$ is convex.

Furthermore, for each $k \in I$ and for all $n \in \mathbb{N}$,

$$H^{-1}_{k,n}(u_k) = A^{-1}_k(u_k) \cap \left\{ y^k \in Y^k\text{: } \min S^{W_k}(u_k, y^k) < \min S^{W_k}(\bar{A}_k(y^k), y^k) + \frac{1}{n} \right\}.$$

It is easy to see from the definition of l.s.c. that a multimap $T : M \multimap N$ is l.s.c. if and only if $T^{-1}(V)$ is open in M for any open set V of N, where M and N are topological spaces. Since for all $k \in I$, $A^{-1}_k(y_k)$ is open for all $y_k \in Y_k$, $A^{-1}_k(W) = \bigcup\{A^{-1}_k(y_k)\text{: } y_k \in W\}$ is open in Y^k for any open set W in Y_k, A_k is l.s.c. Then it is easy to see that $\bar{A}_k$ is also l.s.c. Since $\bar{A}_k$ is an u.s.c. multimap, $\bar{A}_k : Y^k \multimap Y_k$ is a continuous multimap with compact values. By Lemmas 2.3 and 2.4, we see that for each $y^k \in Y^k$, $u_k \in Y_k$, the functions

$$y^k \mapsto \min S^{W_k}(\bar{A}_k(y^k), y^k) \quad \text{and} \quad (u_k, y^k) \mapsto \min S^{W_k}(u_k, y^k)$$

are continuous. Therefore, for each $k \in I$ and for all $n \in \mathbb{N}$, $H^{-1}_{k,n}(u_k)$ is open for all $u_k \in Y_k$. Therefore,

$$\begin{aligned} Y^k &= \bigcup\{H^{-1}_{k,n}(u_k)\text{: } u_k \in Y_k\} \\ &= \bigcup\{\mathrm{int}_{Y^k} H^{-1}_{k,n}(u_k)\text{: } u_k \in Y_k\}. \end{aligned}$$

By Theorem 3.1, there exists ${}_n\bar{y} = ({}_n\bar{y}_k)_{k\in I} \in Y$, where ${}_n\bar{y}_k = ({}_n\bar{x}_{k_j})_{j\in J_k}$, with ${}_n\bar{x}_{k_j} \in \bar{A}_{k_j}({}_n\bar{y}^k)$ such that

$$\min S^{W_k}({}_n\bar{y}_k, {}_n\bar{y}^k) < \min S^{W_k}(A_k({}_n\bar{y}^k), {}_n\bar{y}^k) + \frac{1}{n},$$

where ${}_n\bar{y}^k = ({}_n\bar{y}_l)_{l\in I, l\neq k}$. Let ${}_na_k \in S^{W_k}({}_n\bar{y}_k, {}_n\bar{y}^k)$ be such that ${}_na_k = \min S^{W_k}({}_n\bar{y}_k, {}_n\bar{y}^k)$. Since for each $k \in I$ and $j \in J_k$, X_{k_j} is compact and S^{W_k} and $\bar{A}_k$ are u.s.c. multimaps with compact values, it follows from Lemma 2.2 that $\bar{A}_k(Y^k)$ and $S^{W_k}(\bar{A}_k(Y^k), Y^k)$

are compact. Since for each $n \in \mathbb{N}$, ${}_n\bar{y}_k \in \bar{A}_k(Y^k)$ and ${}_na_k \in S^{W_k}(\bar{A}_k(Y^k), Y^k)$, there exist a subnet $\{{}_{n(\alpha)}\bar{y}_k\}$ of $\{{}_n\bar{y}_k\}$, a subsequence $\{{}_{n(\alpha)}a_k\}$ of $\{{}_na_k\}$ and $\bar{y}_k \in \bar{A}_k(Y^k)$, $a_k \in S^{W_k}(\bar{A}_k(Y^k), Y^k)$ such that

$$_{n(\alpha)}(\bar{y}_k) \to \bar{y}_k, \quad _{n(\alpha)}(\bar{y}^k) \to \bar{y}^k, \quad _{n(\alpha)}a_k \to a_k.$$

As we have seen that $\bar{A}_k$ is a continuous multimap with nonempty compact values, by Lemma 2.3(c), we have the function

$$u^k \to \min S^{W_k}(\bar{A}_k(u^k), u^k)$$

is continuous. Since ${}_{n(\alpha)}a_k < \min S^{W_k}(\bar{A}_k({}_{n(\alpha)}\bar{y}^k), {}_{n(\alpha)}\bar{y}^k) + \frac{1}{n(\alpha)}$, we have

$$a_k \leqslant \min S^{W_k}(\bar{A}_k(\bar{y}^k), \bar{y}^k) \quad \text{for all } k \in I \tag{4.1}$$

by letting $n(\alpha) \to \infty$. Since for all $k \in I$, $\bar{A}^k$ and S^{W_k} are u.s.c. with compact values, it follows from Lemma 2.2(c) that $\bar{A}^k$ and S^{W_k} are closed and thus $\bar{y}_k \in \bar{A}_{k_j}(\bar{y}^k)$ and $a_k \in S^{W_k}(\bar{A}_k(\bar{y}^k), \bar{y}^k)$.

