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AN ITERATIVE METHOD FOR GENERALIZED VARIATIONAL INEQUALITIES

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Abstract. In this paper we introduce notions of continuity and contraction for multivalued mappings on Banach spaces. Applying this notion of contraction we prove a fixed point theorem for multivalued mappings in the setting of Banach spaces. An algorithm (iterative scheme) is given to obtain the approximate solution of a class of generalized variational inequalities. It is proved that the approximate solution obtained by the iterative scheme converges strongly to the exact solution. As a special case, we obtain the corresponding iterative scheme for variational inequalities.

1. Introduction. The study of variational inequalities has become a rich source of inspiration in both Physical and Engineering sciences. Variational inequalities are a tool of vital importance in studying the existence of solutions of constrained problems arising in mechanics, optimization and control, operations research, engineering sciences, etc. Variational inequalities have been extended and generalized to study a wide class of problems in several areas of economics, science and technology. For a comprehensive account of this field we refer to important monographs of Duvaut and Lions [7], Glowinski, Lions and Trémoliéres [10], Kinderlehrer and Stampacchia [12], Barbu [2], Friedman [9] and Cottle, Giannessi and Lions [6]. The convex set in a variational inequality does not depend upon its solution. If in a variational inequality formulation, the convex set does depend upon the solution then this class of variational inequalities is called a quasi-variational inequality. These were introduced and studied by Bensoussan, Goursat and Lions [3]. For further details we refer Bensoussan, Lions [4] and Baiocchi and Capelo [1]. Noor [14] and Noor and Noor [16] have developed an iterative scheme to obtain the approximate solution of a class of variational and quasi variational inequalities. They have shown that the approximate solution obtained by the iterative scheme converges strongly in the Hilbert space to the exact solution. The study of generalized variational inequalities, where the operator involved in the formulation of a variational inequality is replaced by a multivalued mapping, has been pursued by Browder [5], Rockafellar [17], Saigal [18] and Fang and Peterson [8].

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In the present paper we extend the result of Fang and Peterson [8, pp. 382] for multivalued mappings. More pricesely we introduce the notions of continuity and contractivity for multivalued mappings on Banach spaces and prove a fixed point theorem for such a contraction mapping. Furthermore an algorithm (iterative scheme) is given to obtain the approximate solution of a class of generalized variational inequalities.

2. Preliminaries. Let R^n be the n-dimensional Euclidean space, whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$, respectively and let K be a subset of R^n. If F is a multivalued mapping on R^n into itself, then the *generalized variational inequality* problem is to find $u \in K$ such that

$$\left. \begin{array}{l} \langle v, w-u \rangle \geqslant 0 \text{ for all } w \in K \\ \text{and } v \in F(u). \end{array} \right\} \qquad (2.1)$$

If F is a single valued mapping then (2.1) is known as variational inequality.

Definition 2.1. A multivalued mapping F on R^n into itself is said to be *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle w-z, u-v \rangle \geqslant \alpha \|u-v\|^2 \text{ for all } w \in F(u) \text{ and } z \in F(v).$$

Lemma 2.1. [12]. *Let K be a closed convex subset of R^n. Given $w \in R^n$, $u \in K$ satisfies*

$$\langle u-w, v-u \rangle \geqslant 0, \text{ for all } v \in K \qquad (2.2)$$

if and only if

$$u = P_K w, \qquad (2.3)$$

where P_K is the projection of R^n into K.

Lemma 2.2 [12]. *P_K defined by (2.3) is non-expansive, i. e.*

$$\| P_K u - P_K v \| \leqslant \| u-v \|, \text{ for all } u, v \in R^n.$$

Lemma 2.3 [8]. *Let K be a closed convex subset of R^n. Then (u, v) is a solution to the generalized variational inequality (2.1) if and only if*

$$u = P_K(u - \rho v) \text{ and } v \in F(u), \qquad (2.4)$$

for some positive constant ρ.

3. Fixed Point Theorem.

Definition 3.1. Let F be a multivalued mapping on a Banach space X into another Banach space Y. F is called *continuous* at a point $x_0 \in X$ if, for every $\epsilon > 0$

there exists $\delta > 0$ such that

$$\|u - v\|_Y < \epsilon \quad \text{for all} \quad u \in F(x) \quad \text{and} \quad v \in F(y),$$

$$\text{whenever} \quad \|x - y\|_X < \delta .$$

Definition 3. 2. Let X and Y be two Banach spaces and F be a multivalued mapping on X into Y. Then

(i) F is called *Lipchitz continuous* if

$$\|u - v\|_Y \leqslant \beta \|x - y\|_X \quad \text{for all} \ u \in F(x) \ \text{and} \ v \in F(y),$$

where $\beta \geqslant 0$ a fixed real number.

