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General KKM Theorem with Applications to Minimax and Variational Inequalities[1]

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Abstract. In this paper, a general version of the KKM theorem is derived by using the concept of generalized KKM mappings introduced by Chang and Zhang. By employing our general KKM theorem, we obtain a general minimax inequality which contains several existing ones as special cases. As applications of our general minimax inequality, we derive an existence result for saddle-point problems under general setting. We also establish several existence results for generalized variation inequalities and generalized quasi-variational inequalities.

Key Words. Generalized KKM mappings, transfer closed-valued mappings, γ-transfer lower semi-continuous functions, minimax inequalities, generalized variational inequalities, generalized quasi-variational inequalities.

1. Introduction

Let E be a Hausdorff topological vector space, and let X be a nonempty subset of E. A multivalued mapping $F: X \to 2^E$ is called a KKM-map if $\operatorname{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ for each finite subset $\{x_1, \ldots, x_n\} \subset X$, where $\operatorname{co}\{x_1, \ldots, x_n\}$ denotes the convex hull of the set $\{x_1, \ldots, x_n\}$. In Ref. 1, Fan proved the following celebrated lemma which asserts that, given an

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arbitrary set X in E and a KKM mapping $F: X \to 2^E$, if F has closed values and $F(x)$ is compact for at least one $x \in X$, then $\bigcap_{x \in X} F(x) \neq \emptyset$. This lemma generalizes a classical finite-dimensional result of Knaster, Kuratowski, and Mazurkiewicz (Ref. 2). Since then, many results in this direction have been obtained; see for example Refs. 3–8. The purpose of this paper is to derive a more general version of the KKM theorem by using the concept of generalized KKM mappings introduced by Chang and Zhang (Ref. 7). Then, by employing our general KKM theorem, we obtain a general minimax inequality, which contains several existing ones as special cases. As applications of our general minimax inequality, we derive an existence result for saddle-point problems under general setting. We also establish some existence results for generalized variational inequalities and generalized quasi-variational inequalities.

2. General KKM Theorem

First, we recall the following definition due to Chang and Zhang (Ref. 7).

Definition 2.1. Let X be a nonempty convex subset of a topological vector space E. A multivalued mapping $F: X \to 2^E$ is called a generalized KKM mapping if, for any finite set $\{x_1, \ldots, x_n\} \subset X$, there is a finite subset $\{y_1, \ldots, y_n\} \subset E$ such that, for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$, we have

$$\mathrm{co}\{y_{i_1}, \ldots, y_{i_k}\} \subset \bigcup_{j=1}^{k} F(x_{i_j}).$$

In proving the main result of this section, we need the following result from Ref. 7.

Lemma 2.1. See Ref. 7, Theorem 3.1. Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $F: X \to 2^E$ be a multivalued mapping such that, for each $x \in X$, $F(x)$ is finitely closed; that is, for every finite-dimensional subspace L in E, $F(x) \cap L$ is closed in the Euclidean topology in L. Then, the family of sets $\{F(x): x \in X\}$ has the finite intersection property if and only if $F: X \to 2^E$ is a generalized KKM mapping.

Definition 2.2. See Ref. 9. Let Y and Z be two topological spaces. A multivalued mapping $F: Y \to 2^Z$ is said to be transfer closed-valued on Y if, for every $x \in Y$, $y \notin F(x)$, there exists an element $x' \in Y$ such that $y \notin \overline{F(x')}$, where $\bar{A}$ denotes the closure of a subset A of a topological space.

It has been shown in Refs. 9–10 that F is a transfer closed-valued mapping if and only if

$$\bigcap_{x\in Y} F(x) = \bigcap_{x\in Y} \overline{F(x)}.$$

Now, we state and prove the main result of this section which will be used in the sequel.

Theorem 2.1. Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $F: X\to 2^E$ be a transfer closed-valued mapping such that $\overline{F(x_0)} =: K$ is compact for at least one $x_0\in X$, and let $\bar{F}: X\to 2^E$ be a generalized KKM mapping. Then,

$$\bigcap_{x\in X} F(x)\neq\emptyset.$$

Proof. Since $\bar{F}: X\to 2^E$ is defined by $\bar{F}(x) = \overline{F(x)}$, for each $x\in X$, we have that $\bar{F}$ is a generalized KKM mapping with closed values. By Lemma 2.1, the family of sets $\{\overline{F(x)}: x\in X\}$ has the finite intersection property. Since $\overline{F(x_0)}$ is compact, we have

$$\bigcap_{x\in X} \overline{F(x)}\neq\emptyset.$$

Since F is a transfer closed-valued mapping,

$$\bigcap_{x\in X} F(x) = \bigcap_{x\in X} \overline{F(x)}\neq\emptyset. \qquad \square$$

Remark 2.1. Theorem 2.1 generalizes Theorem 3.2 in Ref. 7 in several ways.