Inequality (4.1) shows that for each $k \in I$ and $j \in J_k$, there exist $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$, $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that

$$a_k = \sum_{j \in J_k} W_{k_j} \cdot \bar{z}_{k_j} \leqslant \sum_{j \in J_k} W_{k_j} \cdot z_{k_j} \quad \text{for all } z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k),\ x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k). \tag{4.2}$$

For each $k \in I$ and $j \in J_k$ and for all $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, let

$$x_{ks} = \bar{x}_{ks} \text{ and } z_{ks} = \bar{z}_{ks} \quad \text{for } s \in J_k \text{ and } s \neq j$$

in (4.2), we obtain

$$W_{k_j} \cdot \bar{z}_{k_j} \leqslant W_{k_j} \cdot z_{k_j},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$.

Since $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \setminus \{0\}$, $W_{k_j} \cdot z > 0$ for all $z > 0$. We have

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$. □

Remark 4.1. If $I = \{1, 2\}$, then Theorem 4.1 is reduced to Theorem 3.7 in [8].

Theorem 4.2. *For each $k \in I$ and $j \in J_k$, let $A_{k_j} : Y^k \multimap X_{k_j}$ be a multimap with nonempty convex values such that for each any $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ is open and $\bar{A}_{k_j}$ is an u.s.c. multimap. For each $k \in I$ and $j \in J_k$, let the cost multimap $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be continuous with nonempty compact values and for any $y^k \in Y^k$, the multimap $x_{k_j} \multimap F_{k_j}(x_{k_j}, y^k)$ is $\mathbb{R}_+^{\ell_{k_j}}$-quasiconvex. Then there exists $\bar{y} = (\bar{y}_k)_{k \in I} \in Y$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$, with $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and there exist $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}_k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$.

Proof. For each $k \in I$ and $j \in J_k$ and for all $n \in \mathbb{N}$, we let $a \in \mathbb{R}^{k_j}$ and $e_{k_j} \in \operatorname{int} \mathbb{R}_+^{\ell_{k_j}}$ be fixed, and we define $H_{k_j,n} : Y^k \multimap X_{k_j}$ by

$$H_{k_j,n}(y^k) = \left\{ x_{k_j} \in A_{k_j}(y^k):\ \min p_{k_j} F_{k_j}(x_{k_j}, y^k) < \min p_{k_j} F_{k_j}(\bar{A}_{k_j}(y^k), y^k) + \frac{1}{n} \right\},$$

where the function $p_{k_j} : \mathbb{R}^{\ell_{k_j}} \to \mathbb{R}$ is continuous and strict monotonically increasing with respect to $\mathbb{R}_+^{\ell_{k_j}}$ defined by

$$p_{k_j}(z) = \min\{t \in \mathbb{R}:\ z \in a + te_{k_j} - \mathbb{R}_+^{\ell_{k_j}}\}$$

for all $z \in \mathbb{R}^{\ell_{k_j}}$.

Following the argument as in the proof of Theorem 4.1 and using Lemma 2.6, we can show that for each $k \in I$ and $j \in J_k$ and for all $y^k \in Y^k$ and $n \in \mathbb{N}$, $H_{k_j,n}(y^k)$ is a nonempty convex set and for each $x_{k_j} \in X_{k_j}$, $H_{k_j,n}^{-1}(x_{k_j})$ is open in Y^k.

Since for fixed $k \in I$ and for all $n \in \mathbb{N}$ and $y^k \in Y^k$, $H_{k_j,n}(y^k)$ is nonempty for all $j \in J_k$, we can let $x_{k_j} \in H_{k_j,n}(y^k)$ for all $j \in J_k$, then $y^k \in H_{k_j,n}^{-1}(x_{k_j})$ for all $j \in J_k$. Therefore, $y^k \in \bigcap_{j \in J_k} H_{k_j,n}^{-1}(x_{k_j})$ and we have

$$Y^k = \bigcup \left\{ \bigcap_{j \in J_k} H_{k_j,n}^{-1}(x_{k_j}):\ x_{k_j} \in X_{k_j} \right\}$$

$$= \bigcup \left\{ \operatorname{int}_{Y^k} \bigcap_{j \in J_k} H_{k_j,n}^{-1}(x_{k_j}):\ x_{k_j} \in X_{k_j} \right\}.$$

