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System of vector quasi-variational inclusions with some applications[☆]

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Abstract

In this paper, we consider systems of vector quasi-variational inclusions which include systems of vector quasi-equilibrium problems for multivalued maps, systems of vector optimization problems and several other systems as special cases. We establish existence results for solutions of these systems. As applications of our results, we derive the existence results for solutions of system vector optimization problems, mathematical programs with systems of vector variational inclusion constraints and bilevel problems. Another application of our results provides the common fixed point theorem for a family of lower semicontinuous multivalued maps. Further applications of our results for existence of solutions of systems of vector quasi-variational inclusions are given to prove the existence of solutions of systems of Minty type and Stampacchia type generalized implicit quasi-variational inequalities. The results of this paper can be seen as extensions and generalizations of several known results in the literature.

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1. Introduction and formulations

The (vector) equilibrium problem is a unified model of several problems, namely, (vector) variational inequality problem, (vector) optimization problem, (vector) complementarity problem, fixed point problem, etc. In the last decade, a large number of papers appeared in the literature on several aspects of (vector) equilibrium problems and their generalizations; see for example [2,5,6,8,9,11] and references therein. The (vector) variational inclusion problem and (vector) quasi-variational inequality inclusion problem are the generalizations of (vector) equilibrium problem to multivalued maps. They were proposed and studied in [10,14,18,19,21] and references therein.

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In this paper, we consider the systems of these problems, namely, systems of vector quasi-variational inclusions for multivalued maps which include systems of vector quasi-equilibrium problems for multivalued maps, system of vector optimization problems and several other systems as special cases. We establish existence results for solutions of these systems. Our results and our approach are different from those given by Luc and Tan [14] and Tan [21]. As applications of our results, we derive the existence results for solutions of system vector optimization problems, mathematical programs with systems of vector variational inclusion constraints and bilevel problems. Another application of our results provides the common fixed point theorem for a family of lower semicontinuous multivalued maps. Further applications of our results for existence of solutions of systems of vector quasi-variational inclusions are given to prove the existence of solutions of systems of Minty type and Stampacchia type generalized implicit quasi-variational inequalities. The results of this paper can be seen as extensions and generalizations of several known results in the literature.

Throughout the paper, we adopt the following notations and assumptions, unless otherwise specified. Let I be any index set. For each $i \in I$, let U_i, V_i and Z_i be real topological vector spaces and $X_i \subseteq U_i$ and $Y_i \subseteq V_i$ be nonempty sets. We set $X = \prod_{i\in I} X_i$. For each $i \in I$, let $C_i : X \multimap Z_i$ be a multivalued map such that for each $x \in X$, $C_i(x)$ is a closed convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ and $S_i, Q_i : X \multimap X_i$, $F_i : Y_i \times X \times X_i \multimap Z_i$, $T_i : X_i \times X \multimap Y_i$ be multivalued maps with nonempty values, where $\operatorname{int} C_i(x)$ denotes the interior of $C_i(x)$.

We consider the following *systems of vector quasi-variational inclusions* (in short, SVQVI):

$$\text{(SVQVI)(I)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ F_i(t_i, \bar{x}, y_i) \subseteq F_i(t_i, \bar{x}, \bar{x}_i) + C_i(\bar{x}) \quad \text{for all } y_i \in Q_i(\bar{x}) \text{ and all } t_i \in T_i(y_i, \bar{x}) \end{cases}$$

and

$$\text{(SVQVI)(II)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ F_i(t_i, \bar{x}, \bar{x}_i) \subseteq F_i(t_i, \bar{x}, y_i) - C_i(\bar{x}) \quad \text{for all } y_i \in Q_i(\bar{x}) \text{ and all } t_i \in T_i(y_i, \bar{x}). \end{cases}$$

If for each $i \in I$, $F_i(t_i, x, x_i) \subseteq C_i(x)$ for all $x = (x_i)_{i\in I} \in X$ and $t_i \in Y_i$, then (SVQVI)(I) reduces to the following *system of vector quasi-equilibrium problems*:

$$\text{(SVQEP)(I)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ F_i(t_i, \bar{x}, y_i) \subseteq C_i(\bar{x}) \quad \text{for all } y_i \in Q_i(\bar{x}) \text{ and all } t_i \in T_i(y_i, \bar{x}). \end{cases}$$

If for each $i \in I$, $F_i(t_i, x, x_i) \cap C_i(x) \neq \emptyset$ for all $x = (x_i)_{i\in I} \in X$ and $t_i \in Y_i$, then (SVQVI)(II) reduces to the following *system of vector quasi-equilibrium problems*:

$$\text{(SVQEP)(II)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \neq \emptyset \quad \text{for all } y_i \in Q_i(\bar{x}) \text{ and all } t_i \in T_i(y_i, \bar{x}). \end{cases}$$

When I is a singleton set and $C_i(x)$ is a fixed cone for all $x \in X$, then (SVQVI)(I) and (SVQVI)(II) were considered and studied by Luc and Tan [14]. They established the existence of solutions of these problems, first for scalar case and then by using scalarization technique for vector case. Minh and Tan [18] considered and studied (SVQVIP)(I) and (SVQVIP)(II) when index set I is a singleton set, $T_i(y_i, x) := P_i(x)$ and $C_i(x)$ is a fixed cone for all $x, y \in X$.

2. Preliminaries

Let X and Y be nonempty sets. A multivalued map $T : X \multimap Y$ is a function from X to the power set of Y. Let $A \subseteq X$, $x \in X$ and $y \in Y$, we define $T(A) = \bigcup\{T(a) : a \in A\}$ and $x \in T^{-1}(y)$ if and only if $y \in T(x)$. For topological spaces X and Y, a multivalued map $T : X \multimap Y$ is said to be (i) *upper semicontinuous at* $x \in X$ if for every open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subseteq V$ for all $x' \in U(x)$; (ii) *lower semicontinuous at* $x \in X$ if for every open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U(x)$; (iii) T is *upper semicontinuous* (respectively, *lower semicontinuous*) on X if T is upper semicontinuous (respectively, lower semicontinuous) at every point of X; (iv) *closed* if the graph of T, $\operatorname{Gr}(T) = \{(x, y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$. If A is nonempty subset of a vector space, then $\operatorname{co}A$ denotes the convex hull of A. For a nonempty subset A of a topological space, $\overline{A}$ denotes the closure of A.

Definition 2.1 (*[20]*). Let $\mathcal{Z}$ be a topological vector space ordered by a closed convex cone C and $M \subseteq \mathcal{Z}$ be a nonempty set.

(i) A point $z_0 \in M$ is said to be an *efficient point* of M if there is no $z \in M$ such that $z_0 \in z + C \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero element of $\mathcal{Z}$.
The set of all efficient points of M is denoted by $\mathrm{Min}_C(M)$.

(ii) A point $z_0 \in M$ is said to be an *ideal efficient point* of M if $y - z_0 \in C$ for all $y \in M$.
The set of all ideal efficient points of M is denoted by $\mathrm{IMin}_C(M)$.

