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System of Vector Equilibrium Problems and Its Applications[1]

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Abstract. In this paper, we introduce a system of vector equilibrium problems and prove the existence of a solution. As an application, we derive some existence results for the system of vector variational inequalities. We also establish some existence results for the system of vector optimization problems, which includes the Nash equilibrium problem as a special case.

Key Words. System of vector equilibrium problems, system of vector variational inequalities, system of vector optimization problems, Nash equilibrium problem, fixed points.

1. Introduction and Preliminaries

In 1980, Giannessi (Ref. 1) extended classical variational inequalities to the case of vector-valued functions. Meanwhile, vector variational inequalities have been researched quite extensively; for example, see Ref. 2 and references therein. Inspired by the study of vector variational inequalities, more general equilibrium problems (Refs. 3–4) have been extended to the case of vector-valued bifunctions, known as vector equilibrium problems; see for example Refs. 2 and 5–13.

In this paper, we introduce the system of vector equilibrium problems, that is, a family of equilibrium problems with vector-valued bifunctions defined on a product set, and we prove the existence of solutions for such problems. A special case of a system of vector equilibrium problems, a

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system of (scalar) variational inequalities, was considered earlier by Pang (Ref. 14). He showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Later, this model was studied also by Cohen and Chaplais (Ref. 15) and by Bianchi (Ref. 16). Existence results were derived in Ref. 16 assuming pseudomonotonicity extended to product sets.

In the first part of the present paper, we establish the existence of a solution for the considerably more general problem of a system of vector equilibrium problems (Section 2). Then, in the second part (Section 3), we specialize our results to a system of vector variational inequalities and to a system of vector optimization problems. In the latter case, we obtain also existence results for the Nash equilibrium problem with vector-valued functions, since a solution of a system of vector optimization problems is also a solution of the Nash equilibrium problem.

Let I be an index set; for each $i\in I$, let X_i be a Hausdorff topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i. Let

$$K=\prod_{i\in I} K_i \quad \text{and} \quad X=\prod_{i\in I} X_i.$$

Let Y be a Hausdorff topological vector space, and let C be a nonempty pointed closed convex cone in Y with $\operatorname{int} C\neq\emptyset$, where $\operatorname{int} C$ denotes the interior of C. The cone C induces a partial ordering $\leqslant$ on Y defined by $x\leqslant y$ if and only if $y-x\in C$. Let $\{f_i\}_{i\in I}$ be a family of bifunctions defined on $K\times K_i$ with values in Y. We consider the system of vector equilibrium problems (in short, SVEP), which is to find $\bar{x}\in K$ such that, for each $i\in I$,

$$\text{(SVEP)} \quad f_i(\bar{x}, y_i)\notin -\operatorname{int} C, \qquad \text{for all } y_i\in K_i.$$

If the index set I is a singleton, then the (SVEP) reduces to a vector equilibrium problem studied in Refs. 2 and 5–13, which includes vector variational inequalities as a special case; see for example Refs. 1–2 and the references therein.

Let M be a nonempty convex subset of a topological vector space Z. The function $\Phi: M\to Y$ is called C-quasiconcave (Refs. 17–18) if, for all $\alpha\in Y$, the set $\{x\in M: \Phi(x)-\alpha\in C\}$ is convex. It is called C-quasiconvex if $-\Phi$ is C-quasiconcave.

A function ξ: $Y-\mathbb{R}$ is said to be monotonically increasing [respectively, strictly monotonically increasing] with respect to C (Ref. 17) if $\xi(a)\geq\xi(b)$, for all $a-b\in C$ [respectively, $\xi(a)>\xi(b)$, for all $a-b\in \operatorname{int} C$].

For any fixed $a \in Y$ and $e \in \operatorname{int} C$, the set

$$\zeta_{e,a}(y) = \{t \in \mathbb{R}: y \in a + te + C\}$$

is nonempty closed and bounded for each $y \in Y$; see for example Refs. 17 and 19. Hence, we can define real-valued functions $\xi_{e,a}$ and $\xi'_{e,a}: Y \to \mathbb{R}$ by

$$\xi_{e,a}(y) = \max\{t \in \mathbb{R}: y \in a + te + C\},$$

$$\xi'_{e,a}(y) = \min\{t \in \mathbb{R}: y \in a + te - C\},$$

for all $y \in Y$. These functions have the following properties.