From Theorem 3.3, there exists ${}_n\bar{y} = ({}_n\bar{y}_k)_{k \in I} \in Y$, where ${}_n\bar{y}_k = ({}_n\bar{x}_{k_j})_{j \in J_k}$, with ${}_n\bar{x}_{k_j} \in \bar{A}_{k_j}({}_n\bar{y}^k)$ such that

$$\min p_{k_j} F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k) < \min p_{k_j} F_{k_j}(\bar{A}_{k_j}({}_n\bar{y}^k), {}_n\bar{y}^k) + \frac{1}{n},$$

where ${}_n\bar{y}^k = ({}_n\bar{y}_l)_{l \in I, l \neq k}$.

Let ${}_n a_{k_j} \in F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k)$ be such that $p_{k_j}({}_n a_{k_j}) = \min p_{k_j} F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k)$. Since for each $k \in I$ and $j \in J_k$, X_{k_j} is compact and F_{k_j} and $\bar{A}_{k_j}$ are u.s.c. with compact values, it follows from Lemma 2.2 that $\bar{A}_{k_j}(Y^k)$ and $F_{k_j}(X_{k_j}, Y^k)$ are compact. Since for each $n \in \mathbb{N}$,

$$\min p_{kj} F_{kj}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k) < \min p_{k_j} F_{k_j}(\bar{A}_{k_j}({}_n\bar{y}^k), {}_n\bar{y}^k) + \frac{1}{n},$$

where ${}_n\bar{y}^k = ({}_n\bar{y}_l)_{l \in I, l \neq k}$.

Let ${}_n a_{k_j} \in F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k)$ be such that $p_{k_j}({}_n a_{k_j}) = \min p_{k_j} F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{y}^k)$. Since for each $n \in \mathbb{N}$, ${}_n\bar{x}_{k_j} \in \bar{A}_{k_j}({}_n\bar{y}^k)$ and ${}_n a_k \in F_{k_j}(X_{k_j}, Y^k)$, there exist a subsequence $\{{}_{n(\alpha)} a_{k_j}\}$ of $\{{}_n a_{k_j}\}$, a subnet $\{{}_{n(\alpha)}\bar{x}_{k_j}\}$ of $\{{}_n\bar{x}_{k_j}\}$ and $\bar{a}_{k_j} \in F_{k_j}(X_{k_j}, Y^k)$, $\bar{x}_{k_j} \in \bar{A}_{k_j}(Y^k)$ such that ${}_{n(\alpha)}\bar{x}_{k_j} \to \bar{x}_{k_j}$ and ${}_{n(\alpha)} a_{k_j} \to \bar{a}_{k_j}$.

By Lemma 2.3 and following the argument as in the proof of Theorem 4.1, we note that the function $y^k \mapsto \min p_{k_j} F_{k_j}(\bar{A}_{k_j}(\bar{y}^k), \bar{y}^k)$ is continuous. Since

$$p_{k_j}({}_{n(\alpha)} a_{k_j}) < \min p_{k_j} F_{k_j}(\bar{A}_{k_j}({}_{n(\alpha)}\bar{y}^k), {}_{n(\alpha)}\bar{y}^k) + \frac{1}{n(\alpha)},$$

we have

$$p_{k_j}(\bar{a}_{k_j}) \leqslant \min p_{k_j} F_{k_j}(\bar{A}_{k_j}(\bar{y}^{\,k}), \bar{y}^{\,k}),$$

by letting $n(\alpha) \to \infty$. As in the proof of Theorem 4.1, we can show that $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ and $a_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^{\,k})$. This implies that for each $k \in I$ and $j \in J_k$,

$$p_{k_j}(\bar{a}_{k_j}) \leqslant p_{k_j}(a_{k_j}),$$

for all $a_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^{\,k})$ and $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$.

Since p_{k_j} is strict monotonically increasing with respect to $\mathbb{R}_+^{\ell_{k_j}}$,

$$a_{k_j} - \bar{a}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}},$$

for all $a_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^{\,k})$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ and each $k \in I$ and $j \in J_k$. □

From Theorem 4.2, we can easily derive the following result.

Corollary 4.1. *For each $k \in I$ and $j \in J_k$, let $A_{k_j} : Y^k \multimap X_{k_j}$ be a multimap with nonempty convex values such that for any $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ is open in Y^k and $\bar{A}_{k_j}$ is an u.s.c. multimap. For each $k \in I$ and $j \in J_k$, let the cost function $F_{k_j} : X_{k_j} \times Y^k \to \mathbb{R}$ be continuous and for any $y^k \in Y^k$, the function $x_{k_j} \mapsto F_{k_j}(x_{k_j}, y^k)$ be $\mathbb{R}_+$-quasiconvex. Then there exists $\bar{y} = (\bar{y}_k)_{k \in I} \in Y$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$, with $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ such that*

$$F_{k_j}(x_{k_j}, \bar{y}_k) - F_{k_j}(\bar{x}_{k_j}, \bar{y}^{\,k}) \geqslant 0$$

for all $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ and for each $k \in I$ and $j \in J_k$.

