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On generalized vector equilibrium problems*

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1. Introduction and preliminaries

Let X and Y be two topological vector spaces. Let K be a non-empty convex subset of X and $F : K \times K \to \Pi(Y)$ be a multifunction, where $\Pi(Y)$ denotes the set of all non-empty subsets of Y. Let $C : K \to \Pi(Y)$ be a multifunction such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with int $C(x) \neq \emptyset$ and $P = \bigcap_{x\in K} C(x)$, where int $C(x)$ denotes the interior of $C(x)$. We consider the following *generalized vector equilibrium problem*:

$$\text{Find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K. \tag{GVEP}$$

For $F : K \times K \to Y$ is a single-valued map, (GVEP) is known as *vector equilibrium problem* [13], which includes vector variational inequalities [12,13], vector optimization problems, vector saddle point problems and Nash equilibrium problem for vector-valued functions as special cases; see, for example [3, 27]. Fu [11] also considered vector equilibrium problem and its simultaneous problems. For further details on vector equilibrium problems, we refere to [3, 4, 7, 11, 13, 14, 19, 23, 27] and references therein.

We denote by $L(X, Y)$ the space of all continuous linear operators from X to Y. Let $\psi : L(X, Y) \times K \times K \to Y$ be a function and $T : K \to \Pi(L(X, Y))$ be a multifunction. We define

$$F(x, y) = \psi(T(x), x, y) = \cup_{u\in T(x)}\psi(u, x, y).$$

Then (GVEP) reduces to the following *implicit vector variational inequality problem*:

$$\text{Find } \bar{x} \in K \text{ such that } \psi(T(\bar{x}), \bar{x}, y) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K, \tag{IVVIP}$$

equivalently, to find $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\psi(\bar{u}, \bar{x}, y) \notin -\text{int } C(\bar{x}).$$

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This problem was considered by Lee and Kum [20], which includes generalized vector variational and variational-like inequalities as special cases; see, for example [1, 2, 5, 9, 13, 14, 15, 17, 18, 21, 22, 26] and references therein.

The (GVEP) was first proposed by Ansari et al [4] for non-moving cone and they established some existence results on compact set K by using the partition of unity. Oettli and Schläger [24] extended and generalized results of [4] for moving cone and derive some existence results for generalized vector variational inequalities. Konnov and Yao [16] extended and modified results of [24] and [11] for C_x-pseudomonotone multifunctions by using Fan-KKM Theorem. Recently in [6], (GVEP) has been studied by using a fixed point theorem due to Tarafdar [28]. The main purpose of this paper is to establish some existence results for solutions to (GVEP) for C_x-pseudomonotone as well as C_x-quasimonotone multifunctions by using Fan-Browder type fixed point theorem of Park [25]. We also prove an existence result without any kind of monotonicity assumption. As special cases, we derive some existence results for solutions to the implicit vector variational inequalities. Our results generalizes known results in the literature.

In the rest of this section, we present some notation, definitions and results which will be used in the sequel.

Let $A : X \to \Pi(Y)$ be a multifunction. The *graph* of A, denoted by $\mathcal{G}(A)$, is

$$\mathcal{G}(A) = \{(x, y) \in X \times Y : x \in X, y \in A(x)\}.$$

The *inverse* A^{-1} of A is a multifunction from the range of A to X defined by

$$x \in A^{-1}(y) \quad \text{if and only if} \quad y \in A(x).$$

We first mention a particular form of Theorem 1 in [25] which is modified by Lee and Kum [20] into convenient shape. This theorem is a generalization of well-known fixed point theorem of Fan-Browder [8, 10].

Theorem A. *Let K be a convex subset of a Hausdorff topological vector space X and D be a non-empty compact subset of K. Let $A, B : K \to \Pi(K) \cup \{\emptyset\}$ be two multifunctions such that*

(a) *for each $x \in K$, $A(x) \subseteq B(x)$;*
(b) *for each $x \in K$, $B(x)$ is convex;*
(c) *for each $x \in D$, $A(x)$ is non-empty;*
(d) *for each $y \in K$, $A^{-1}(y)$ is open in K;*
(e) *for each finite subset N of K, there exists a non-empty compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, $A(x) \cap L_N \neq \emptyset$.*

Then B has a fixed point x_0, that is, $x_0 \in B(x_0)$.