Definition 2.2. Let X be a nonempty convex subset of a topological vector space $\mathcal{U}$, $\mathcal{Z}$ a topological vector space and C a closed convex cone in $\mathcal{Z}$. A multivalued map $F : X \multimap \mathcal{Z}$ is said to be C-*convex* if for every $x_1, x_2 \in X$ and $t \in [0, 1]$,

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + C.$$

Definition 2.3. Let X be a convex subset of a topological vector space and let $C : X \multimap \mathcal{Z}$ be a multivalued map such that for all $x \in X$, $C(x)$ is a closed convex cone with nonempty interior. For all $x \in X$, a multivalued function $F : X \times X \multimap \mathcal{Z}$ is said to be

(i) $C(x)$-*quasiconvex* if for all $y_1, y_2 \in X$ and $\lambda \in [0, 1]$, we have either

$$F(x, y_1) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x)$$

or

$$F(x, y_2) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x);$$

(ii) $C(x)$-*quasiconvex-like* if for all $y_1, y_2 \in X$ and $\lambda \in [0, 1]$, we have either

$$F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x, y_1) - C(x)$$

or

$$F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x, y_2) - C(x).$$

By induction, we have the following proposition.

Proposition 2.1. *Let X be a nonempty subset of a topological vector space, $\mathcal{Z}$ a topological vector space, $F : X \times X \multimap \mathcal{Z}$ and $C : X \multimap \mathcal{Z}$ multivalued maps such that for all $x \in X$, $C(x)$ is a convex cone.*

(i) *F is $C(x)$-quasiconvex* [3] *if and only if for any $x \in X$, $y_i \in X_i$, $t_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n t_i = 1$, then there exists $1 \leq j \leq n$ such that $F(x, y_j) \subseteq F\left(x, \sum_{i=1}^n t_i y_i\right) + C(x)$.*

(ii) *F is $C(x)$-quasiconvex-like* [12] *if and only if for any $x \in X$, $y_i \in X_i$, $t_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n t_i = 1$, then there exists $1 \leq j \leq n$ such that $F\left(x, \sum_{i=1}^n t_i y_i\right) \subseteq F(x, y_j) - C(x)$.*

The following maximal element theorem for a family of multivalued maps will be used in the main result of this paper.

Theorem 2.1 (*[7]*). *For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff topological vector space U_i and $S_i : X = \prod_{i \in I} X_i \multimap X_i$ be a multivalued map such that*

(i) *for all $x = (x_i)_{i \in I} \in X$, $x_i \notin \mathrm{co} S_i(x)$;*

(ii) *for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X_i;*

(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exists $j \in I$ such that $B_j \cap S_j(x) \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

3. Existence results

Definition 3.1. For each $i \in I$, a multivalued map $T_i : X_i \times X \multimap Y_i$ is said to be

(i) *type* I *properly F_i-quasimonotone* on X_i if for any finite subset $\{x_{i_1}, \ldots, x_{i_n}\} \subseteq X_i$ and for all $x = (x_i)_{i \in I} \in X$ with $x_i \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$, there exists $j \in \{1, 2, \ldots, n\} = J$ such that $F_i(t_{i_j}, x, x_{i_j}) \subseteq F_i(t_{i_j}, x, x_i) + C_i(x)$ for all $t_{i_j} \in T_i(x_{i_j}, x)$;

(ii) *type* II *properly F_i-quasimonotone* on X_i if for any finite subset $\{x_{i_1}, \ldots, x_{i_n}\} \subseteq X_i$ and for all $x = (x_i)_{i \in I} \in X$ with $x_i \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$, there exists $j \in \{1, 2, \ldots, n\} = J$ such that $F_i(t_{i_j}, x, x_i) \subseteq F_i(t_{i_j}, x, x_{i_j}) - C_i(x)$ for all $t_{i_j} \in T_i(x_{i_j}, x)$;

(iii) *type* I *pseudomonotone w.r.t.* F_i if for all $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in X$,

$$F_i(s_i, x, y_i) \subseteq F_i(s_i, x, x_i) + C_i(x) \quad \text{for some } s_i \in T_i(x_i, x)$$
$$\Rightarrow F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x) \quad \text{for all } t_i \in T_i(y_i, x);$$

(iv) *type* II *pseudomonotone w.r.t.* F_i, if for each $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in X$,

$$F_i(s_i, x, x_i) \subseteq F_i(s_i, x, y_i) - C_i(x) \quad \text{for some } s_i \in T_i(x_i, x)$$
$$\Rightarrow F_i(t_i, x, x_i) \subseteq F_i(t_i, x, y_i) - C_i(x) \quad \text{for all } t_i \in T_i(y_i, x).$$

Remark 3.1. If I is a singleton set, $F_i : Y_i \times X \times X_i \to \mathbb{R}$ is a single-valued map and $C_i(x) = \mathbb{R}_+$ for all $x \in X$, then the definition (i)–(iv) can be found in [14].

Applying Proposition 2.1 and following the similar argument as in the proof of Lemma 2.5 in [14], we have the following results for properly F_i-quasimonotone maps on X_i.

Lemma 3.1. *For each $i \in I$, let $T_i : X_i \times X \multimap Y_i$ and $F_i : Y_i \times X \times X_i \multimap Z_i$ be multivalued maps with nonempty values such that*

(i) *T_i is type* I *pseudomonotone w.r.t. F_i;*
(ii) *for each $t_i \in Y_i$ and $x \in X$, $y_i \multimap F_i(t_i, x, y_i)$ is $C_i(x)$-quasiconvex.*

Then T_i is type I *properly F_i-quasimonotone on X_i.*

Remark 3.2. If I is a singleton set, $F_i : Y_i \times X \times X_i \to \mathbb{R}$ is a single-valued map and $C_i(x) = \mathbb{R}_+$, then Lemma 3.1 reduces to Lemma 2.5 in [14].

Lemma 3.2. *For each $i \in I$, let $T_i : X_i \times X \multimap Y_i$ and $F_i : Y_i \times X \times X_i \multimap Z_i$ be multivalued maps with nonempty values such that*

(i) *T_i is type* II *pseudomonotone w.r.t. F_i;*
(ii) *for all $t_i \in Y_i$ and $x \in X$, $y_i \multimap F_i(t_i, x, y_i)$ is $C_i(x)$-quasiconvex-like.*

Then T_i is type II *properly F_i-quasimonotone on X_i.*

Now we establish the existence result for a solution of a system of vector quasi-variational inclusions under properly F_i-quasimonotonicity. From now onward, all topological spaces are assumed to be Hausdorff.