Lemma 1.1. See Ref. 17.

(i) The functions $\xi_{e,a}$ and $\xi'_{e,a}$ are continuous and strictly monotonically increasing with respect to C.
(ii) A function $p: K \to Y$ is C-quasiconcave [respectively, C-quasiconvex] if and only if the composite mapping $\xi_{e,a} \circ p: K \to \mathbb{R}$ is $\mathbb{R}_+$-quasiconcave [respectively, $\xi'_{e,a} \circ p: K \to \mathbb{R}$ is $\mathbb{R}_+$-quasiconvex], where $\mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\}$.

We need the following result to prove the main result of this paper.

Lemma 1.2. See Ref. 20. Let X and Y be Hausdorff topological vector spaces; and let Y be compact. Let g be a real-valued function defined on $X \times Y$ such that:

(i) g is lower semicontinuous on $X \times Y$;
(ii) for each fixed $y \in Y$, the function $x \mapsto g(x, y)$ is upper semicontinuous on X.

Then, the function $\Psi: X \to \mathbb{R}$ defined by

$$\Psi(x) = \min_{y \in Y} g(x, y), \qquad \text{for each } x \in X,$$

is continuous on X.

Let Z be a topological vector space, and let $F: Z \to 2^Y$ be a multivalued map, where 2^Y denotes the family of all subsets of Y. The inverse F^{-1} of F is the multivalued map from $\mathscr{R}(F)$, the range of F, to Z defined by

$$z \in F^{-1}(y), \qquad \text{if and only if } y \in F(z).$$

We shall use the following particular form of a fixed-point theorem given in Ref. 21 to prove the existence of a solution of the (SVEP).

Theorem 1.1. For each $i \in I$, let $S_i: K \to 2^{K_i}$ be a multivalued map. Assume that the following conditions hold:

(i) For each $i \in I$ and each $x \in K$, $S_i(x)$ is nonempty and convex.
(ii) For each $i \in I$, $K = \cup\{\text{int}_K S_i^{-1}(x_i): x_i \in K_i\}$.
(iii) If K is not compact, assume that there exist a nonempty compact convex subset B_i of K_i for each $i \in I$ and a nonempty compact subset D of K such that, for each $x \in K \backslash D$, there exists $\tilde{y}_i \in B_i$ such that $x \in \text{int}_K S_i^{-1}(\tilde{y}_i)$ for each $i \in I$.

Then, there exists $\bar{x} \in K$ such that

$$\bar{x} \in S(\bar{x}) = \prod_{i \in I} S_i(\bar{x}),$$

that is,

$$\bar{x}_i \in S_i(\bar{x}), \qquad \text{for each } i \in I,$$

where $\bar{x}_i$ is the projection of $\bar{x}$ onto K_i.

2. Existence Results

An element of the set

$$X^i = \prod_{j \in I, j \neq i} X_j$$

will be represented by x^i; therefore, $x \in X$ can be written as

$$x = (x^i, x_i) \in X^i \times X_i.$$

Theorem 2.1. For each $i \in I$, let K_i be a nonempty compact convex subset of X_i, and let $f_i: K \times K_i \to Y$ be a bifunction such that $f_i(x, x_i) = 0$, for all $x = (x^i, x_i) \in K$. Assume that the following conditions are satisfied.

(i) For each $i \in I$ and for all $x \in K$, the function $y_i \mapsto f_i(x, y_i)$ is C-quasiconvex.
(ii) For each $i \in I$, f_i is continuous on $K \times K_i$.

Then, the solution set of the (SVEP) is nonempty and compact.

Proof. For fixed $a \in Y$ and $e \in \text{int } C$, we define a real-valued function $\xi: Y \to \mathbb{R}$ by

$$\xi(y) = \min\{t \in \mathbb{R}: y \in a + te - C\},$$

for all $y \in Y$. Then, by Lemma 1.1, ξ is strictly monotonically increasing and continuous, and for each $i \in I$, the function $y_i \mapsto \xi \circ f_i(x, y_i)$ is $\mathbb{R}_+$-quasiconvex. From assumption (ii), we see that, for each $i \in I$, $\xi \circ f_i$ is continuous on $K \times K_i$.