Now we mention some definition and a result which will be used in the sequel.

Definition 1. A multifunction $F : K \times K \to \Pi(Y)$ is called

(i) *C_x-quasiconvex-like* [6] if, for all x, y', $y'' \in K$ and $\alpha \in (0, 1)$, we have either

$$F(x, \alpha y' + (1 - \alpha)y'') \subseteq F(x, y') - C(x)$$

or

$$F(x, \alpha y' + (1 - \alpha)y'') \subseteq F(x, y'') - C(x);$$

(ii) *explicitly C_x-quasiconvex − like* if, it is C_x-quasiconvex-like and, in case $F(x,y'') - F(x,y') \not\subseteq -\text{int } C(x)$, for all $x,\ y',\ y'' \in K$ and $\alpha \in (0,1)$, we have

$$F(x, \alpha y' + (1-\alpha)y'') \subseteq F(x,y'') - \text{int } C(x);$$

(iii) *explicitly $\delta(C_x)$-quasiconvex* [16] if, for all $y',\ y'' \in K$ and $\alpha \in (0,1)$, we have either

$$F(y_\alpha, y') \subseteq F(y_\alpha, y_\alpha) + C(y')$$

or

$$F(y_\alpha, y'') \subseteq F(y_\alpha, y_\alpha) + C(y'),$$

and, in case $F(y_\alpha, y') - F(y_\alpha, y'') \subseteq \text{int } C(y')$, for all $\alpha \in (0,1)$, we have

$$F(y_\alpha, y') \subseteq F(y_\alpha, y_\alpha) + \text{int } C(y'),$$

where $y_\alpha = \alpha y' + (1-\alpha)y''$;

(iv) *C_x-pseudomonotone* [16] if, for all $x, y \in K$, we have

$$F(x,y) \not\subseteq -\text{int } C(x) \quad \text{implies} \quad -F(y,x) \not\subseteq -\text{int } C(x).$$

(v) *C_x-quasimonotone* if, for all $x, y \in K$, we have

$$F(x,y) \not\subseteq -C(x) \quad \text{implies} \quad -F(y,x) \not\subseteq -\text{int } C(x).$$

Remark 1. The concept of C_x-quasimonotonicity can be viewed as an extension of that of quasimonotonicity in [7] on the multivalued and moving case.

Definition 2. A multifunction $T : K \to \Pi(Y)$ is called

(i) *upper semicontinuous on K* if, for each $x_0 \in K$ and any open set V in Y containing $T(x_0)$, there exists an open neighbourhood U of x_0 in K such that $T(x) \subseteq V$ for all $x \in U$;

(ii) *u-hemicontinuous* [15] if, for any $x, y \in K$ and $\alpha \in [0,1]$, the multifunction $\alpha \mapsto T(\alpha x + (1-\alpha)y)$ is upper semicontinuous at 0^+;

(iii) *P-convex* if, for all $x, y \in K$ and $\alpha \in (0,1)$,

$$T(\alpha x + (1-\alpha)y) \subseteq \alpha T(x) + (1-\alpha)T(y) - P.$$

Definition 3. Let $\psi : L(X,Y) \times K \times K \to Y$ be a given function. A multifunction $T : K \to \Pi(L(X,Y))$ is called

(i) *generalized weakly C_x-pseudomonotone w.r.t. ψ* [20] if, for any $x, y \in K$ and for any $u \in T(x)$, we have

$$\psi(u,x,y) \notin -\text{int } C(x) \quad \text{implies} \quad -\psi(v,y,x) \notin -\text{int } C(x) \quad \text{for some } v \in T(y);$$

(ii) *generalized weakly C_x-quasimonotone w.r.t. ψ* if, for any $x, y \in K$ and for any $u \in T(x)$, we have

$$\psi(u,x,y) \notin -C(x) \quad \text{implies} \quad -\psi(v,y,x) \notin -\text{int } C(x) \quad \text{for some } v \in T(y);$$