Theorem 3.1. *For each $i \in I$, let $X_i \subseteq U_i$ and $Y_i \subseteq V_i$ be nonempty closed convex sets, and $C_i : X \multimap Z_i$ be an upper semicontinuous multivalued map such that for all $x \in X$, $C_i(x)$ is a proper closed convex cone with* int $C_i(x) \neq \emptyset$. *For each $i \in I$, let $T_i : X_i \times X \multimap Y_i$, $S_i, Q_i : X \multimap X_i$, $F_i : Y_i \times X \times X_i \multimap Z_i$ be multivalued maps with nonempty values and $\mathscr{F}_i = \{x \in X : x_i \in S_i(x)\}$ be closed in X. For each $i \in I$, assume that the following conditions are satisfied.*

(i) *For all $y_i \in X_i$, $T_i(y_i, \cdot)$ is lower semicontinuous and T_i is type I properly F_i-quasimonotone on X_i;*
(ii) *For all $x \in X$,* co$Q_i(x) \subseteq S_i(x)$*;*
(iii) *For all $y_i \in X_i$, $Q_i^{-1}(y_i)$ is open in X;*
(iv) *For all $y_i \in X_i$, $(t_i, x) \multimap F_i(t_i, x, y_i)$ is lower semicontinuous and $(t_i, x) \multimap F_i(t_i, x, x_i)$ is upper semicontinuous with compact values;*
(v) *There exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for each $i \in I$ such that for all $x \in X \setminus K$ there exist $j \in I$ and $y_j \in B_j$ with $y_j \in Q_j(x)$ such that $F_j(t_j, x, y_j) \not\subseteq F_j(t_j, x, x_j) + C_j(x)$ for some $t_j \in T_j(y_j, x)$.*

Then there exists a solution $\bar{x} \in X$ of (SVQVI)(I).

Proof. For each $i \in I$ and for all $x \in X$, define a multivalued map $\mathbb{A}_i : X \multimap X_i$ as

$$\mathbb{A}_i(x) = \{y_i \in X_i : \text{for some } t_i \in T_i(y_i, x),\ F_i(t_i, x, y_i) \nsubseteq F_i(t_i, x, x_i) + C_i(x)\}.$$

For every $y_i \in X_i$, $\mathbb{A}_i^{-1}(y_i)$ is open in X. Indeed,

$$X \setminus \mathbb{A}_i^{-1}(y_i) = \{x \in X : \text{ for all } t_i \in T_i(y_i, x),\ F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x)\}.$$

If $x \in \overline{X \setminus \mathbb{A}_i^{-1}(y_i)}$, then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $X \setminus \mathbb{A}_i^{-1}(y_i)$ such that $x_\alpha \to x$. Therefore, for all $t_i \in T_i(y_i, x_\alpha)$,

$$F_i(t_i, x_\alpha, y_i) \subseteq F_i(t_i, x_\alpha, (x_\alpha)_i) + C_i(x_\alpha). \tag{3.1}$$

By lower semicontinuity of $T_i(y_i, \cdot)$, for every $t_i \in T_i(y_i, x)$, there exists $t_i^\alpha \in T_i(y_i, x_\alpha)$ such that $t_i^\alpha \to t_i$ [19]. Therefore, by (3.1) we have

$$F_i(t_i^\alpha, x_\alpha, y_i) \subseteq F_i(t_i^\alpha, x_\alpha, (x_\alpha)_i) + C_i(x_\alpha). \tag{3.2}$$

Since $x_i^\alpha \to x_i$, condition (iv) implies that for each $z_i \in F_i(t_i, x, y_i)$, there exists a net $z_i^\alpha \in F_i(t_i^\alpha, x_\alpha, y_i)$ such that $z_i^\alpha \to z_i$. By (3.2), we have

$$z_i^\alpha \in F_i(t_i^\alpha, x_\alpha, (x_\alpha)_i) + C_i(x_\alpha).$$

So there exist $u_i^\alpha \in F_i(t_i^\alpha, x_\alpha, (x_\alpha)_i)$ and $d_i^\alpha \in C_i(x_\alpha)$ such that

$$z_i^\alpha = u_i^\alpha + d_i^\alpha. \tag{3.3}$$

For each $i \in I$, let $L_i = \{t_i^\alpha\} \cup \{t_i\}$, $M_i = \{x_i^\alpha\} \cup \{x_i\}$ and $M = \{x_\alpha\} \cup \{x\}$. Then L_i, M_i and M are compact sets. Condition (iv) implies that $F_i(L_i, M, M_i)$ is compact [1], and $u_i^\alpha \in F_i(t_i^\alpha, x_\alpha, (x_\alpha)_i) \subseteq F_i(L_i, M, M_i)$. Hence without loss of generality, we may assume that there exists $u_i \in F_i(L_i, M, M_i)$ such that $u_i^\alpha \to u_i$. Since $(t_i, x) \multimap F_i(t_i, x, x_i)$ is upper semicontinuous with compact values, it follows that $(t_i, x) \multimap F_i(t_i, x, x_i)$ is closed [1], so $u_i \in F_i(t_i, x, x_i)$. Hence we have $z_i^\alpha \to z_i$ and $u_i^\alpha \to u_i$. From (3.3), we have $d_i^\alpha \to z_i - u_i$. Since $C_i : X \multimap Z_i$ is upper semicontinuous with closed values, C_i is closed. From (3.3), we have $d_i^\alpha \in C_i(x_\alpha)$, $z_i - u_i \in C_i(x)$ and

$$z_i \in u_i + C_i(x) \subseteq F_i(t_i, x, x_i) + C_i(x).$$

Hence $F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x)$. Therefore $x \in X \setminus A_i^{-1}(y_i)$ and so $A_i^{-1}(y_i)$ is open in X.

For each $i \in I$, define a multivalued map $\Psi_i : X \multimap X_i$ as

$$\Psi_i(x) = \begin{cases} Q_i(x) \cap A_i(x) & \text{if } x \in \mathscr{F}_i \\ Q_i(x) & \text{if } x \in X \setminus \mathscr{F}_i. \end{cases}$$

By type I, T_i is properly F_i-quasimonotonicity of T_i, it is easy to see $x_i \notin \operatorname{co} \Psi_i(x)$ for all $i \in I$ and for all $\in X$.

For each $i \in I$ and all $y_i \in X_i$,

$$\Psi_i^{-1}(y_i) = \left(Q_i^{-1}(y_i) \cap \mathbb{A}_i^{-1}(y_i)\right) \cup \left(Q_i^{-1}(y_i) \cap (X \setminus \mathscr{F}_i)\right)$$

is open in X.

From condition (v), for each $x \in X \setminus K$, there exist $j \in I$ and $y_j \in B_j$ with $y_j \in Q_j(x)$ such that $B_j \cap \Psi_j(x) \neq \emptyset$. Applying Theorem 2.1, there exists $\bar{x} \in X$ such that for each $i \in I$, $\Psi_i(\bar{x}) = \emptyset$, that is, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, y_i) \subseteq F_i(t_i, \bar{x}, \bar{x}_i) + C_i(\bar{x}) \quad \text{for all } t_i \in T_i(y_i, x) \text{ and all } y_i \in Q_i(\bar{x}). \quad \square$$

Remark 3.3. (a) If for each $i \in I$, X_i is a nonempty compact convex subset of U_i, then Theorem 3.1 holds good without condition (v).

(b) If I is a singleton set and $C_i(x) = C$, then the existence result for a solution of vector variational inclusion problem with constraints considered in [14], can be easily derived from Theorem 3.1. In this case, Luc and Tan [14] established the existence results by using scalarization method but under compactness assumption.

Now we establish the existence result for a solution of (SVQVI)(II).