Remark 4.2. For each $k \in I$, let $J_k = \{k\}$ be a singleton set. If for each $k \in I$, $A_k : X^k \multimap X_k$ is defined by $A_k(x^k) = X_k$ for all $x^k \in X^k$ and $F_k : X_k \times X^k \to \mathbb{R}$ is a single-valued function, then Corollary 4.1 is reduced to the Nash equilibrium theorem [14].

Theorem 4.3. *For each $k \in I$ and $j \in J_k$, let F_{k_j}, A_{k_j} be the same as in Theorem* 4.1 *and also let $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \setminus \{0\}$. Assume that*

(i) *for each $k \in I$ and $j \in J_k$, the multimap $W_{k_j} \cdot F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}$ is continuous with nonempty compact values,*
(ii) *for each $y^k \in Y^k$, the multimap $x_{k_j} \multimap W_{k_j} \cdot F_{k_j}(x_{k_j}, y^k)$ is $\mathbb{R}_+$-quasiconvex.*

Then there exists $\bar{y} = (\bar{y}_k)_{k \in I}$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$, with $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ and there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^{\,k})$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^{\,k})$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^{\,k})$ and for each $k \in I$ and $j \in J_k$.

Proof. From Theorem 4.2, there exists $\bar{y} = (\bar{y}_k)_{k \in I}$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$, with $\bar{x}_{k_j} \in A_{k_j}(\bar{y}^{\,k})$ and there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^{\,k})$ such that

$$W_{k_j} \cdot z_{k_j} \geqslant W_{k_j} \cdot \bar{z}_{k_j},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$. Since $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \setminus \{0\}$, $W_{k_j} \cdot z > 0$ for all $z > 0$. Therefore,

$$z_{k_j} - \bar{z}_{k_j} \notin -\mathrm{int}\,\mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$. □

Now we adopt the fixed point approach to prove the existence of a solution of equilibrium problem with finite number of families of players and finite number of families of constraints on the strategy sets.

Proposition 4.1. *Let F_{k_j}, A_{k_j}, A_k, W_{k_j} and S^{W_k} be the same as in Theorem* 4.1. *For each $k \in I$ and $j \in J_k$, assume that F_{k_j} is a continuous multimap with nonempty compact values. Then $\bar{y} = (\bar{y}_k)_{k\in I} \in \prod_{k\in I} Y_k$ is a fixed point of M if and only if $\bar{y}_k \in A_k(\bar{y}^k)$ and there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that*

$$W_{k_j} \cdot \bar{z}_{k_j} \leqslant W_{k_j} \cdot z_{k_j},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$, where M is defined in the beginning of this section.

Proof. Suppose that $\bar{y}$ is a fixed point of M. Then for each $k \in I$, $\bar{y}_k \in M^{W_k}(\bar{y}^k)$. Therefore, $\bar{y}_k \in A_k(\bar{y}^k)$ and $\min S^{W_k}(\bar{y}_k, \bar{y}^k) = \min S^{W_k}(A_k(\bar{y}^k), \bar{y}^k)$. This implies that for each $k \in I$,

$$\min \sum_{j\in J_k} W_{k_j} \cdot F_{k_j}(\bar{x}_{k_j}, \bar{y}^k) \leqslant \min \sum_{j\in J_k} W_{k_j} \cdot F_{k_j}(x_{k_j}, \bar{y}^k),$$

for all $y_k = (x_{k_j})_{j\in J_k} \in A_k(\bar{y}^k)$. Therefore, for each $k \in I$ and $j \in J_k$

$$\sum_{j\in J_k} \min W_{k_j} \cdot F_{k_j}(\bar{x}_{k_j}, \bar{y}^k) \leqslant \sum_{j\in J_k} \min W_{k_j} \cdot F_{k_j}(x_{k_j}, \bar{y}^k), \quad \text{for all } x_{k_j} \in A_{k_j}(\bar{y}^k). \tag{4.3}$$

Fix any $k \in I$ $j \in J_k$, let $y_k = (x_{k_j})_{j\in J_k} \in A_k(\bar{y}^k)$ and $x_{ks} = \bar{x}_{ks}$ for $s \in J_k$, $s \neq j$ in (4.3). Then (4.3) becomes,

$$\min W_{k_j} \cdot F_{k_j}(\bar{x}_{k_j}, \bar{y}^k) \leqslant \min W_{k_j} \cdot F_{k_j}(x_{k_j}, \bar{y}^k),$$

for all $x_{k_j} \in A_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$. This implies that there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that

$$W_{k_j} \cdot \bar{z}_{k_j} \leqslant W_{k_j} \cdot z_{k_j},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in A_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$.