(iii) *generalized weakly C_x-pseudomonotone$^+$ w.r.t.* ψ if, for any $x, y \in K$ and for some $u \in T(x)$, we have

$$\psi(u,x,y) \notin -\text{int } C(x) \quad \text{implies} \quad -\psi(v,y,x) \notin -\text{int } C(x) \quad \text{for some } v \in T(y);$$

(iv) *generalized weakly C_x-quasimonotone$^+$ w.r.t.* ψ if, for any $x, y \in K$ and for some $u \in T(x)$, we have

$$\psi(u,x,y) \notin -C(x) \quad \text{implies} \quad -\psi(v,y,x) \notin -\text{int } C(x) \quad \text{for some } v \in T(y);$$

(v) *generalized hemicontinuous w.r.t.* ψ [20] if, for any $x, y \in K$ and $\alpha \in [0,1]$, the multifunction

$$\alpha \mapsto \psi(T(x+\alpha(y-x)), x+\alpha(y-x), y)$$

is upper semicontinuous at 0^+, where

$$\psi(T(x+\alpha(y-x)), x+\alpha(y-x), y) = \{\psi(t, x+\alpha(y-x), y) \ : \ t \in T(x+\alpha(y-x))\}.$$

Remark 2. It is clear from the definitions that every generalized weakly C_x- pseudomonotone$^+$ (C_x - quasimonotone$^+$) is generalized weakly C_x- pseudomonotone (C_x- quasimonotone) but the reverse assertion is not true in general.

Definition 4. A function $\psi : L(X,Y) \times K \times K \to Y$ is called

(i) *C_x-quasiconvex-like* if, for all $x,\ y',\ y'' \in K$ and $\alpha \in (0,1)$, we have either for all $u \in T(x)$,

$$\psi(u,x,\alpha y' + (1-\alpha)y'') \in \psi(u,x,y') - C(x)$$

or

$$\psi(x,\alpha y' + (1-\alpha)y'') \in \psi(u,x,y'') - C(x);$$

(ii) *explicitly C_x-quasiconvex$-$like* if, it is C_x-quasiconvex-like and, in case $\psi(u,x,y'') - \psi(u,x,y') \notin -\text{int } C(x)$, for all $x,\ y',\ y'' \in K$, $u \in T(x)$ and $\alpha \in (0,1)$, we have

$$\psi(u,x,\alpha y' + (1-\alpha)y'') \in \psi(u,x,y'') - \text{int } C(x);$$

(iii) *explicitly $\delta(C_x)$-quasiconvex* if, for all $y',\ y'' \in K$, $y_\alpha = \alpha y' + (1-\alpha)y''$ and $\alpha \in (0,1)$, we have either for all $v_\alpha \in T(y_\alpha)$

$$\psi(v_\alpha, y_\alpha, y') \in \psi(v_\alpha, y_\alpha, y_\alpha) + C(y')$$

or

$$\psi(v_\alpha, y_\alpha, y'') \in \psi(v_\alpha, y_\alpha, y_\alpha) + C(y'),$$

and, in case $\psi(v_\alpha, y_\alpha, y') - \psi(v_\alpha, y_\alpha, y'') \in \text{int } C(y')$ for all $\alpha \in (0,1)$, we have

$$\psi(v_\alpha, y_\alpha, y') \in \psi(v_\alpha, y_\alpha, y_\alpha) + \text{int } C(y').$$

Lemma 1. *Let K be a non-empty convex subset of X. Let $C : K \to \Pi(Y)$ be a multifunction such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with* int $C(x) \neq \emptyset$ *and $P = \bigcap_{x \in K} C(x)$. Let $F : K \times K \to \Pi(Y)$ be a multifunction. Consider the following problems:*

(I) *Find* $x \in K : F(x,y) \not\subseteq -\text{int } C(x)$, *for all* $y \in K$.
(II) *Find* $x \in K : -F(y,x) \not\subseteq -\text{int } C(x)$, *for all* $y \in K$.

Then,

(i) *Problem* (I) *implies Problem* (II) *if* F *is* C_x*-pseudomonotone;*
(ii) *Problem* (II) *implies Problem* (I) *if* F *is explicitly* $\delta(C_x)$*-quasiconvex,* $F(y,y) \subseteq P$ *and* $F(\cdot,y)$ *is u-hemicontinuous for any* $y \in K$.