(ii) F is called *contraction* if the positive real number $\beta < 1$.

(iii) F is called *nonexpansive* if $\beta = 1$.

Remark 3. 1. Every contraction multivalued mapping is continuous.

Definition 3. 3. Let $\{A_n\}$ be a sequence of sets in X. We say that A_n *converges to* A if for every $\epsilon > 0$ there exists a positive integer N such that

$$\|x_n - x\| < \epsilon \ \text{for all} \ x_n \in A_n \ \text{and} \ x \in A, \ \text{and for all} \ n \geqslant N.$$

Lemma 3. 1. *A multivalued mapping* F *on* X *into* Y *is continuous at a point* x_0 *if and only if* $F(x_n) \to F(x_0)$ *for all sequences* $\{x_n\}$ *in* X *with* $x_n \to x_0$.

Proof. Suppose that F is continuous at x_0 and $\{x_n\}$ is a sequence such that $x_n \to x_0$. Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - x\|_X < \delta \Rightarrow \|u - v\|_Y < \epsilon \ \text{for all} \ u \in F(x_0) \ \text{and} \ v \in F(x),$$

and there exists a positive integer N such that

$$\|x_0 - x_n\| < \delta \ \text{for all} \ n \geqslant N.$$

Thus, if $n \geqslant N$ we have

$$\|u_n - u\| < \epsilon \ \text{for all} \ u_n \in F(x_n) \ \text{and} \ u \in F(x).$$

Therefore $F(x_n) \to F(x)$.

Conversely, suppose that $F(x_n) \to F(x)$ i. e. if $n \geqslant N$,

$$\|u_n - u\| < \epsilon \ \text{for all} \ u_n \in F(x_n) \ \text{and} \ u \in F(x), \ \text{and} \ \|x_n - x\| < \delta .$$

Now, suppose that F is not continuous at x_0. Then there exists $\epsilon > 0$, for each $\delta > 0$ there exists $x \in X$ such that

$$\|x_0 - x\| < \delta \Rightarrow \|u - v\| > \epsilon \text{ for all } u \in F(x_0) \text{ and } v \in F(x).$$

In particular for each positive integer n there exists $x_n \in X$ such that

$$\|x_0 - x_n\| < \frac{1}{n} \text{ and } \|u_n - u_0\| > \epsilon \text{ for all } u_n \in F(x_n) \text{ and } u_0 \in F(x_0).$$

Clearly $x_n \to x_0$ but $F(x_n) \not\to F(x_0)$, which is a contradiction. This proves the lemma.

Definition 3.4. If F is a multivalued mapping on X into itself then a point $x \in X$ is called a *fixed point* of F if $x \in F(x)$.

Now we prove a fixed point theorem for a multivalued mapping on a Banach space, which is a natural generalization of Banach Contraction Theorem.

Theorem 3.1. *Let X be a Banach space. If F is a multivalued contraction mapping on X into itself then F has a fixed point.*

Proof. Let $\beta < 1$ be a Lipchitz constant for F, and let $x_0 \in X$. Choose $x_1 \in F(x_0)$. Since $F(x_0)$ and $F(x_1)$ are subsets of X and $x_1 \in F(x_0)$. There is a $x_2 \in F(x_1)$ such that

$$\|x_1 - x_2\| \leqslant \beta \|x_0 - x_1\|.$$

Now since $F(x_1)$ and $F(x_2)$ are subsets of X and $x_2 \in F(x_1)$, there is a point $x_3 \in F(x_2)$ such that

$$\|x_2 - x_3\| \leqslant \beta \|x_1 - x_2\| \leqslant \beta^2 \|x_0 - x_1\|.$$

Continuing in this fashion, we produce a sequence $\{x_n\}$ of points of X such that $x_{n+1} \in F(x_n)$ and

$$\begin{aligned}\|x_n - x_{n+1}\| &\leqslant \beta \|x_{n-1} - x_n\| \\ &\leqslant \beta^n \|x_0 - x_1\|, \text{ for all } n \geqslant 1.\end{aligned}$$