Conversely, let $\bar{y} = (\bar{y}_k)_{k\in I}$, where $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in A_k(\bar{y}^k)$ and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}_k)$ be such that

$$W_{k_j} \cdot \bar{z}_{k_j} \leqslant W_{k_j} \cdot z_{k_j},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in A_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$. Then for each $k \in I$,

$$\sum_{j \in J_k} \min W_{k_j} \cdot F_{k_j}(\bar{x}_{k_j}, \bar{y}^k) \leqslant \sum_{j \in J_k} \min W_{k_j} \cdot F_{k_j}(x_{k_j}, \bar{y}^k),$$

for all $y_k = (x_{k_j})_{j \in J_k} \in A_k(\bar{y}^k)$. Therefore, $\min S^{W_k}(\bar{y}_k, \bar{y}^k) = \min S^{W_k}(A_k(\bar{y}^k), \bar{y}^k)$ and $\bar{y}_k \in M^{W_k}(\bar{y}^k)$ for each $k \in I$. Hence

$$\bar{y} = (\bar{y}_k)_{k \in I} \in \prod_{k \in I} M^{W_k}(\bar{y}^k) = M(\bar{y}).$$

This shows that $\bar{y}$ is a fixed point of M. □

Remark 4.3. If $I = \{1, 2\}$, then Proposition 4.1 is reduced to Theorem 3.1 in [8].

As a consequence of Proposition 4.1, we have the following result.

Theorem 4.4. *For each $k \in I$ and $j \in J_k$, let X_{k_j} be a nonempty compact convex subset of a locally convex topological vector space E_{k_j}, $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \backslash \{0\}$ and let $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be a multimap with nonempty values. For each $k \in I$, assume that*

(i) *the multimap S^{W_k} is continuous with compact values,*
(ii) *for each $j \in J_k$, the multimap $A_{k_j} : Y^k \multimap X_{k_j}$ is continuous with nonempty closed values,*
(iii) *for all $y^k \in Y^k$, $M^{W_k}(y^k)$ is an acyclic set.*

Then there exists $\bar{y} = (\bar{y}_k)_{k \in I} \in Y$, where $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$ with $\bar{x}_{k_j} \in A_k(\bar{y}^k)$ and there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{y}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\text{int}\, \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{y}^k)$, $x_{k_j} \in A_{k_j}(\bar{y}^k)$ and for each $k \in I$ and $j \in J_k$.

Proof. Since for each $k \in I$ and for all $y^k \in Y^k$, $A_k(y^k) = \prod_{j \in J_k} A_{k_j}(y^k)$ is a closed subset of a compact set Y_k, $A_k(y^k)$ is compact. Since $S^{W_k}(\cdot, y^k)$ is a continuous multimap with nonempty compact values, it follows from Lemma 2.2 that $S^{W_k}(A_k(y^k), y^k)$ is a compact set and there exists a $u_k \in A_k(y^k)$ such that

$$\min S^{W_k}(A_k(y^k), y^k) = \min S^{W_k}(u_k, y^k),$$

that is, $M^{W_k}(y^k)$ is nonempty for all $y^k \in Y^k$ and for each $k \in I$. By Lemmas 2.3 and 2.4, the functions $(u_k, y^k) \mapsto \min S^{W_k}(u_k, y^k)$ and $y^k \mapsto \min S^{W_k}(A_k(y^k), y^k)$ are continuous. It is easy to see that for each $k \in I$ and for all $y^k \in Y^k$, $M^{W_k}(y^k)$ is a closed subset of Y^k and $M^{W_k} : Y^k \multimap Y_k$ is closed. Since Y^k is compact, it follows from Lemma 2.2 and condition (iii) that $M^{W_k} : Y^k \multimap Y_k$ is an u.s.c. multimap with compact acyclic values. By Kunneth formula [13] and Lemma 3 of Fan [5], $M = \prod_{k \in I} M^{W_k} : Y \multimap Y$ is an u.s.c multimap with compact acyclic values, that is, $M \in V(Y, Y)$. Then it follows from Lemma 2.5 that M has a fixed point and the conclusion follows from Proposition 4.1 with $W_{k_j} \in \mathbb{R}_+^{\ell_{k_j}} \setminus \{0\}$ for each $k \in I$ and $j \in J_k$. □