Proof. See Corollary 2.1 in [16]. □

Remark 3. The assumption "F is explicitly $\delta(C_x)$-quasiconvex" in Lemma 1 (ii) can be replaced by the following condition.

(ii)′ For each $x \in K$, $F(x,\cdot)$ is P-convex.

2. Existence results

We first establish an existence result for solutions to (GVEP) under C_x- pseudomonotonicity assumption.

Theorem 1. *Let* Y *be a topological vector space. Let* K *be a convex subset of a Hausdorff topological vector space* X *and* D *be a non-empty weakly compact subset of* K. *Assume that:*

(i) $C : K \to \Pi(Y)$ *is a multifunction such that for each* $x \in K$, $C(x)$ *is a proper, closed and convex cone in* Y *with* $\text{int } C(x) \neq \emptyset$ *and* $P = \bigcap_{x \in K} C(x)$;
(ii) $F : K \times K \to \Pi(Y)$ *is a multifunction such that*

  (a) *for each* $x \in K$, $F(x,x) \subseteq P$,
  (b) F *is* C_x*-pseudomonotone,*
  (c) F *is* C_x*-quasiconvex-like and explicitly* $\delta(C_x)$*-quasiconvex,*
  (d) *for each* $y \in K$, $Q(y) = \{x \in K : -F(y,x) \not\subseteq -\text{int } C(x)\}$ *is weakly closed in* K,
  (e) *for each* $y \in K$, $F(\cdot,y)$ *is u-hemicontinuous;*

(iii) *for each finite subset* N, *there exists a non-empty weakly compact convex subset* L_N *of* K *containing* N *such that for each* $x \in L_N \backslash D$, *there is a* $y \in L_N$ *satisfying* $-F(y,x) \subseteq -\text{int } C(x)$.

Then there exists a solution $\bar{x} \in D$ *to* (GVEP).

Proof. Let K be equipped with the weak topology from X. We define two multifunctions $A, B : K \to \Pi(K) \cup \{\emptyset\}$ by

$$A(x) = \{y \in K : -F(y,x) \subseteq -\text{int } C(x)\}$$

and

$$B(x) = \{y \in K : F(x,y) \subseteq -\text{int } C(x)\},$$

for all $x \in K$.
1^0. *For each* $x \in K$, $A(x) \subseteq B(x)$: Let $z \in A(x)$, then we have

$$-F(z,x) \subseteq -\text{int } C(x). \tag{1}$$

Assume that $z \notin B(x)$, then we have

$$F(x,z) \not\subseteq -\text{int } C(x).$$

By C_x-pseudomonotonicity of F, we get

$$-F(z,x) \not\subseteq -\text{int } C(x),$$

which contradicts to (1). Hence $A(x) \subseteq B(x)$, for all $x \in K$.
2^0. *For each* $x \in K$, $B(x)$ *is convex*: Let $y', y'' \in B(x)$, then for all $x \in K$,

$$F(x,y') \subseteq -\text{int } C(x) \tag{2}$$

and

$$F(x,y'') \subseteq -\text{int } C(x). \tag{3}$$

Since F is C_x-quasiconvex-like, for all $\alpha \in [0,1]$, we have either

$$\begin{aligned} F(x,\alpha y' + (1-\alpha)y'') &\subseteq F(x,y') - C(x) \\ &\subseteq -\text{int } C(x) - C(x) \text{ by (2)} \\ &\subseteq -\text{int } C(x), \end{aligned}$$

or

$$\begin{aligned} F(x,\alpha y' + (1-\alpha)y'') &\subseteq F(x,y'') - C(x) \\ &\subseteq -\text{int } C(x) - C(x) \text{ by (3)} \\ &\subseteq -\text{int } C(x). \end{aligned}$$