For each $i \in I$ and each $n_i = 1, 2, \ldots$, we define the multivalued map $S_{i,n_i}: K \to 2^{K_i}$ by

$$S_{i,n_i}(x) = \left\{ y_i \in K_i : \xi \circ f_i(x, y_i) < \min_{z_i \in K_i} \xi \circ f_i(x, z_i) + 1/n_i \right\},$$

for all $x \in K$. Then, for each $i \in I$ and for all $x \in K$, $S_{i,n_i}(x)$ is nonempty because $\xi \circ f_i$ is continuous on $K \times K_i$ and each K_i is compact. Since for each $i \in I$, the function $y_i \mapsto \xi \circ f_i(x, y_i)$ if $\mathbb{R}_+$-quasiconvex, $S_{i,n_i}(x)$ is convex for all $x \in K$.

From Lemma 1.2 and assumption (ii), we see that the set

$$S^{-1}_{i,n_i}(y_i) = \left\{ x \in K : \xi \circ f_i(x, y_i) < \min_{z_i \in K_i} \xi \circ f_i(x, z_i) + 1/n_i \right\}$$

is open in K for each $i \in I$ and for all $y_i \in K_i$. Since $S_{i,n_i}(x)$ is nonempty for each $i \in I$ and for all $x \in K$, we have

$$K = \bigcup_{y_i \in K_i} S^{-1}_{i,n_i}(y_i) = \bigcup_{y_i \in K_i} \operatorname{int}_K S^{-1}_{i,n_i}(y_i).$$

Hence, by Theorem 1.1, there exists $x^*_{n_i} \in K$ such that, for each

$$x^*_{i,n_i} \in S_{i,n_i}(x^*_{n_i}), \qquad \text{for all } n_i = 1, 2, \ldots,$$

that is,

$$\xi \circ f_i(x^*_{n_i}, x^*_{i,n_i}) < \min_{z_i \in K_i} \xi \circ f_i(x^*_{n_i}, z_i) + 1/n_i, \qquad \text{for all } n_i = 1, 2, \ldots .$$

Since for each $i \in I$, K_i is compact, we may assume that $x^*_{n_i} \to \bar{x}$, that is

$$x^*_{i,n_i} \to \bar{x}_i \in K_i \qquad \text{for each } i \in I.$$

Therefore,

$$\lim_{n_i \to \infty} \xi \circ f_i(x^*_{n_i}, x^*_{i,n_i}) \le \min_{z_i \in K_i} \lim_{n_i \to \infty} \xi \circ f_i(x^*_{n_i}, z_i).$$

Hence, for each $i \in I$,

$$\xi \circ f_i(\bar{x}, \bar{x}_i) = \min_{z_i \in K_i} \xi \circ f_i(\bar{x}, z_i).$$

Since ξ is strictly monotonically increasing with respect to C, we have

$$f_i(\bar{x}, \bar{x}_i) - f_i(\bar{x}, z_i) \notin \operatorname{int} C, \qquad \text{for all } z_i \in K_i.$$

Since for each $i \in I$ and for all $x = (x^i, x_i) \in K$,

$$f_i(x, x_i) = 0,$$

we have that, for each $i \in I$,

$$f_i(\bar{x}, z_i) \notin -\operatorname{int} C, \qquad \text{for all } z_i \in K_i.$$

By (ii), the solution set of the (SVEP) is a closed subset of a compact set K, and hence it is compact. □

In case K_i is not necessarily compact, we have the following result.

Theorem 2.2. For each $i \in I$, let K_i be a nonempty convex subset of X_i and let $f_i : K \times K_i \to Y$ be a bifunction such that $f_i(x, x_i) = 0$, for all $x = (x^i, x_i) \in K$. Assume that the following conditions are satisfied:

(i) For each $i \in I$ and for all $x \in K$, the function $y_i \mapsto f_i(x, y_i)$ is C-quasiconvex.

(ii) For each $i \in I$, f_i is continuous on each compact convex subset of $K \times K_i$.

(iii) For each $i \in I$, there exists a nonempty compact convex subset B_i of K_i; let $B = \prod_{i \in I} B_i \subset K$ such that, for each $x \in K \setminus B$, there exists $\tilde{y}_i \in B_i$ such that

$$f_i(x, \tilde{y}_i) \in -\operatorname{int} C.$$

Then, there exists a solution $\bar{x} \in B$ of the (SVEP).