Corollary 4.2. *Let I be an index set and for each $k \in I$,* let *X_k be a nonempty compact convex subset of a locally convex topological vector space E_k and let $A_k : X^k \multimap X_k$ be a continuous multimap with nonempty closed values. For each $k \in I$, assume that the single valued function $F_k : X_k \times X^k \to \mathbb{R}^{\ell_k}$ is continuous with nonempty compact values and for any $x^k \in X^k$, the function $x_k \to F_k(x_k, x^k)$ is $\mathbb{R}_+^{\ell_k}$-convex. Then there exists $\bar{x} = (\bar{x}_k)_{k\in I}$ with $\bar{x}_k \in A_k(\bar{x}^k)$ such that*

$$F_k(x_k, \bar{x}_k) - F_k(\bar{x}_k, \bar{x}^k) \notin -\operatorname{int} \mathbb{R}_+^{\ell_k},$$

for each $k \in I$.

Proof. It is easy to see that all conditions of Theorem 4.4 are satisfied and the conclusion of Corollary 4.2 follows immediately from Theorem 4.4. □

Remark 4.4. If for each $k \in I$ and $j \in J_k$, $A_{k_j} : Y^k \multimap X_{k_j}$ is a continuous multimap with nonempty closed convex values, then the condition (iii) in Theorem 4.4 is replaced by the following condition:

(iii)′ For each $k \in I$ and for each fixed $y^k \in Y^k$, the multimap $u_k \multimap S^{W_k}(u_k, y^k)$ is $\mathbb{R}_+$-quasiconvex.

Proof. It is sufficient to show that for each $k \in I$ and for all $y^k \in Y^k$, $M^{W_k}(y^k)$ is a convex subset of Y_k. Indeed, let u_k, $v_k \in M^{W_k}(y^k)$ and $\lambda \in [0, 1]$, then $u_k, v_k \in A_k(y^k)$ and there exist $a \in S^{W_k}(u_k, y^k)$, $b \in S^{W_k}(v_k, y^k)$ such that $a = b = \min S^{W_k}(A_k(y^k), y^k)$. By condition (iii)′, for each $y^k \in Y^k$ and each $\alpha \in \mathbb{R}$, the set

$$H(\alpha) = \{y_k \in Y_k: \text{ there is a } c \in S^{W_k}(y_k, y^k) \text{ such that } c \leqslant \alpha\}$$

is convex. Take $\beta = \min S^{W_k}(A_k(y^k), y^k)$, we see that $u_k, v_k \in H(\beta)$. Therefore, $\lambda u_k + (1-\lambda)v_k \in H(\beta)$. This implies that there is a $c \in S^{W_k}(\lambda u_k + (1-\lambda)v_k, y^k)$ such that $c \leqslant \beta = \min S^{W_k}(A_k(y^k), y^k)$. Since for each $k \in I$, $A_k(y^k)$ is convex, $\lambda u_k + (1-\lambda)v_k \in A_k(y^k)$ and

$$\min S^{W_k}(\lambda u_k + (1-\lambda)v_k, y^k) = \min S^{W_k}(A_k(y^k), y^k).$$

This implies that for each $k \in I$, $\lambda u_k + (1-\lambda)v_k \in M^{w_k}(y^k)$ and $M^{W_k}(y^k)$ is convex. Hence for each $k \in I$, $M^{W_k}(y^k)$ is acyclic and the conclusion follows from Theorem 4.4. □

Remark 4.5. In Theorem 4.1, if we assume that for each $k \in I$ and $j \in J_k$, X_{k_j} is a nonempty compact convex subset of a locally convex topological vector space E_{k_j}, then Theorem 4.1 is a simple consequence of Theorem 4.4 along with Remark 4.4.

Proof. Since for each $k \in I$ and $j \in J_k$, A_{k_j} is l.s.c., it is easy to see that $\bar{A}_{k_j} : Y^k \multimap X_{k_j}$ is l.s.c. and since $\bar{A}_{k_j} : Y^k \multimap X_{k_j}$ is u.s.c., we have $\bar{A}_{k_j} : Y^k \multimap X_{k_j}$ is continuous. Thus the conclusion follows from Theorem 4.4 along with Remark 4.4. □

Remark 4.6. If for each $k \in I$, $F_k : X_k \times X^k \multimap \mathbb{R}$ is a single-valued function, then Corollary 4.2 is reduced to Debreu social equilibrium theorem [5].

5. Equilibrium problems with finite families of players and two finite families of constraints on strategy sets

In this section, we establish existence results for a solution of the equilibrium problems with finite number of families of players and two finite number of families of constraints on the strategy sets by using our coincidence Theorem 3.2 for two families of multimaps.