In both the cases, we get

$$F(x,\alpha y' + (1-\alpha)y'') \subseteq -\text{int } C(x).$$

Hence $\alpha y' + (1-\alpha)y'' \in B(x)$ and so $B(x)$ is convex.
3^0: By (ii)(d), for each $y \in K$, $A^{-1}(y)$ is weakly open in K.
4^0: By hypothesis (iii), for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N$ satisfying $-F(y,x) \subseteq -\text{int } C(x)$. Thus $A(x) \cap L_N \neq \emptyset$.
5^0: *B has no fixed point*: Assume that B has a fixed point $x \in K$. Then $F(x,x) \subseteq -\text{int } C(x)$. By assumption (ii)(a), we have $F(x,x) \subseteq -\text{int } C(x) \cap P \subseteq -\text{int } C(x) \cap C(x) = \emptyset$, a contradiction. Indeed, if there were a $v \in -\text{int } C(x) \cap C(x)$, then $0 = -v + v \in \text{int } C(x) + C(x) \subset \text{int } C(x)$. This implies that $C(x) = Y$ because $\text{int } C(x) \ni 0$ is an absorbing set in Y, which contradicts the assumption that $C(x)$ is proper. Therefore, B has no fixed point.

Since B has no fixed point, we reach to a conclusion that either A or B would not satisfy at least one of the hypotheses of Theorem A. But, as we have seen above that A and B satisfy all the hypotheses of Theorem A except (c), that is, for each $x \in D$, $A(x)$ is non-empty. Hence, there must be an $\bar{x} \in D$ such that $A(\bar{x}) = \emptyset$, i.e.

$$-F(y,\bar{x}) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K.$$

By Lemma 1 (b), we have

$$F(\bar{x}, y) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K,$$

as desired. □

Remark 4. The assumption (ii)(c) and (ii)(d), respectively, in Theorem 1 can be replaced by the following conditions:

(c)′ For each $x \in K$, $F(x, \cdot)$ is P-convex;

(d)′ The multifunction $W : K \to \Pi(Y)$ is defined as $W(x) = Y \backslash \{-\text{int } C(x)\}$, for each $x \in K$ such that its graph is weakly closed in $K \times Y$, and

(d)″ for each $x \in K$, $F(x, \cdot)$ is upper semicontinuous with non-empty compact values on K.

Proof. It is sufficient to show that for each $y \in K$, $A^{-1}(y)$ is weakly open in K, where A is defined as in the proof of Theorem 1.

Indeed, if $\{A^{-1}(y)\}^c = \emptyset$ then $A^{-1}(y) = K$ which is weakly open in K. So, we assume that $A^{-1}(y) \neq K$. Let $\{x_\lambda\}_{\lambda \in \Gamma}$ be a net in $\{A^{-1}(y)\}^c$ weakly convergent to $x \in K$. Then $-F(y, x_\lambda) \not\subseteq -\text{int } C(x_\lambda)$, that is, there exists $z_\lambda \in -F(y, x_\lambda)$ such that $z_\lambda \notin -\text{int } C(x_\lambda)$, or $z_n \in W(x_\lambda)$, for all λ. Let $M = \{x_\lambda\} \bigcup \{x\}$. Then M is compact and $z_\lambda \in -F(y, M)$ which is compact. Therefore $\{z_\lambda\}$ has a convergent subnet with limit z. Without loss of generality we may assume that $\{z_\lambda\}$ converges to z. Then by upper semicontinuity of $F(y, \cdot)$, we have $z \in -F(y, x)$. Also since W has a weak closed graph in $K \times Y$, we have $z \in W(x)$. Consequently, $z \in -F(y, x)$ and $z \notin -\text{int } C(x)$, i.e., $-F(y, x) \not\subseteq -\text{int } C(x)$. Thus $x \in \{A^{-1}(y)\}^c$. Therefore, $\{A^{-1}(y)\}^c = \{x : x \notin A^{-1}(y)\} \subseteq K$ is weakly closed hence $A^{-1}(y)$ is weakly open in K with respect to the relative topology. □

Corollary 1. *Let Y be a topological vector space. Let K be a convex subset of a Hausdorff topological vector space X and D be a non-empty weakly compact subset of K. Assume that:*

(i) *$C : K \to \Pi(Y)$ is a multifunction such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone in Y with* $\text{int } C(x) \neq \emptyset$ *and $P = \bigcap_{x \in K} C(x)$;*

(ii) *$W : K \to \Pi(Y)$ is a multifunction defined as $W(x) = Y \backslash \{-\text{int } C(x)\}$, for each $x \in K$ such that its graph is weakly closed in $K \times Y$;*