Now,

$$\begin{aligned}\|x_n - x_{n+m}\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots\cdots + \|x_{n+m-1} - x_{n+m}\| \\ &\leqslant \beta^n \|x_0 - x_1\| + \beta^{n+1} \|x_0 - x_1\| + \cdots\cdots + \beta^{n+m-1} \|x_0 - x_1\| \\ &= (\beta^n + \beta^{n+1} + \cdots + \beta^{n+m-1}) \|x_0 - x_1\| \\ &\leqslant (\beta^n \sum_{i=1}^{\infty} \beta^i) \|x_0 - x_1\|, \text{ for all } n, m \geqslant 1 \\ &= \left(\frac{\beta^n}{1-\beta}\right) \|x_0 - x_1\|, \text{ for all } n, m \geqslant 1.\end{aligned}$$

If $m, n \to \infty$, then the sequence $\{x_n\}$ is a Cauchy sequence. Since X is a Banach space, the sequence $\{x_n\}$ converges to some point $p_0 \in X$. Therefore the sequence $\{F(x_n)\}$ converges to $F(p_0)$ and, since $x_n \in F(x_{n-1})$ for all n, it follows that $p_0 \in F(p_0)$.

4. Generalized Variational Inequalities.

Algorithm 4. 1. For any given $u_0 \in K \subseteq R^n$, compute

$$\left.\begin{aligned} &u_{n+1} = P_K(u_n - \rho v_n) \\ &\text{and } v_{n+1} \in F(u_{n+1}). \end{aligned}\right\} \tag{4.1}$$

for some positive constnat ρ.

Now we show that the approximate solution (u_n, v_n) obtained from the iterative scheme (4. 1) does converge strongly to (u, v), the exact solution of (2. 1), under certain conditions on the constant ρ.

Theorem 4. 1. *Let K be a closed convex subset of R^n and F be a strongly monotone and Lipchitz continuous multivalued mapping on R^n into itself. If (u, v) and (u_{n+1}, v_{n+1}) are solution of (2. 1) and (4. 1), respectively, then*

u_{n+1} converges to u strongly in R^n,

and v_{n+1} converges to v strongly in R^n,

for

$$0 < \rho < \frac{2\alpha}{\beta^2}.$$

Proof. From Lemma 2.3, we know that (u, v) satisfying (2. 1) is also a solution of (2. 4) and conversely. Thus from (2. 4) and (4. 1), we obtain

$$u_{n+1} - u = P_K(u_n - \rho v_n) - P_K(u - \rho v).$$

By using Lemma 2. 2, we get

$$\begin{aligned} \| u_{n+1} - u \| &= \| P_K(u_n - \rho v_n) - P_K(u - \rho v) \| \\ &\leq \| u_n - \rho v_n - u + \rho v \| = \| u_n - u - \rho(v_n - v) \|. \end{aligned}$$

Now by using Lipschitz continuity and strong monotonicity, we get

$$\begin{aligned} \| u_n - u - \rho(v_n - v) \|^2 &= \| u_n - u \|^2 + \rho^2 \| v_n - v \|^2 - 2\rho \langle v_n - v, u_n - u \rangle \\ &\leq \| u_n - u \|^2 + \rho^2 \beta^2 \| u_n - u \|^2 - 2\rho\alpha \| u_n - u \|^2 \\ &= (1 + \rho^2 \beta^2 - 2\rho\alpha) \| u_n - u \|^2. \end{aligned}$$

Therefore,

$$\|u_{n+1} - u\| \leqslant \sqrt{(1+\rho^2\beta^2 - 2\rho\alpha)}\,\|u_n - u\| = \theta\,\|u_n - u\|,$$

where $\theta = \sqrt{(1+\rho^2\beta^2 - 2\rho\alpha)} < 1$ for $0 < \rho < \dfrac{2\alpha}{\beta^2}$.

Applying this relation n times, we get $\|u_{n+1} - u\| < \theta^n \|u_1 - u\|$. Since $\theta < 1$, it follows that u_{n+1} converges strongly to u in R^n. Since F is Lipschitz continuous and $u_{n+1} \to u$ strongly, we have $v_{n+1} \to v$ strongly.

Remark 4. 1. This theorem includes as a special case a result of Fang and Peterson [8, p.382].

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