Theorem 3.2. *For each $i \in I$, let $X_i, U_i, Y_i, V_i, X, Z_i, C_i$ and $\mathscr{F}_i$ be the same as in Theorem 3.1 and let $F_i : Y_i \times X \times X_i \multimap Z_i$ be a multivalued map with nonempty values. For each $i \in I$, assume that the following conditions are satisfied.*

(i) *For all $y_i \in X_i$, $T_i(y_i, \cdot)$ is lower semicontinuous;*
(ii) *T_i is type II properly F_i-quasimonotone on X_i;*
(iii) *For all $x \in X$, $\mathrm{co}Q_i(x) \subseteq S_i(x)$;*
(iv) *For all $y_i \in X_i$, $Q_i^{-1}(y_i)$ is open in X;*
(iv) *For each $y_i \in X_i$, $(t_i, x) \multimap F_i(t_i, x, y_i)$ is upper semicontinuous with compact values and $(t_i, x) \multimap F_i(t_i, x, x_i)$ is lower semicontinuous;*
(v) *There exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for all $i \in I$ such that for all $x \in X \setminus K$ there exist $j \in I$ and $y_j \in B_j$ with $y_i \in Q_j(x)$ such that $F_j(t_j, x, x_j) \nsubseteq F_j(t_j, x, y_j) - C_j(x)$, for some $t_j \in T_j(y_j, x)$.*

Then there exists a solution $\bar{x} \in X$ of (SVQVI)(II).

Proof. For each $i \in I$ and for all $x \in X$, define a multivalued map $\mathbb{A}_i : X \multimap X_i$ as

$$\mathbb{A}_i(x) = \{y_i \in X_i : \text{ for some } t_i \in T_i(y_i, x),\ F_i(t_i, x, x_i) \nsubseteq F_i(t_i, x, y_i) - C_i(x)\}.$$

By using the same argument as in the proof of Theorem 3.1, we get the conclusion. □

Remark 3.4. (a) When I is a singleton set and $T_i(y_i, x) = P_i(x)$ for all $x \in X$, where $P_i : X \multimap Y_i$ is a multivalued map, then Theorem 3.2 generalizes a result of [18].

(b) In Theorem 3.2, if we assume further that $F_i(t_i, x, x_i) \cap C_i(x) \neq \emptyset$ for all $x \in X$, $t_i \in Y_i$ and for each $i \in I$, then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, x, y_i) \cap C_i(\bar{x}) \neq \emptyset \quad \text{for all } t_i \in T_i(y_i, \bar{x}) \text{ and all } y_i \in Q_i(\bar{x}).$$

Indeed, by Theorem 3.2, there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, \bar{x}_i) \subseteq F_i(t_i, \bar{x}, y_i) - C_i(\bar{x}) \quad \text{for all } t_i \in T_i(y_i, x) \text{ and all } y_i \in Q_i(\bar{x}). \tag{3.4}$$

Let $z_i \in F_i(t_i, \bar{x}, \bar{x}_i) \cap C_i(\bar{x})$, hence $z_i \in F_i(t_i, \bar{x}, \bar{x}_i)$ and $z_i \in C_i(\bar{x})$. By (3.4),

$$z_i \in F_i(t_i, \bar{x}, y_i) - C_i(\bar{x}) \quad \text{for all } t_i \in T_i(y_i, x) \text{ and all } y_i \in Q_i(\bar{x}).$$

Let $t_i \in T_i(y_i, x)$ and $y_i \in Q_i(\bar{x})$ be fixed, then there exist $u_i \in F_i(t_i, \bar{x}, y_i)$ and $c_i \in C_i(\bar{x})$ such that $z_i = u_i - c_i$, hence

$$u_i = z_i + c_i \subseteq C_i(\bar{x}) + C_i(\bar{x}) \subseteq C_i(\bar{x}).$$

Therefore, $F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \neq \emptyset$.

(c) Under the assumptions of Theorem 3.2 and remark (b), there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, y_i) \nsubseteq -C_i(\bar{x}) \setminus \{\mathbf{0}\} \quad \text{for all } t_i \in T_i(y_i, \bar{x}) \text{ and all } y_i \in Q_i(\bar{x}).$$

Indeed, assume that $F_i(t_i, x, y_i) \subseteq -C_i(\bar{x}) \setminus \{\mathbf{0}\}$ for some $t_i \in T_i(y_i, \bar{x})$ and some $y_i \in Q_i(\bar{x})$. Then

$$F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \subseteq -C_i(\bar{x}) \setminus \{\mathbf{0}\} \cap C_i(\bar{x}) = \emptyset,$$

a contradiction.

4. Mathematical program with a system of vector variational inclusion constraints

Let $\mathcal{Z}$ be a real topological vector space ordered by a closed convex cone C and let $h : X \multimap \mathcal{Z}$ and $P_i : X \multimap Y_i$ be a multivalued map for each $i \in I$.

We consider the following optimization problems:

$$(\text{OP})(\text{I}) \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \\ F_i(t_i, \bar{x}, \bar{x}_i) \bigcap \text{Min}_{C_i(\bar{x})} F_i(t_i, \bar{x}, S_i(\bar{x})) \neq \emptyset \quad \text{for all } t_i \in P_i(\bar{x}), \end{cases}$$

and

$$(\text{OP})(\text{II}) \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \\ F_i(t_i, \bar{x}, \bar{x}_i) \bigcap \text{IMin}_{C_i(\bar{x})} F_i(t_i, \bar{x}, S_i(\bar{x})) \neq \emptyset \quad \text{for all } t_i \in P_i(\bar{x}). \end{cases}$$

We also consider the following mathematical program with a system of variational inclusion constraints and bilevel program:

$$(\text{MP}) \begin{cases} \text{Find } \bar{x} \in M \text{ such that } h(\bar{x}) \cap \text{Min}_D h(M) \neq \emptyset, \\ \text{where } M = \{x \in X : \text{ for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ \quad F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x) \text{ for all } y_i \in Q_i(x) \text{ for all } t_i \in P_i(x)\} \end{cases}$$

and

$$(\text{BP}) \begin{cases} \text{Find } \bar{x} \in M \text{ such that } h(\bar{x}) \cap \text{Min}_D h(M) \neq \emptyset, \\ \text{where } M = \{x \in X : \text{ for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ \quad F_i(t_i, x, y_i) \cap \text{IMIN}_{C_i(x)} F_i(t_i, x, S_i(x)) \neq \emptyset \text{ for all } t_i \in P_i(x)\}. \end{cases}$$

If $\mathcal{Z} = Z_i = \mathbb{R}$, $C = C_i(x) = [0, \infty)$ for all $x \in X$, h and F_i are single-valued functions, then (MP) and (BP) reduce to the following problem:

$$\begin{cases} \min_{x \in X} h(x) \\ \text{such that for each } i \in I, x_i \in S_i(x) \text{ and} \\ F_i(t_i, x, x_i) \leq F_i(t_i, x, y_i) \quad \text{for all } y_i \in Q_i(x) \text{ and } t_i \in P_i(x). \end{cases}$$

By using the results of the previous section, we establish existence results for a solution of (OP)(I) and (OP)(II).