(iii) *$\psi : L(X, Y) \times K \times K \to Y$ is a function and $T : K \to \Pi(L(X, Y))$ is a multifunction with compact values such that*

  (a) *for each $x \in K$ and for all $u \in T(x)$, $\psi(u, x, x) \subseteq P$,*
  (b) *T is generalized weakly C_x-pseudomonotone$^+$ w.r.t. ψ,*
  (c) *ψ is C_x-quasiconvex-like and explicitly $\delta(C_x)$-quasiconvex,*
  (d) *for each $x \in K$, $\psi(\cdot, x, \cdot)$ is continuous,*
  (e) *T is generalized hemicontinuous w.r.t. ψ;*

(iv) *for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N$ such that for all $v \in T(y)$,*

$$-\psi(v, y, x) \in -\text{int } C(x).$$

Then there exists a solution $\bar{x} \in D$ to (IVVIP).

Proof. We set

$$F(x,y) = \psi(T(x),x,y) = \cup_{u\in T(x)}\psi(u,x,y), \quad \text{for all } x,y \in K.$$

Then it is easy to verify all the conditions of Theorem 1. Hence by Theorem 1, there exists $\bar{x} \in D$ such that

$$F(\bar{x},y) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K,$$

which is equivalent to say that there exists $\bar{x} \in D$ such that for each $y \in K$, there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\psi(\bar{u},\bar{x},y) \notin -\text{int } C(\bar{x}). \quad \square$$

Remark 5. The assumption (iii)(d) in Corollary 1 can be replaced by the following condition:

(d)′ For each $x \in K$ and for all $u \in T(x)$, $\psi(u,x,\cdot)$ is P-convex.

In this case, Corollary 1 improves Theorem 3.1 in [20].

We now provide existence results for C_x-quasimonotone multifunctions, which strictly contain the class of C_x-pseudomonotone multifunctions.

Let us give some notation and recall some definitions. Given x and y in a vector space E, we denote by $[x,y]$ and $]x,y[$ the closed and open line segments joining x and y, respectively. If K is a nonempty convex subset of E, then $x \in K$ is a *relative algebraic interior point of* K if for any $u \in E$ such that $x+u \in K$, there exists $\epsilon > 0$ such that $]x-\epsilon u, x+\epsilon u[\subseteq K$. The set of relative algebraic interior points of K is denoted by ri$K$. We note that if $0 < \alpha < \epsilon$, then $]x-\alpha u, x+\alpha u[\subseteq]x-\epsilon u, x+\epsilon u[$.

Lemma 2. *Let K be a non-empty convex subset of X. Let $C : K \to \Pi(Y)$ be a multifunction such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with* int $C(x) \neq \emptyset$. *Let $F : K \times K \to \Pi(Y)$ be a C_x-quasimonotone, explicitly C_x-quasiconvex-like multifunction and $F(\cdot,z)$ be u-hemicontinuous for any $z \in K$. Then, for each pair of points $x \in K$, $y \in$* ri$K$, *at least one of the following must hold:*

(i) $F(y,x) \subseteq$ int $C(x)$ *implies* $F(x,y) \subseteq -$int $C(x)$,

or

(ii) $F(x,z) \not\subseteq -$int $C(x)$, *for all* $z \in K$.

Proof. Let $F(y,x) \subseteq$ int $C(x)$ for some $x \in K$, $y \in$ ri$K$, and let there exist $z \in K$ such that $F(x,z) \subseteq -$int $C(x)$. Since $y+(z-y) = z \in K$, it follows from the definition of relative algebraic interior point that there is $\epsilon > 0$ such that $]y-\epsilon(z-y), y+\epsilon(z-y)[\subseteq K$. Then for any β with $0 < \beta < 1$, we have

$$\gamma_\beta = \beta(y-\epsilon(z-y)) + (1-\beta)(y+\epsilon(z-y)) = y + (2\beta-1)\epsilon(y-z) \in K.$$

Therefore by choosing β with $\frac{1}{2} < \beta < 1$ and letting ϵ to be sufficient small if necessary, we conclude that there exists $\gamma_0 \in (0,1)$ such that $y+\gamma(y-z) \in K$ for all $\gamma \in (0,\gamma_0)$.