Theorem 5.1. *For each $k \in I$ and $j \in J_k$, let X_{k_j} be a nonempty compact convex subset of a locally convex topological vector space E_{k_j} and $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{\ell_{k_j}}$ be a multimap with nonempty values. Let $A_{k_j} : Y^k \multimap X_{k_j}$ be a multimap with nonempty convex values such that for each $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ is open in Y^k and $\bar{A}_{k_j}$ is an u.s.c. multimap. For each $k \in I$, let $B_k : Y_k \multimap Y^k$ be a closed multimap with nonempty convex values. For each $k \in I$, assume that*

(i) $Y_k = \bigcup\{\mathrm{int}_{Y_k} B_k^{-1}(y^k) :\ y^k \in Y^k\}$,
(ii) $S^{W_k} : Y_k \times Y^k \multimap \mathbb{R}$ *is continuous multimap with compact values,*
(iii) *for any* $y^k \in Y^k$, $u_k \multimap S^{W_k}(u_k, y^k)$ *is* $\mathbb{R}_+$*-quasiconvex.*

Then there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$ with $\bar{y}_k \in \bar{A}_k(\bar{u}^k)$ and $\bar{u}^k \in \bar{B}_k(\bar{y}_k)$ and there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\mathrm{int}\, \mathbb{R}_+^{\ell_{k_j}},$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for each $k \in I$ and $j \in J_k$.

Proof. For each $k \in I$ and for all $n \in \mathbb{N}$, we define $H_{k,n} : Y^k \multimap Y_k$ by

$$H_{k,n}(u^k) = \left\{ y_k \in A_k(u^k) :\ \min S^{W_k}(y_k, u^k) < \min S^{W_k}(\bar{A}_k(u^k), u^k) + \frac{1}{n} \right\}.$$

As in the proof of Theorem 4.1, for all $y^k \in Y^k$, $H_{k,n}(y^k)$ is a nonempty convex set and

$$Y^k = \bigcup\{\mathrm{int}_{Y^k} H_{k,n}^{-1}(u_k) :\ u_k \in Y_k\}.$$

Since $Y_k = \bigcup\{\mathrm{int}_{Y_k} B_k^{-1}(y^k) :\ y^k \in Y^k\}$, it follows from Theorem 3.2 that there exist ${}_n\bar{y} = ({}_n\bar{y}_k)_{k\in I} \in Y = \prod_{j\in I} Y_j$ and ${}_n\bar{u} = ({}_n\bar{u}_k)_{k\in I} \in Y$ such that

$${}_n\bar{y}_k \in H_{k,n}({}_n\bar{u}^k) \quad \text{and} \quad {}_n\bar{u}^k \in B_k({}_n\bar{u}_k).$$

Therefore, ${}_n\bar{y}_k \in A_k({}_n\bar{u}^k)$ and

$$\min S^{W_k}({}_n\bar{y}_k, {}_n\bar{u}^k) < \min S^{W_k}(\bar{A}_k({}_n\bar{u}^k), {}_n\bar{u}^k) + \frac{1}{n}.$$

Let ${}_n a_k \in S^{W_k}({}_n\bar{y}_k, {}_n\bar{u}^k)$ be such that ${}_n a_k = \min S^{W_k}({}_n\bar{y}_k, {}_n\bar{u}^k)$. As in the proof of Theorem 4.1, $\bar{A}_k(Y^k)$ and $S^{W_k}(\bar{A}_k(Y^k), Y^k)$ are compact and thus there exist a subnet $\{{}_{n(\alpha)}\bar{y}_k\}$ of $\{{}_n\bar{y}_k\}$, a subsequence $\{{}_{n(\alpha)}a_k\}$ of $\{{}_n a_k\}$ and $y_k \in \bar{A}_k(Y^k)$, $a \in S^{W_k}(\bar{A}_k(Y^k), Y^k)$ such that

$$ {}_{n(\alpha)}(\bar{y}_k) \to \bar{y}_k \quad \text{and} \quad {}_{n(\alpha)}a_k \to a_k. $$

Since for each $k \in I$, $B_k : Y_k \multimap Y^k$ is closed and Y^k is compact, it follows from Lemma 2.2 that $B_k : Y_k \multimap Y^k$ is an u.s.c. multimap with compact values and $B_k(Y_k)$ is compact. Therefore, there exist a subnet $\{{}_{n(\alpha)}\bar{u}^k\}$ of $\{{}_n\bar{u}^k\}$ and $\bar{u}^k \in B_k(Y_k)$ such that

$$ {}_{n(\alpha)}\bar{u}^k \to \bar{u}^k. $$

As in the proof of Theorem 4.1, for each $k \in I$, $\bar{A}_k : Y^k \multimap Y_k$ is a continuous multimap with compact values. By Lemma 2.3, the function $u^k \mapsto \min S^{W_k}(\bar{A}_k(u^k), u^k)$ is continuous for each $k \in I$. Since ${}_{n(\alpha)}a_k < \min S^{W_k}(\bar{A}_k({}_{n(\alpha)}\bar{u}^k), {}_{n(\alpha)}\bar{u}^k) + 1/n(\alpha)$, letting $n(\alpha) \to \infty$, we have

$$ a_k \leqslant \min S^{W_k}(\bar{A}_k(u^k), u^k), \quad \text{for each } k \in I. $$