Lemma 4.1. *For each $i \in I$, let X_i, X, Y_i, C_i be the same as in Theorem 3.1, $P_i : X \multimap Y_i$ be a lower semicontinuous and properly F_i-quasimonotone multivalued map and $S_i : X \multimap X_i$ be a multivalued map with nonempty values such that for all $x \in X$, $S_i(x)$ is convex and for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X. For each $i \in I$, let $\mathscr{F}_i = \{x \in X : x_i \in S_i(x)\}$ be closed in X. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *For all $y_i \in X_i$, $(t_i, x) \multimap F_i(t_i, x, y_i)$ is lower semicontinuous and $(t_i, x) \multimap F_i(t_i, x, x_i)$ is upper semicontinuous with compact values;*
(ii) *There exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$ there exist $j \in I$ and $y_j \in B_j$ with $y_i \in S_j(x)$ such that $F_j(t_j, x, y_j) \nsubseteq F_j(t_j, x, x_j) + C_j(x)$ for some $t_j \in P_j(x)$.*

Then there exists a solution $\bar{x} \in X$ of OP(I).

Proof. It follows from Theorem 3.1 with $T_i(y, x) = P_i(x)$, there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, y_i) \subseteq F_i(t_i, \bar{x}, \bar{x}_i) + C_i(\bar{x}) \quad \text{for all } y_i \in S_i(\bar{x}) \text{ and all } t_i \in P_i(\bar{x}). \tag{4.1}$$

For each $t_i \in P_i(\bar{x})$, since $F_i(t_i, \bar{x}, \bar{x}_i)$ is nonempty compact, $\text{Min}_{C_i(\bar{x})} F_i(t_i, \bar{x}, \bar{x}_i) \neq \emptyset$. Let $v_i^0 \in \text{Min}_{C_i(\bar{x})} F_i(t_i, \bar{x}, \bar{x}_i) \subseteq F_i(t_i, \bar{x}, \bar{x}_i)$.

Suppose to the contrary that for some $i \in I$,

$$F_i(t_i, \bar{x}, \bar{x}_i) \cap \text{Min}_{C_i(\bar{x})} F_i(t_i, \bar{x}, S_i(\bar{x})) = \emptyset \quad \text{for some } t_i \in P_i(\bar{x}).$$

Then

$$v_i^0 \notin \text{Min}_{c_i(\bar{x})} F_i(t_i, \bar{x}, S_i(\bar{x})). \tag{4.2}$$

There exists $v_i^* \in F_i(t_i, \bar{x}, y_i)$ for some $y_i \in S_i(\bar{x})$ such that $v_i^0 \in v_i^* + C_i(\bar{x}) \setminus \{\mathbf{0}\}$. By (4.1), there exists $v_i \in F_i(t_i, \bar{x}, \bar{x}_i)$ such that $v_i^* \in v_i + C_i(\bar{x})$ and hence

$$v_i^0 - v_i = v_i^0 - v_i^* + v_i^* - v_i \in C_i(\bar{x}) \setminus \{0\} + C_i(\bar{x}) \subseteq C_i(\bar{x}) \setminus \{\mathbf{0}\}$$

which contradicts the fact that $v_i^0 \in \mathrm{Min}_{C_i(\bar{x})} F_i(t_i, \bar{x}, \bar{x}_i)$. This completes the proof. □

By using the same argument as in [21], we have the following Lemma.

Lemma 4.2. *Assume that all the conditions of Lemma 4.1 and the following condition are satisfied:*

(i) *For each $i \in I$, $t_i \in Y_i$, $x \in X$,* $\mathrm{IMin}_{C_i(x)} F_i(t_i, x, x_i) \neq \emptyset$.

Then there exists a solution $\bar{x} \in X$ of (OP)(II).

As applications of the results of previous section, we establish the existence results for solutions of (MP) and (BP).

Theorem 4.1. *For each $i \in I$, let X_i, U_i, V_i, X, Z_i, C_i be the same as in Theorem 3.1 and $\mathcal{Z}$ be a real topological vector space ordered by a closed convex cone D. Let $h : X \multimap \mathcal{Z}$ be a multivalued map with nonempty values. Assume that all conditions of Theorem 3.1 with $T_i(y, x) = P_i(x)$ and the following conditions are satisfied:*

(i′) *h is upper semicontinuous with compact values;*
(ii′) *for each $t_i \in Y_i$ and $x \in X$, $y_i \multimap F_i(t_i, x, y_i)$ is lower semicontinuous.*

Then there exists a solution $\bar{x} \in M$ of (MP)(I).

Proof. For each $i \in I$, let

$$\begin{aligned} M_i &= \{x \in X : x_i \in S_i(x) \text{ and } F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x) \text{ for all } y_i \in Q_i(x) \text{ and all } t_i \in P_i(x)\} \\ &= \{x \in X : x_i \in S_i(x)\} \cap \{x \in X : F_i(t_i, x, y_i) \\ &\quad \subseteq F_i(t_i, x, x_i) + C_i(x) \text{ for all } y_i \in Q_i(x) \text{ and all } t_i \in P_i(x)\} \\ &= \mathscr{F}_i \cap \{x \in X : F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x) \text{ for all } y_i \in Q_i(x) \text{ and all } t_i \in P_i(x)\}. \end{aligned}$$

By Theorem 3.1 with $T_i(y, x) = P_i(x)$, gives that feasible set $M = \bigcap_{i \in I} M_i$ is nonempty. Let

$$B_i = \{x \in X : F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x) \text{ for all } y_i \in Q_i(x) \text{ and all } t_i \in P_i(x)\}.$$

If $x \in \overline{B_i}$, there exists a net $\{x_\alpha\}_{\alpha \in \Lambda} \in B_i$ such that $x_\alpha \to x$. Then $x_\alpha \in X$ and

$$F_i(t_i, x_\alpha, y_i) \subseteq F_i(t_i, x_\alpha, x_i^\alpha) + C_i(x_\alpha) \quad \text{for all } t_i \in P_i(x_\alpha) \text{ and all } y_i \in Q_i(x_\alpha). \tag{4.3}$$

Since X is closed, $x \in X$. For each $t_i \in P_i(x)$ and $y_i \in Q_i(x)$, since P_i is lower semicontinuous and $Q_i^{-1}(y_i)$ is open in X, Q_i is lower semicontinuous [22], and there exist $t_i^\alpha \in P_i(x_\alpha)$ and $y_i^\alpha \in Q_i(x_\alpha)$ such that $t_i^\alpha \to t_i$, $y_i^\alpha \to y_i$. Hence by (4.3),

$$F_i(t_i^\alpha, x_\alpha, y_i^\alpha) \subseteq F_i(t_i^\alpha, x_\alpha, x_i^\alpha) + C_i(x_\alpha). \tag{4.4}$$