By u-hemicontinuity of $F(\cdot,x)$, it follows that, for some $\alpha \in (0,1)$,

$$F(z_\alpha,x) \subseteq \text{int } C(x),$$

where $z_\alpha = y + \alpha(y - z) \in K$. Since F is C_x-quasimonotone, we have

$$F(x, z_\alpha) \subseteq -C(x).$$

Since $z_\alpha = (1+\alpha)y - \alpha z$, we have $y = \beta z + (1-\beta)z_\alpha \in \ ]z_\alpha, z[$ with $\beta = \frac{\alpha}{1+\alpha}$. Besides, $y \in \ ]z_\alpha, z[$ and F is explicitly C_x-quasiconvex-like. In case $F(x, z_\alpha) \subseteq -\text{int } C(x)$ it follows that either

$$F(x, y) \subseteq F(x, z) - C(x) \subseteq -\text{int } C(x),$$

or

$$F(x, y) \subseteq F(x, z_\alpha) - C(x) \subseteq -\text{int } C(x).$$

Otherwise, if $F(x, z_\alpha) \not\subseteq -\text{int } C(x)$, then $F(x, z_\alpha) - F(x, z) \not\subseteq -\text{int } C(x)$, and we must have

$$F(x, y) \subseteq F(x, z_\alpha) - \text{int } C(x) \subseteq -\text{int } C(x).$$

In all the cases, we obtain the desired result. □

Remark 6. Lemma 2 can be viewed as some extension of the negative formulation of Lemma 2.1 in [14].

Theorem 2. *Let X, Y, K, D, C, P and F be the same as in Theorem 1 except the assumptions* (ii)(b), (ii)(c), (ii)(d) *and* (iii) *which are replaced with the following conditions:*

(ii)(b)′ *F is C_x-quasimonotone,*

(ii)(c)′ *F is explicitly C_x-quasiconvex-like and explicitly $\delta(C_x)$-quasiconvex,*

(ii)(d)′ *the multifunction $W : K \to \Pi(Y)$ is defined as $W(x) = Y \setminus \{-\text{int } C(x)\}$, for each $x \in K$ such that its graph is weakly closed in $K \times Y$,*

(ii)(d)″ *for each $x \in K$, $F(x, \cdot)$ is upper semicontinuous with non-empty compact values on K,*

and

(iii)′ *for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \setminus D$, there is a $y \in L_N \cap \text{ri}K$ satisfying $-F(y, x) \subseteq -\text{int } C(x)$.*

In addition, suppose that $\text{ri}K \neq \emptyset$.

Then there exists a solution $\bar{x} \in D$ to (GVEP).

Proof. We define two multifunctions $A, B : K \to 2^K \cup \{\emptyset\}$ by

$$A(x) = \{y \in \text{ri}K : -F(y, x) \subseteq -\text{int } C(x)\}$$

and

$$B(x) = \{y \in \text{ri}K : F(x, y) \subseteq -\text{int } C(x)\},$$

for all $x \in K$.

First we note that from Lemma 2 it follows that, for each $x \in K$, either $A(x) \subseteq B(x)$, or

$$F(x, y) \not\subseteq -\text{int } C(x), \quad \text{for all } \ y \in K.$$

Next, following steps 2^0 – 5^0 in Theorem 1 we see that the corresponding conclusions are true for these multifunctions A and B as well. It follows that there exists an $\bar{x} \in D$ such that $A(\bar{x}) = \emptyset$, i.e.

$$-F(y, \bar{x}) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } \ y \in \text{ri}K.$$

Take any $z \in K$ and $y' \in \text{ri}K$ and suppose that

$$-F(z, \bar{x}) \subseteq -\text{int } C(\bar{x}).$$

Then, by u-hemicontinuity of $F(\cdot, \bar{x})$, there is an $y \in \ ]z, y'[\ \subseteq \text{ri}K$ such that

$$-F(y, \bar{x}) \subseteq -\text{int } C(\bar{x}),$$

a contradiction. Hence,

$$-F(y, \bar{x}) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } \ y \in K.$$

By Lemma 1 (b), we now have

$$F(\bar{x}, y) \not\subseteq -\text{int } C(\bar{x}), \quad \text{for all } \ y \in K,$$

as desired. □

Corollary 2. *Let X, Y, K, D, C, W, P, ψ and T be the same as in Corollary 1 except the assumptions* (iii) (b), (iii) (c) *and* (iv) *which are replaced with the following conditions:*