Proof. For each $i \in I$, let $\{y_{i_1}, \ldots, y_{i_k}\}$ be a finite subset of K_i. Let

$$Q_i = \operatorname{co}(B_i \cup \{y_{i_1}, \ldots, y_{i_k}\}),$$

where $\operatorname{co}(M)$ denotes the convex hull of M. Then, for each $i \in I$, Q_i is compact and convex. By Theorem 2.1, there exists $\bar{x} \in Q = \prod_{i \in I} Q_i$ such that, for each $i \in I$,

$$f_i(\bar{x}, y_i) \notin -\operatorname{int} C, \qquad \text{for all } y_i \in Q_i.$$

From assumption (iii), $\bar{x} \in B$. In particular, we have $\bar{x} \in B$ such that, for each $i \in I$,

$$f_i(\bar{x}, y_{i_k}) \notin -\operatorname{int} C, \qquad \text{for all } k.$$

Since B is compact and convex, by (ii) we have that, for each $i \in I$ and for all $y_i \in K_i$,

$$G(y_i) = \{x \in B : f_i(x, y_i) \notin -\operatorname{int} C\}$$

is closed in B. As shown above, every finite subfamily of closed sets $G(y_i)$ has a nonempty intersection. Since B is compact, for each $i \in I$,

$$\bigcap_{y_i \in K_i} G(y_i) \neq \emptyset.$$

Thus, there exists $\bar{x} \in B$ such that, for each $i \in I$,

$$f_i(\bar{x}, y_i) \notin -\text{int}\, C, \qquad \text{for all } y_i \in K_i.$$

This completes the proof. □

Remark 2.1. Let I be a finite index set and, for each $i \in I$, let X_i be a reflexive Banach space with norm $\|\cdot\|_i$ equipped with the weak topology. Consider a Banach space Y equipped with the norm topology. The norm on $X = \prod_{i \in I} X_i$ will be denoted by $\|\cdot\|$. Then, assumption (iii) in Theorem 2.2 can be replaced by the following condition:

(iii)′ There exists $r > 0$ such that, for all $x \in K$, $\|x\| > r$, there exists $\tilde{y}_i \in K_i$, $\|\tilde{y}_i\|_i \leq r$, such that

$$f_i(x, \tilde{y}_i) \in -\text{int}\, C.$$

Proof. Define

$$B_i^r = \{x_i \in K_i : \|x_i\|_i \leq r\}.$$

Then, B_i^r is a nonempty compact convex subset of X_i. By taking $B_i = B_i^r$, we obtain the conclusion. □

3. Applications

For each $i \in I$, let $\varphi_i : K \to Y$ be a given function. The system of vector optimization problems (in short, SVOP) is to find $\bar{x} \in K$ such that, for each $i \in I$,

$$\text{(SVOP)} \quad \varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int}\, C, \qquad \text{for all } y \in K.$$

We can choose $y \in K$ in such a way that $y^i = \bar{x}^i$. Then, we have the Nash equilibrium problem for vector-valued functions, which is to find $\bar{x} \in K$ such that, for each $i \in I$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}\, C, \qquad \text{for all } y_i \in K_i.$$

It is clear that every solution of the (SVOP) is also a solution of the Nash equilibrium problem for vector-valued functions. But the converse is not true.

For each $i \in I$, let $A_i: K \to L(X_i, Y)$ be a given map, where $L(X_i, Y)$ denotes the space of all continuous linear operators from X_i into Y. Then, we consider the system of vector variational inequalities (in short, SVVI), which is to find $\bar{x} \in K$ such that, for each $i \in I$,

$$\text{(SVVI)} \quad \langle A_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \qquad \text{for all } y_i \in K_i,$$

where $\langle s, x_i \rangle$ denotes the evaluation of $s \in L(X_i, Y)$ at $x_i \in X_i$.

If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then the (SVVI) becomes the system of variational inequalities studied before in Refs. 14–16 and 21.

In case the index set I is a singleton, the (SVVI) reduces to a vector variational inequality first considered in Ref. 1; see also Ref. 2 and references therein.

In the following, we make use of a result given in Ref. 22.

Lemma 3.1. Let E and Y be topological vector spaces, and let $L(E, Y)$ be equipped with the uniform convergence topology δ; see Ref. 22, pp. 79–81. Then, the bilinear form $\langle \cdot, \cdot \rangle: L(E, Y) \times E \to Y$ is continuous on $(L(E, Y), \delta) \times E$.