For each $z_i \in F_i(t_i, x, y_i)$, since F_i is lower semicontinuous there exists $z_i^\alpha \in F_i(t_i^\alpha, x_\alpha, y_i^\alpha)$ such that $z_i^\alpha \to z_i$. By (4.4), $z_i^\alpha \in F_i(t_i^\alpha, x_\alpha, x_i^\alpha) + C_i(x_\alpha)$. Therefore, there exist $z_i'^\alpha \in F_i(t_i^\alpha, x_\alpha, x_i^\alpha)$ and $d_i^\alpha \in C_i(x_\alpha)$ such that

$$z_i^\alpha = z_i'^\alpha + d_i^\alpha. \tag{4.5}$$

Let $N_i = \{t_i^\alpha\} \cup \{t_i\}$ and $L_i = \{x_i^\alpha\} \cup \{x_i\}$ and $L = \{x_\alpha\} \cup \{x\}$, then N_i, L_i and L are compact. By assumption, $F_i(N_i, L, L_i)$ is compact [1], without loss of generality there exists $z_i' \in F_i(N_i, L, L_i)$ such that $z_i'^\alpha \to z_i'$. Since $(t_i, x) \multimap F_i(t_i, x, x_i)$ is upper semicontinuous with closed values, $(t_i, x) \multimap F_i(t_i, x, x_i)$ is closed. Therefore $z_i' \in F_i(t_i, x, x_i)$. Thus by (4.5), we have $d_i^\alpha \to z_i - z_i'$. By assumption, C_i is closed and $z_i - z_i' \in C_i(x)$, $z_i \in z_i' + C_i(x) \subseteq F_i(s_i, x, x_i) + C_i(x)$. Therefore, $F_i(s_i, x, y_i) \subseteq F_i(s_i, x, x_i) + C_i(x)$. So B_i is closed in X.

By assumption for each $i \in I$, $\mathscr{F}_i$ is closed in X, hence $M_i = \mathcal{F}_i \cap B_i$ is closed in X. The feasible set $M = \bigcap_{i \in I} M_i$ is also closed. By assumption (iv′) of Corollary 3.1, $M \subseteq K$. Indeed, suppose not, there exists $\bar{x} \in M \setminus K$, then there exist $j \in J$ and $y_j \in B_j$ with $y_j \in Q_j(\bar{x})$ such that $F_j(s_j, \bar{x}, y_j) \not\subseteq F_j(s_j, \bar{x}, \bar{x}_j) + C_j(\bar{x})$, it contradicts with $\bar{x} \in M$. Hence $M \subseteq K$. Since K is compact, $M \subseteq K$, M is compact.

Since $h : X \multimap \mathcal{Z}$ is upper semicontinuous with compact values, $h(M)$ is compact. Thus $\mathrm{Min}_D h(M) \neq \emptyset$ [13]. There exists $\bar{x} \in M$ such that $\bar{u} \in h(\bar{x})$, $\bar{u} \in \mathrm{Min}_D h(M)$. Therefore, $h(\bar{x}) \bigcap \mathrm{Min}_D h(M) \neq \emptyset$.
where $M = \{x \in X;$ for each $i \in I, x_i \in S_i(x)$ and $F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x)$ for all $y_i \in Q_i(x)$ for all $t_i \in P_i(x)\}$. □

Theorem 4.2. *In addition to assumptions of Theorem 4.1, we further assume that*

(i′) *for each* $i \in I, t_i \in Y_i$, $\mathrm{IMin}_{C_i(x)} F_i(t_i, x, x_i) \neq \emptyset$.

Then there exists a solution $\bar{x} \in M$ *of* (MP)(II).

Proof. For each $i \in I$, let

$$M_i = \{x \in X : x_i \in S_i(x) \text{ and } F_i(t_i, x, x_i) \cap \mathrm{IMin}_{C_i(x)} F_i(t_i, x, S_i(x)) \neq \emptyset \text{ for all } t_i \in P_i(x)\}.$$

By Theorem 3.1 and Lemma 4.3, the feasible set $M = \bigcap_{i \in I} M_i$ is nonempty. By Lemma 3.4,

$$\begin{aligned} &\{x \in X | x_i \in S_i(x), F_i(t_i, x, x_i) \cap \mathrm{IMin}_{C_i(x)} F_i(t_i, x, S_i(x)) \neq \emptyset \text{ for all } t_i \in P_i(x)\} \\ &\quad = \{x \in X | x_i \in S_i(x), F_i(t_i, x, y_i) \subseteq F_i(t_i, x, x_i) + C_i(x)\ \forall y_i \in S_{2_i}(x) \forall t_i \in P_i(x)\}. \end{aligned}$$

Then the conclusion follows from Theorem 3.2. □

Remark 4.1. By considering appropriate choices of the spaces and maps involved in the formulations of (OP(I), (OP)(II), (MP)(I) and (MP)(II), we can easily derive the existence results for several problems which are particular cases of these problems.

5. Common fixed point theorems

Throughout this section, unless otherwise specified, we adopt the following notations and assumptions.
Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

We denote $CB(X)$ the class of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, let $\delta(A, B) = \sup\{d(x, B) : x \in A\}$ and

$$\mathcal{H}(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

$\mathcal{H}$ is called the Hausdorff metric on $CB(X)$.

Definition 5.1 (*[22]*). A multivalued map $T : X \multimap CB(X)$ is said to be a *k-contractive map* if there exists $k \in (0, 1)$ such that

$$\mathcal{H}(T(x), T(y)) \leq kd(x, y) \quad \text{for all } x, y \in X.$$

Definition 5.2 (*[15]*). Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed spaces. A multivalued map $F : U \multimap V$ is said to be *nonexpansive* if for all $x, y \in X$, $u \in F(x)$, there exists $w \in F(y)$ such that $\|u - w\|_V \leqslant \|x - y\|_U$.

We recall the following lemmas which will be used in the proof of the main result of this section.

Lemma 5.1 (*[16]*). *Let* (X, d) *be a complete metric space and let* T *be a* k*-contractive mapping from* X *into* $CB(X)$. *Then* T *is a lower semicontinuous multivalued map.*

Lemma 5.2 (*[15]*). *Let* $(U, \|\cdot\|_U)$ *and* $(V, \|\cdot\|_V)$ *be normed spaces. Every nonexpansive multivalued map* $T : U \multimap V$ *is lower semicontinuous.*

Lemma 5.3 (*[22]*). *Let* U *be a Banach space with a Fréchet differentiable norm. Then the duality mapping* $J : E \to U^*$ *is norm to norm continuous, where* U^* *is the dual space of* U.

Theorem 5.1. *Let I be a countable index set. For each $i \in I$, let U_i be a Banach space with a Fréchet differentiable norm and X_i be a nonempty closed convex subset of U_i. Let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $T_i : X \multimap X_i$ be a lower semicontinuous multivalued map with nonempty values, $S_i : X \multimap X_i$ a multivalued map with nonempty convex values such that for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X and $\mathscr{F}_i = \{x \in X : x_i \in S_i(x)\}$ is closed and for all $x \in X$, $S_i(x) \cap T_i(x) \neq \emptyset$. For each $i \in I$, assume that there exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for all $i \in I$ such that for all $x \in X \setminus K$ there exist $j \in I$, and $y_j \in B_j$ with $y_j \in S_j(x)$ such that $\langle J_j(x_j - t_j), y_j - x_j\rangle < 0$ for some $t_j \in T_j(x)$. Then there exists $\bar{x} \in X$ such that $\bar{x}_i \in S_i(\bar{x}) \cap T_i(\bar{x})$ for each $i \in I$.*

Proof. For each $i \in I$ and for all $(x_i)_{i\in I} \in X$, $y_i, t_i, \in X_i$, define a multivalued map $F_i : X_i \times X \times X_i \multimap \mathbb{R}$ as

$$F_i(t_i, x, y_i) = \{\langle J_i(x_i - t_i), y_i - x_i\rangle\},$$

where J_i is duality map of U_i. Let $Z_i = \mathbb{R}$, $C_i(x) = [0, \infty)$, $Q_i(x) = S_i(x)$. We want to show that all the conditions of Theorem 3.1 for $T_i(y, x) = T_i(x)$ hold.