As in the proof of Theorem 4.1, we see that for each $k \in I$, $a_k \in S^{W_k}(\bar{A}_k(\bar{u}^k), \bar{u}^k)$, $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k} \in \bar{A}_k(\bar{u}^k)$ and $\bar{u}^k \in B_k(\bar{y}_k)$. Therefore, for each $k \in I$ and $j \in J_k$, there exist $\bar{x}_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$, $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$ a_k = \sum_{j \in J_k} W_{k_j} \cdot \bar{z}_{k_j} \leqslant \sum_{j \in J_k} W_{k_j} \cdot z_{k_j}, \tag{5.1} $$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$. For each fixed $k \in I$ and fixed $j \in J_k$, let $y_k = (x_{hs})_{s \in J_k} \in \bar{A}_k(\bar{u}^k)$ and $z_k = (z_{ks})_{s \in J_k} \in \prod_{s \in J_k} F_{ks}(x^k_{ks}, \bar{u}^k)$ with $x_{ks} = \bar{x}_{ks}$ and $z_{ks} = \bar{z}_{ks}$ for $s \in J_k$, $s \neq j$, in (5.1), we have

$$ W_{k_j} \cdot \bar{z}_{k_j} \leqslant W_{k_j} \cdot z_{k_j}, $$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for each $k \in I$ and $j \in J_k$. Therefore,

$$ z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}}, $$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for each $k \in I$ and $j \in J_k$. □

Theorem 5.2. *Under the assumption of Theorem* 4.2, *we further assume that for each* $k \in I$, $B_k : Y_k \multimap Y^k$ *is a closed multimap with nonempty convex values and* $Y_k = \bigcup\{\operatorname{int}_{Y_k} B_k^{-1}(y^k) \colon y^k \in Y^k\}$. *Then there exist* $\bar{y} = (\bar{y}_k)_{k \in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k \in I} \in Y$ *with* $\bar{y}_k \in \bar{A}_k(\bar{u}^k)$ *and* $\bar{u}^k \in B_k(\bar{y}_k)$ *and there exists* $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ *such that*

$$ z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int} \mathbb{R}_+^{\ell_{k_j}}, $$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ *and for each* $k \in I$ *and* $j \in J_k$, *where* $\bar{y}_k = (\bar{x}_{k_j})_{j \in J_k}$.

Proof. For each $k \in I$ and $j \in J_k$, let $H_{k_j,n} : Y^k \multimap X_{k_j}$ and $p_{k_j} : Z_{k_j} \to \mathbb{R}$ be defined as in the proof of Theorem 4.2 and let

$$ G_{k,n}(y^k) = \prod_{j \in J_k} H_{k_j,n}(y^k). $$

As in the proof of Theorem 4.2, we can show that $Y^k = \bigcup\{\operatorname{int} G_{k,n}^{-1}(y_k)\colon\ y_k \in Y_k\}$. Since $Y_k = \bigcup\{\operatorname{int} B_k^{-1}(y^k)\colon\ y^k \in Y^k\}$, it follows from Theorem 3.2 that there exist ${}_n\bar{y} = ({}_n\bar{y}_k)_{k\in I} \in Y$ and ${}_n\bar{u} = ({}_n\bar{u}_k)_{k\in I} \in Y$ such that ${}_n\bar{y}_k \in G_{k,n}({}_n\bar{u}^k)$ and ${}_n\bar{u}_k \in B_k({}_n\bar{y}^k)$. Therefore, for each $k \in I$,

$$ {}_n\bar{y}_k = ({}_n\bar{x}_{k_j})_{j\in J_k} \in A_k({}_n\bar{u}^k) $$

and

$$ \min p_{k_j} \cdot F_{k_j}({}_n\bar{x}_{k_j}, {}_n\bar{u}^k) < \min p_{k_j} \cdot F_{k_j}((\bar{A}_{k_j}({}_n\bar{u}^k), {}_n\bar{u}^k), {}_n\bar{u}^k) + \frac{1}{n}. $$

where ${}_n\bar{y}^k = ({}_n\bar{y}_j)_{j\in I, j\neq k}$. Following the arguments of Theorems 4.2 and 5.1, we get the conclusion. □

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