(iii)(b)′ *T is generalized weakly C_x-quasimonotone$^+$ w.r.t. ψ,*
(iii)(c)′ *ψ is explicitly C_x-quasiconvex-like and explicitly $\delta(C_x)$-quasiconvex,*

and

(iv)′ *for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N \cap \text{ri}K$ $y \in L_N$ such that for all $v \in T(y)$,*

$$-\psi(v, y, x) \in -\text{int } C(x).$$

In addition, suppose that $riK \neq \emptyset$.
Then there exists a solution $\bar{x} \in D$ to (IVVIP).

Remark 7. When $\psi(u, x, y) = \langle u, y - x \rangle$, for all $x, y \in K$ and $u \in T(x)$, where $\langle l, x \rangle$ denotes the evaluation of $l \in L(X, Y)$ at $x \in X$, Corollary 2 generalizes Theorems 1 and 3 in [9] and Theorem 4.1 in [14].

Now we provide an existence result of solution to (GVEP) without any kind of monotonicity assumption.

Theorem 3. *Let X, Y, K, D, C and P be the same as in Theorem 1. Let $F : K \times K \to \Pi(Y)$ be a multifunction satisfying the following properties:*

(i) *for each $x \in K$, F is C_x-quasiconvex-like;*
(ii) *for each $y \in K$, $Q(y) = \{x \in K : F(x, y) \not\subseteq -\text{int } C(x)\}$ is weakly closed in K;*
(iii) *for each $x \in K$, $F(x, x) \subseteq P$;*

(iv) *for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N$ satisfying $-F(y,x) \subseteq -\text{int } C(x)$.*

Then there exists a solution $\bar{x} \in D$ to (GVEP).

Proof. We define a multifunction $B : K \to 2^K$ by

$$B(x) = \{y \in K : F(x,y) \subseteq -\text{int } C(x)\}, \quad \text{for all } \ x \in K.$$

1^0. For each $x \in K$, $B(x)$ is convex and B has no fixed point as we have seen in the proof of Theorem 1.
2^0. By (ii), for each $y \in K$, $B^{-1}(y) = \{x \in K : y \in B(x)\} = \{x \in K : F(x,y) \subseteq -\text{int } C(x)\}$ is weakly open in K.
3^0. By the hypothesis (iv), for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N$ satisfying $-F(y,x) \subseteq -\text{int } C(x)$. Thus $B(x) \cap L_N \neq \emptyset$.

By the same argument as in Theorem 1, there must be an $\bar{x} \in D$ such that $B(\bar{x}) = \emptyset$, i.e. $\bar{x}$ is a solution to (GVEP). □

Remark 8. The assumption (ii) in Theorem 3 can be replaced by the following conditions:

(d)′ The multifunction $W : K \to \Pi(Y)$ is defined as $W(x) = Y \setminus \{-\text{int } C(x)\}$, for each $x \in K$ such that its graph is weakly closed in $K \times Y$; and

(d)″ For each $y \in K$, $x \mapsto F(x,y)$ is upper semicontinuous with non-empty compact values on K.

Corollary 3. *Let X, Y, K, D, C, W and P be the same as in Corollary 1. Let $\psi : L(X,Y) \times K \times K \to Y$ be a function and $T : K \to \Pi(Y)$ be a multifunction with compact values such that*

(i) *for each $x \in K$, ψ is C_x-quasiconvex-like;*
(ii) *for each $y \in K$, $\psi(\cdot,\cdot,y)$ is continuous;*
(iii) *for each $x \in K$ and for all $u \in T(x)$, $\psi(u,x,x) \subseteq P$;*
(iv) *for each finite subset N, there exists a non-empty weakly compact convex subset L_N of K containing N such that for each $x \in L_N \backslash D$, there is a $y \in L_N$ $y \in L_N$ such that for all $v \in T(y)$,*

$$-\psi(v,y,x) \in -\text{int } C(x).$$

Then there exists a solution $\bar{x} \in D$ to (IVVIP).

Remark 9. Corollary 3 generalizes Theorem 3.2 in [20].

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