By Lemma 5.3, $(t_i, x, y_i) \mapsto \langle J_i(x_i - t_i), y_i - x_i\rangle$ is a continuous function. Hence condition (iv) of Theorem 3.1 holds.

For each $x = (x_i)_{i\in I}$, define $\|x\| = \sum_{i\in I} \|x_i\|$, then $(X, \|\cdot\|)$ is a normed linear space.

We claim that T_i is properly F_i-quasimonotone. Suppose there exists a finite subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\} \subseteq X_i$ and $x = (x_i)_{i\in I}$ with $x_i \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ such that

$$\langle J(x_i - t_i), x_{i_j} - x_i\rangle < 0 \quad \text{for some } t_i \in T_i(x).$$

Since $x_i \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$, there exist $\lambda_j \geq 0$ for all $j = 1, \ldots, n$ and $\sum \lambda_j = 1$, $x_i = \sum_{j=1}^n \lambda_j x_{i_j}$. Therefore,

$$\sum \lambda_j \langle J(x_i - t_i), x_{i_j} - x_i\rangle < 0, \quad \text{that is, } \langle J(x_i - t_i), \sum \lambda_j x_{i_j} - \sum \lambda_j x_i\rangle < 0,$$

and thus $\langle J(x_i - t_i), x_i - x_i\rangle < 0$, a contradiction.

By hypothesis, for each $x \in X \setminus K$, there exist $j \in I$ and $y_j \in B_j$ with $y_j \in S_j(x)$ such that $F_j(t_j, x, y_j) \not\subseteq F_j(t_j, x, x_j) + C_i(x)$ for some $t_j \in T_j(x)$. Then by Theorem 3.1, there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, y_i) \subseteq F_i(t_i, \bar{x}, \bar{x}_i) + C_i(\bar{x}) \quad \text{for all } y_i \in S_i(\bar{x}) \text{ and all } t_i \in T_i(\bar{x}),$$

that is,

$$\langle J_i(\bar{x}_i - t_i), y_i - \bar{x}_i\rangle \geq 0 \quad \text{for all } y_i \in S_i(\bar{x}) \text{ and all } t_i \in T_i(\bar{x}).$$

Let $y_i = t_i \in T_i(\bar{x}) \cap S_i(\bar{x})$. Then

$$\|\bar{x}_i - t_i\|^2 = \langle J_i(\bar{x}_i - t_i), \bar{x}_i - t_i\rangle \leq 0,$$

therefore, $\bar{x}_i = t_i \in T_i(\bar{x}) \cap S_i(\bar{x})$. □

Remark 5.1. The lower semicontinuity of T_i in Theorem 5.1 can be replaced by any one of the following conditions.

(a) $T_i : X \multimap CB(X_i)$ is a k_i-contractive map with nonempty value.
(b) $T_i : X \multimap X_i$ is a nonexpansive multivalued map with nonempty values.
(c) $T_i : X \multimap X_i$ has open lower section.

Remark 5.2. (a) Since every Hilbert space has a Fréchet differentiable norm and for each $i \in I$, the duality map $J_i : U_i \to U_i^*$ is defined by $J_i(x_i) = \{x_i\}$, Theorem 5.1 holds good under the setting of Hilbert space U_i.

(b) For each $i \in I$, if U_i is a reflexive Banach space and X_i is a closed bounded convex subset of U_i, then X_i is weak compact and convex, and $X = \prod_{i\in I} X_i$ is weakly compact. Then Theorem 5.1 holds good when each X_i is a bounded closed convex subset of U_i but without the last condition in Theorem 5.1.

6. System of Minty and Stampacchia type generalized implicit quasi-variational inequalities

For each $i \in I$, let $T_i : X_i \times X \multimap Y_i$, $S_i, Q_i : X \multimap X_i$ and $F_i : Y_i \times X \times X_i \multimap \mathbb{R}$ be multivalued maps with nonempty values. We consider the following *system of Minty type generalized implicit quasi-variational inequalities*:

$$\text{(SMGIQVI)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ \inf F_i(T_i(y_i, \bar{x}), \bar{x}, y_i) \geq 0 \quad \text{for all } y_i \in Q_i(\bar{x}). \end{cases} \tag{6.1}$$

We also study the following *system of Stampacchia type generalized implicit quasi-variational inequalities*:

$$\text{(SSGIQVIP)} \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \bar{x}_i \in S_i(\bar{x}) \text{ and} \\ \sup F_i(T_i(\bar{x}_i, \bar{x}), \bar{x}, y_i) \geq 0 \quad \text{for all } y_i \in S_i(\bar{x}). \end{cases} \tag{6.2}$$

When I is a singleton set and $S_i(x) = X$ for all $x \in X$, the above problems were considered and studied in [17]. The existence of solutions of these problems was investigated in [17] under densely pseudomonotonicity assumption.

As an application of Theorem 3.1, we derive the existence result for solutions of above mentioned problems. When I is a singleton set and $S_i(x) = X$ for all $x \in X$, our assumptions are different from those in [17].

Theorem 6.1. *For each $i \in I$, let X_i, Y_i, X, T_i, S_i, Q_i and $\mathscr{F}_i$ be the same as in Theorem 3.1, and let $F_i : Y_i \times X \times X_i \multimap \mathbb{R}$ be multivalued with nonempty values. For each $i \in I$, assume that the conditions* (i) *and* (ii) *of Theorem 3.1 and the following conditions are satisfied.*

(i) *For each $y_i \in X_i$, $(t_i, x) \multimap F_i(t_i, x, y_i)$ is lower semicontinuous;*
(ii) *There exist a nonempty compact subset K of X and a nonempty compact convex subset B_i of X_i for each $i \in I$ such that for all $x \in X \setminus K$, there exist $j \in I$ and $y_j \in B_j$ with $y_j \in Q_j(x)$ such that* $\inf F_i(T_i(y_i, \bar{x}), y, \bar{x}_i) < 0$.

Then there exists a solution $\bar{x} \in X$ of (SMGIQVI).

