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A Generalization of Vectorial Equilibria

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Abstract: A generalized form of vectorial equilibria is proposed, and, using an abstract monotonicity condition, an existence result is demonstrated.

Key Words: Equilibrium problem, multivalued mapping, pseudomonotone mapping.

1 Introduction

Let X be a real topological vector space, $K \subseteq X$ nonempty, and $f: K \times K \to \mathbb{R}$ such that $f(x,x) = 0$ for all $x \in K$. Then the *scalar equilibrium problem* consists in finding $\bar{x} \in X$ such that

$$\bar{x} \in K \;, \qquad f(\bar{x}, y) \geq 0 \qquad \forall y \in K \;.$$

This problem has many diverse applications, in particular optimization problems and variational inequalities (see [4]).

Now let Z be another topological vector space. Let $P \subseteq Z$ be a closed convex cone (not necessarily pointed) with nonempty interior and $P \neq Z$. Then P defines an ordering on Z by means of

$$z \succeq 0 :\Longleftrightarrow z \in P \;, \qquad z \succ 0 :\Longleftrightarrow z \in \operatorname{int} P \;.$$

We extend this notation to arbitrary subsets $S \subseteq Z$ by setting

$$S \succeq 0 :\Longleftrightarrow S \subseteq P \;, \qquad S \succ 0 :\Longleftrightarrow S \subseteq \operatorname{int} P \;,$$

$$S \preceq 0 :\Longleftrightarrow S \subseteq -P \;, \qquad S \prec 0 :\Longleftrightarrow S \subseteq -\operatorname{int} P \;.$$

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Let $F: K \times K \rightrightarrows Z$ be a multivalued mapping. The *vectorial equilibrium problem* may be formulated either as

$$\bar{x} \in K\,, \qquad F(\bar{x}, y) \not\prec 0 \qquad \forall y \in K\,, \tag{1}$$

or as

$$\bar{x} \in K\,, \qquad F(\bar{x}, y) \succeq 0 \qquad \forall y \in K\,. \tag{2}$$

Both problems constitute a legitimate extension of the scalar equilibrium problem. For the case of F being single-valued the vectorial equilibrium problem has been considered in [1], [3], [6], and vector-valued variational inequalities have been considered in [5]. Clearly, it is useful to have a theory for the multivalued case, too. But, in connection with existence problems for (1) and (2), it turns out that the order structure of Z (i.e., the property of P being a convex cone) plays only a minor role. Therefore, in what follows, we propose to consider a more general form of vectorial equilibria.

2 Generalized Equilibria

Let X be a locally convex Hausdorff topological vector space, Y and Z real topological vector spaces, $A \subseteq X$ nonempty, convex, compact, $B \subseteq Y$ nonempty, convex, and $C \subseteq Z$. Let $F: A \times B \rightrightarrows Z$ be a multivalued mapping.

We consider the *generalized equilibrium problem* of finding $\bar{x} \in X$ such that

$$\bar{x} \in A\,, \qquad F(\bar{x}, y) \cap C \neq \emptyset \qquad \forall y \in B\,. \tag{3}$$

We consider also the related problem

$$\bar{x} \in A\,, \qquad F(\bar{x}, y) \subseteq C \qquad \forall y \in B\,. \tag{4}$$

With $C := Z \backslash -\operatorname{int} P$ it is obvious that (3) contains (1), and with $C := P$ it is obvious that (4) contains (2). Using the lower inverse $F^{-}(C)$ and the upper inverse $F^{+}(C)$ of the multivalued mapping F, which are defined as (see [2])

$$F^{-}(C) := \{(x, y) \in A \times B | F(x, y) \cap C \neq \emptyset\}\,,$$

$$F^{+}(C) := \{(x, y) \in A \times B | F(x, y) \subseteq C\}\,,$$

both problems (3) and (4) can be written as

$$\bar{x} \in A , \qquad (\bar{x}, y) \in F^{-1}(C) \qquad \forall y \in B , \tag{5}$$

with $F^{-1} := F^{-}$ for (3) and $F^{-1} := F^{+}$ for (4).

In order to obtain existence results for problem (5), we introduce another mapping $G : A \times B \rightrightarrows Z$, and a set $D \subseteq Z$. Again, we write G^{-1} for G^{-} or G^{+}. Further, an upper semicontinuous mapping $T : B \rightrightarrows A$ with nonempty, closed, convex values is given.

Theorem 1: Let $F^{-1}(C)$ and $G^{-1}(D)$ have the following properties:

(i) $(x, y) \in F^{-1}(C)$ *for all* $y \in B$, $x \in T(y)$;
(ii) $\{x \in A | (x, y) \notin G^{-1}(D)\}$ *is open in* A *for all* $y \in B$;
(iii) $\{y \in B | (x, y) \notin F^{-1}(C)\}$ *is convex for all* $x \in A$;
(iv) $F^{-1}(C) \subseteq G^{-1}(D)$;
(v) *for every* $x \in A$, $(x, y) \in G^{-1}(D)$ $\forall y \in B$ *implies* $(x, y) \in F^{-1}(C)$ $\forall y \in B$.