Throughout this section, we shall assume that, for each $i \in I$, $L(X_i, Y)$ is equipped with the uniform convergence topology.

The following existence results for the (SVVI) can be derived easily from Theorems 2.1 and 2.2.

Theorem 3.1. For each $i \in I$, let K_i be a nonempty compact convex subset of X_i; and let A_i be continuous on K. Then, there exists a solution $\bar{x} \in K$ of the (SVVI).

Theorem 3.2. For each $i \in I$, let K_i be a nonempty convex subset of X_i; and let A_i be continuous on each compact convex subset of K. Assume that, for each $i \in I$, there exists a nonempty compact convex subset B_i of K_i; and let $B = \prod_{i \in I} B_i \subset K$ such that, for each $x \in K \backslash B$, there exists $\tilde{y}_i \in B_i$ such that $\langle A_i(x), \tilde{y}_i - x_i \rangle \in -\operatorname{int} C$. Then, there exists a solution $\bar{x} \in B$ of the (SVVI).

In case I is a singleton, we have the following result.

Corollary 3.1. Let K be a nonempty convex subset of a Hausdorff topological vector space E; and let $A: K \to L(E, Y)$ be continuous on each compact convex subset of K. Assume that there exists a nonempty compact convex subset B of K such that, for each $x \in K \backslash B$, there exists $\tilde{y} \in B$ such that $\langle A(x), \tilde{y} - x \rangle \in -\operatorname{int} C$. Then, there exists $\bar{x} \in B$ such that

$$\langle A(\bar{x}), y - \bar{x} \rangle \notin -\operatorname{int} C, \qquad \text{for all } y \in K.$$

To prove the existence of a solution of the (SVOP), we introduce the following concept. We call the function $\phi_i: K \to Y$ differentiable on K_i if the set

$$\partial\phi_i(x) = \{T_i \in L(X_i, Y): \phi_i(y) - \phi_i(x) \geqslant T_i(y_i - x_i), \text{ for all } y \in K\}$$

is a singleton denoted by $D\phi_i(x)$. When the index set I is a singleton, the above definition is the same as in Ref. 13.

Proposition 3.1. For each $i \in I$, let $\varphi_i: K \to Y$ be differentiable on K_i. Then, any solution of the (SVVI) is a solution of the (SVOP) with $A_i(x) = D\varphi_i(x)$, for all $x \in K$.

Proof. Suppose that $\bar{x} \in K$ is a solution of the (SVVI), but not a solution of the (SVOP). Then, for some $i \in I$, there exists a point $\hat{y} \in K$ such that

$$\varphi_i(\hat{y}) - \varphi_i(\bar{x}) \in -\text{int } C.$$

Since φ_i is differentiable on K_i, we have

$$\varphi_i(\hat{y}) - \varphi_i(\bar{x}) - D\varphi_i(\bar{x})(\hat{y}_i - \bar{x}_i) \in C.$$

Hence, we have

$$D\varphi_i(\bar{x})(\hat{y}_i - \bar{x}_i) \in -\text{int } C,$$

which contradicts our assumption. This proves the result. □

From Theorems 3.1 and 3.2 and from Proposition 3.1, we have the following existence results for the (SVOP) and hence for the Nash equilibrium problem for vector-valued functions.

Theorem 3.3. For each $i \in I$, let K_i be a nonempty compact convex subset of X_i; and let $\varphi_i: K \to Y$ be differentiable on K_i such that $D\varphi_i$ is continuous on K. Then, there exists a solution $\bar{x} \in K$ of the (SVOP).

Theorem 3.4. For each $i \in I$, let K_i be a nonempty convex subset of X_i; and let $\varphi_i: K \to Y$ be differentiable on K_i such that $D\varphi_i$ is continuous on K. Assume that, for each $i \in I$, there exists a nonempty compact convex subset B_i of K_i,; and let $B = \prod_{i \in I} B_i \subset K$ such that, for each $x \in K \setminus B$, there exists $\tilde{y}_i \in B_i$ such that $\langle D\varphi_i(x), \tilde{y}_i - x_i \rangle \in -\text{int } C$. Then, there exists a solution $\bar{x} \in B$ of the (SVOP).

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