Proof. For each $i \in I$, let $Z_i = \mathbb{R}$, $C_i(x) = [0, \infty)$ and $F_i(t_i, x, x_i) = \{0\}$ for all $t_i \in Y_i$ and $x = (x_i)_{i \in I} \in X$. Then all the conditions of Theorem 3.1 are satisfied, and, hence there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$F_i(t_i, \bar{x}, y_i) \subseteq [0, \infty) \quad \text{for all } t_i \in T_i(y_i, \bar{x}) \text{ and all } y_i \in Q_i(\bar{x}),$$

that is,

$$F_i(T_i(y_i, \bar{x}), \bar{x}, y_i) \subseteq [0, \infty) \quad \text{for all } y_i \in Q_i(\bar{x})$$

which implies that $\inf F_i(T_i(y_i, \bar{x}), \bar{x}, y_i) \geq 0$ for all $y_i \in Q_i(\bar{x})$. □

By using Theorem 6.1, we derive the following existence result for a solution of (SSGIVI).

Theorem 6.2. *For each $i \in I$, assume that all conditions of Theorem 5.1 and the following conditions hold:*

(iii) *For all $t_i \in Y_i$, $x = (x_i)_{i \in I} \in X$, the multivalued map $y_i \multimap F_i(t_i, x, y_i)$ is $\mathbb{R}_+$-convex and $F_i(t_i, x, x_i) = \{0\}$;*
(iv) *For all $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I} \in X$ and $\lambda \in (0, 1)$*

$$\inf F_i(T_i((x_\lambda)_i, x), x, y_i) \geq 0 \Rightarrow \sup F_i(T_i(x_i, x), x, y_i) \geq 0,$$

where $(x_\lambda)_i = (1 - \lambda)x_i + \lambda y_i$;
(v) *$S_i : X \multimap X_i$ is a multivalued map with nonempty convex values such that for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X and $\mathscr{F}_i = \{x \in X : x_i \in S_i(x)\}$ is closed in X.*

Then there exists a solution $\bar{x} \in X$ of (SSGIVI).

Proof. Since all the conditions of Theorem 5.1 are satisfied, there exists $\bar{x}$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$\inf F_i(T_i(y_i, \bar{x}), \bar{x}, y_i) \geq 0 \quad \text{for all } y_i \in S_i(\bar{x}). \tag{6.3}$$

For each $i \in I$, let $(x_\lambda)_i = (1 - \lambda)\bar{x}_i + \lambda y_i$, where $\lambda \in (0, 1)$ and $y_i \in S_i(\bar{x})$ is fixed. Since $S_i(\bar{x})$ is convex, $(x_\lambda)_i \in S_i(\bar{x})$. Therefore,

$$\inf F_i(T_i((x_\lambda)_i, \bar{x}), \bar{x}, (x_\lambda)_i) \geq 0,$$

that is, for all $z_i \in F_i(T_i((x_\lambda)_i, \bar{x}), \bar{x}, (x_\lambda)_i)$, we have $z_i \geq 0$. By condition (iii), we have

$$\lambda F_i(t_i, \bar{x}, y_i) + (1-\lambda) F_i(t_i, \bar{x}, \bar{x}_i) \subseteq F_i(t_i, \bar{x}, (x_\lambda)_i) + \mathbb{R}_+.$$

Since $F_i(t_i, x, x_i) = \{0\}$ for all $t_i \in Y_i$ and $x = (x_i)_{i\in I} \in X$, the above inclusion becomes

$$\lambda F_i(t_i, \bar{x}, y_i) \subseteq F_i(t_i, \bar{x}, (x_t)_i) + \mathbb{R}_+. \tag{6.4}$$

For every $u_i \in F_i(T_i((x_\lambda)_i, \bar{x}), \bar{x}, y_i)$, there exists $t_i \in T_i((x_\lambda)_i, \bar{x})$ such that $u_i \in F_i(t_i, \bar{x}, y_i)$. From (6.4), there exist $w_i \in F_i(t_i, \bar{x}, (x_\lambda)_i)$ and $a > 0$ such that $\lambda u_i = w_i + a$, and so $u_i = \frac{1}{\lambda}(w_i + a) \geq 0$. Therefore,

$$\inf F_i(T_i((x_\lambda)_i, \bar{x}), \bar{x}, y_i) \geq 0 \quad \text{for all } \lambda \in (0,1) \text{ and } i \in I.$$

Hence condition (iv) implies the desired conclusion. □

Remark 6.1. If I is a singleton set, $Y = X^*$ is the dual space of X, $T(y, x) = \mathbb{T}(y)$ is a multivalued map, and $F : Y \times X \times X \to \mathbb{R}$ is a single-valued map defined by $F(t, x, y) = \langle t, y - x\rangle$ for all $t \in Y$ and $x, y \in X$. Then the condition (iv) of Theorem 6.2 reduces the following condition:

(iv′) for every $x, y \in X$ and for all $\lambda \in (0, 1)$

$$\inf_{t\in\mathbb{T}(x_\lambda)} \langle t, y - x\rangle \geq 0 \Rightarrow \sup_{t\in\mathbb{T}(x)} \langle t, y - x\rangle \geq 0,$$

where $x_\lambda = (1-\lambda)x + \lambda y$.

A multivalued map $\mathbb{T}$ satisfying condition (iv′) is said to be *upper-sign continuous* [4]. Aussel and Hadjisavvas [4] proved that a local solution of Minty type variational inequality is a solution of Stampacchia type variational inequality under the upper-sign continuous condition.

Remark 6.2. In Theorem 6.2, if for all $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in X$,

$$\lambda \multimap F_i(T_i((x_\lambda)_i, x), x, y_i) \text{ is upper semicontinuous at } 0^+,$$

where $(x_\lambda)_i = (1-\lambda)x_i + \lambda y_i$. Then condition (iv) is satisfied.

Proof. For each $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in X$, and each $i \in I$, we claim that for all $\lambda \in (0, 1)$

$$\inf F_i(T_i((x_\lambda)_i, x), x, y_i) \geq 0 \Rightarrow \sup F_i(T_i(x_i, x), x, y_i) \geq 0,$$

where $(x_\lambda)_i = (1-\lambda)x_i + \lambda y_i$.

Suppose on the contrary that

$$\sup F_i(T_i(x_i, x), x, y_i) < 0 \Rightarrow F_i(T_i(x_i, x), x, y_i) \subseteq (-\infty, 0). \tag{6.5}$$

For each $i \in I$, let $H_i : [0, 1] \to \mathbb{R}$ be defined by

$$H_i(\lambda) = F_i(T_i((x_\lambda)_i, x), x, y_i) \quad \text{for all } \lambda \in [0, 1],$$

where $(x_\lambda)_i = (1-\lambda)x_i + \lambda y_i$. By (6.5), we have $H_i(0) \subseteq (-\infty, 0)$. Since $\lambda \multimap F_i(T_i((x_\lambda)_i, x), x, y_i)$ is upper semicontinuous at 0^+, there exists $\delta > 0$ such that for all $\widetilde{\delta} \in (0, \delta)$ we have $H_i(\widetilde{\delta}) \subseteq (-\infty, 0)$. Therefore $F_i(T_i((x_{\widetilde{\delta}})_i, x), x, y_i) \subseteq (-\infty, 0)$. Thus $\inf F_i(T_i((x_{\widetilde{\delta}})_i, x), x, y_i) < 0$, a contradiction. □

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