Then there exists $\bar{x} \in A$ *such that* $(\bar{x}, y) \in F^{-1}(C)$ *for all* $y \in B$.

Proof: In view of (v) it suffices to find $\bar{x} \in A$ such that $(\bar{x}, y) \in G^{-1}(D)$ for all $y \in B$. If such an $\bar{x}$ does not exist, then the compact set A is covered by the sets

$$V(y) := \{x \in A | (x, y) \notin G^{-1}(D)\} , \qquad y \in B ,$$

which are open by (ii). Let $V(y_1), \ldots, V(y_n)$ be a finite subcover of A, and let $\beta_1(\cdot), \ldots, \beta_n(\cdot)$ be a continuous partition of unity subordinate to this open cover. The functions $\beta_i\colon A \to \mathbb{R}$ are continuous, nonnegative, add up to unity, and from $\beta_i(x) > 0$ follows $x \in V(y_i)$. Then $p(x) := \sum_{i=1}^{n} \beta_i(x) y_i$ defines a continuous function $p\colon A \to B$. The mapping $T(p(\cdot))\colon A \rightrightarrows A$ is upper semicontinuous with nonempty, closed, convex values, hence has a fixed point $x^0 \in T(p(x^0))$. Let $I := \{i | \beta_i(x^0) > 0\}$. Then, for all $i \in I$, $x^0 \in V(y_i)$, hence $(x^0, y_i) \notin G^{-1}(D)$, hence $(x^0, y_i) \notin F^{-1}(C)$ by (iv). Then from $p(x^0) \in \operatorname{conv}\{y_i | i \in I\}$ and (iii) it follows that $(x^0, p(x^0)) \notin F^{-1}(C)$. Since $x^0 \in T(p(x^0))$, this contradicts (i). □

Note that condition (iv) may be considered as an abstract monotonicity requirement. Let us single out two prototypical cases of Theorem 1. First we consider the case $G(x, y) := F(x, y)$, $D := C$. Then conditions (iv) and (v) of Theorem 1 are automatically satisfied, and we obtain

Theorem 2: Let $F^{-1}(C)$ have the following properties:

(i) $(x, y) \in F^{-1}(C)$ *for all* $y \in B$, $x \in T(y)$;
(ii) $\{x \in A | (x, y) \notin F^{-1}(C)\}$ *is open in* A *for all* $y \in B$;
(iii) $\{y \in B | (x, y) \notin F^{-1}(C)\}$ *is convex for all* $x \in A$.

Then there exists $\bar{x} \in A$ *such that* $(\bar{x}, y) \in F^{-1}(C)$ *for all* $y \in B$.

Now we turn to the case $Y := X$, $B := A$, $T := id$, $G(x, y) := F(y, x)$. Then we obtain

Theorem 3: Let for all $x, y \in A$ *the following properties hold:*

(i) $(y, y) \in F^{-1}(C)$;
(ii) $\{\xi \in A | (x, \xi) \notin F^{-1}(D)\}$ *is open in* A;
(iii) $\{\xi \in A | (x, \xi) \notin F^{-1}(C)\}$ *is convex*;
(iv) $(x, y) \in F^{-1}(C)$ *implies* $(y, x) \in F^{-1}(D)$;
(v) *for every* $u \in]x, y[$, *if* $(u, x) \in F^{-1}(D)$ *and* $(u, y) \notin F^{-1}(C)$, *then* $(u, \xi) \in F^{-1}(\operatorname{int} D)$ *for all* $\xi \in]x, y[$;
(vi) $\{\xi \in [x, y] | (\xi, y) \notin F^{-1}(C)\}$ *is open in* $[x, y]$;
(vii) $(y, y) \notin F^{-1}(\operatorname{int} D)$.

Then there exists $\bar{x} \in A$ *such that* $(\bar{x}, y) \in F^{-1}(C)$ *for all* $y \in A$.

Proof: Theorem 3 follows from Theorem 1 upon choosing $G(x, y) := F(y, x)$ and $T := id$. It only remains to verify condition (v) of Theorem 1. To this end, let $x \in A$ with $(y, x) \in F^{-1}(D)$ for all $y \in A$. Assume, for contradiction, that $(x, y) \notin F^{-1}(C)$ for some $y \in A$. By (i), $y \neq x$. From (vi) there exists $u \in]x, y[$ such that $(u, y) \notin F^{-1}(C)$. Since $(u, x) \in F^{-1}(D)$, we obtain from (v) that $(u, u) \in F^{-1}(\operatorname{int} D)$, contradicting (vii). □

Observe that we may use two different inverses for $F^{-1}(C)$ on one hand and $F^{-1}(D)$, $F^{-1}(\operatorname{int} D)$ on the other hand.

3 Examples

The structure of Theorems 2 and 3 is perhaps better understood by considering a special case. Let $P \subseteq Z$ be an ordering cone, as presented in the Introduction. We turn first to Theorem 2. We choose $C := Z \setminus -\operatorname{int} P$, and

$F^{-1} := F^{-}$. Then

$$(x, y) \notin F^{-1}(C) \Longleftrightarrow F(x, y) \prec 0 \ .$$

Thus we obtain from Theorem 2 the following multivalued extension of [6], Corollary 3.

Corollary 1: Let $F : A \times B \rightrightarrows Z$ have the following properties:

(i) $F(x, y) \not\prec 0$ *for all* $y \in B$, $x \in T(y)$;
(ii) $\{x \in A | F(x, y) \prec 0\}$ *is open in* A *for all* $y \in B$;
(iii) $\{y \in B | F(x, y) \prec 0\}$ *is convex for all* $x \in A$.

Then there exists $\bar{x} \in A$ such that $F(\bar{x}, y) \not\prec 0$ for all $y \in B$.

Now we turn to Theorem 3. We choose $C := Z \backslash -\operatorname{int} P$, $D := Z \backslash \operatorname{int} P$, and $F^{-1} := F^{-}$. Then

$$(x, y) \notin F^{-1}(C) \Longleftrightarrow F(x, y) \prec 0 \ ,$$
$$(x, y) \notin F^{-1}(D) \Longleftrightarrow F(x, y) \succ 0 \ ,$$
$$(x, y) \notin F^{-1}(\operatorname{int} D) \Longleftrightarrow F(x, y) \succeq 0 \ ,$$

and $F(y, y) \succeq 0$ implies $F(y, y) \not\prec 0$, provided $F(y, y) \neq \emptyset$. Thus we obtain from Theorem 3 the following multivalued extension of [6], Corollary 1.

Corollary 2: Let $F: A \times A \rightrightarrows Z$ be such that, for all $x, y \in A$, the following properties hold:

(i) $\emptyset \neq F(y, y) \succeq 0$;
(ii) $\{\xi \in A | F(x, \xi) \succ 0\}$ *is open in* A;
(iii) $\{\xi \in A | F(x, \xi) \prec 0\}$ *is convex*;
(iv) $F(x, y) \not\prec 0$ *implies* $F(y, x) \not\succ 0$;
(v) *for every* $u \in]x, y[$, *if* $F(u, x) \not\succ 0$ *and* $F(u, y) \prec 0$, *then* $F(u, \xi) \not\succeq 0$ *for all* $\xi \in]x, y[$;
(vi) $\{\xi \in [x, y] | F(\xi, y) \prec 0\}$ *is open in* $[x, y]$.

Then there exists $\bar{x} \in A$ such that $F(\bar{x}, y) \not\prec 0$ for all $y \in A$.

Concerning the assumptions of Corollary 2 we observe the following: (ii) is satisfied, if the multivalued mapping $F(x, \cdot)$ is upper semicontinuous (i.e., the upper inverse of every open set is open [2]). (vi) is satisfied, if the mapping $F(\cdot, y)$ is upper semicontinuous along line segments in A. (iii) is satisfied, if for every $u \in A$ the mapping $t(\cdot) := F(u, \cdot)$ has the property that

$$(\alpha) \quad t(\lambda x + (1-\lambda)y) \subseteq \lambda t(x) + (1-\lambda)t(y) - P \qquad \forall x, y \in A \ , \lambda \in [0,1] \ .$$

(v) is satisfied, if for every $u \in A$ the mapping $t(\cdot) := F(u, \cdot)$ has the property that

$$(\beta) \quad \begin{cases} t(x) \neq \emptyset & \forall x \in A \ , \\ t(\lambda x + (1-\lambda)y) + P \supseteq \lambda t(x) + (1-\lambda)t(y) & \forall x, y \in A \ , \lambda \in [0,1] \ . \end{cases}$$

To verify that (β) implies (v), let $a \in t(x)$, $a \not\succ 0$, and $b \in t(y)$, $b \prec 0$. Let $\xi := \lambda x + (1-\lambda)y$ with $0 < \lambda < 1$. Setting $c := \lambda a + (1-\lambda)b$ it follows from $(1-\lambda)b \prec 0$ and $\lambda a \not\succ 0$ that $c \not\succeq 0$. From (β) there exists $d \in t(\xi)$ such that $d \preceq c$, hence $d \not\succeq 0$. Thus $t(\xi) \not\succeq 0$, and (v) is true. Both (α) and (β) reduce to P-convexity of t, if t is single-valued.

A mapping F which satisfies condition (iv) of Corollary 2 was termed *pseudomonotone* in [3]. On the other hand, condition (v) of Theorem 1, adapted to the setting of Corollary 2, requires for each $x \in A$ that

$$F(y, x) \not\succ 0 \quad \forall y \in A \quad \text{implies} \quad F(x, y) \not\prec 0 \quad \forall y \in A \ .$$

We propose to call this property *maximal pseudomonotone.*

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