3. General Minimax Inequality

To state the main result of this section, we recall some definitions. Let E be a topological vector space, and let X be a nonempty convex subset of E.

A function $\phi: X\to(-\infty, +\infty)$ is said to be quasiconvex if, for each $\lambda\in(-\infty, +\infty)$, the set $\{x\in X: \phi(x)\leq\lambda\}$ is convex; ϕ is said to be quasiconcave if $-\phi$ is quasiconvex. We note that ϕ is quasiconcave [resp., quasiconvex] if and only if, for each $\lambda\in(-\infty, +\infty)$, the set $\{x\in X: \phi(x)>\lambda\}$ [resp., $\{x\in X: \phi(x)<\lambda\}$] is convex.

A function $\phi(x, y): X\times X\to(-\infty, +\infty)$ is said to be diagonally quasiconvex in y (Ref. 11) if, for any finite subset $\{y_1, \ldots, y_n\}\subset X$ and any

$y_0 \in \mathrm{co}\{y_1, \ldots, y_n\}$, we have

$$\phi(y_0, y_0) \leq \max_{1 \leq i \leq n} \phi(y_0, y_i);$$

$\phi(x, y)$ is said to be diagonally quasiconcave in y if $-\phi(x, y)$ is diagonally quasiconvex in y.

A function $\phi(x, y): X \times X \to (-\infty, +\infty)$ is said to be γ-diagonally quasiconvex in y (Ref. 11) for some $\gamma \in (-\infty, +\infty)$ if, for any finite subset $\{y_1, \ldots, y_n\} \subset X$ and any $y_0 \in \mathrm{co}\{y_1, \ldots, y_n\}$, we have

$$\gamma \leq \max_{1 \leq i \leq n} \phi(y_0, y_i);$$

$\phi(x, y)$ is said to be γ-diagonally quasiconcave in y for some $\gamma \in (-\infty, +\infty)$ if $-\phi(x, y)$ is $-\gamma$-diagonally quasiconvex in y.

Definition 3.1. See Ref. 7. Let E be a topological vector space, and let X be a nonempty convex subset of E. A function $\phi(x, y)$: $X \times X \to (-\infty, +\infty)$ is said to be γ-generalized quasiconvex in y for some $\gamma \in (-\infty, +\infty)$ if, for any finite subset $\{y_1, \ldots, y_n\} \subset X$, there is a finite subset $\{x_1, \ldots, x_n\} \subset X$ such that, for any subset $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ and any $x_0 \in \mathrm{co}\{x_{i_1}, \ldots, x_{i_k}\}$, we have

$$\gamma \leq \max_{1 \leq j \leq k} \phi(x_0, y_{i_j});$$

$\phi(x, y)$ is said to be γ-generalized quasiconcave in y for some $\gamma \in (-\infty, +\infty)$ if $-\phi(x, y)$ is $-\gamma$-generalized quasiconvex in y.

The relation between generalized KKM mappings and γ-generalized convexity (concavity) is the following.

Proposition 3.1. See Ref. 7, Proposition 2.1. Let E be a topological vector space, and let X be a nonempty convex subset of E. Let $\phi(x, y): X \times X \to (-\infty, +\infty)$ and $\gamma \in (-\infty, +\infty)$. Then, the following conclusions are equivalent:

(i) The mapping $G: X \to 2^X$ defined by

$$G(y) = \{x \in X: \phi(x, y) \leq \gamma\} \text{ [resp., } G(y) = \{x \in X: \phi(x, y) \geq \gamma\}]$$

is a generalized KKM mapping.

(ii) $\phi(x, y)$ is γ-generalized quasiconcave [resp., quasiconvex] in y.

Let Y and Z be two topological spaces. A function $\phi(x, y)$: $Y \times Z \to (-\infty, +\infty)$ is said to be γ-transfer lower semicontinuous function in x for some $\gamma \in (-\infty, +\infty)$ (Ref. 9) if, for all $x \in Y$ and $y \in Z$ with

$\phi(x, y) > \gamma$, there exist some point $y' \in Z$ and some neighborhood $N(x)$ of x such that $\phi(z, y') > \gamma$ for all $z \in N(x)$.

Let A be a nonempty subset of a topological vector space E. A subset B of A is called precompact if B is contained in some compact subset of A.

We now state and prove the following general minimax inequality.

Theorem 3.1. Let E be a Hausdorff topological vector space, and let X be a nonempty closed convex subset of E. Let $\gamma \in (-\infty, +\infty)$ be a given number, and let $\phi, \psi: X \times X \to (-\infty, +\infty)$ satisfy the following conditions:

(i) For any fixed $y \in X$, $\phi(x, y)$ is a γ-transfer lower semicontinuous function in x.
(ii) For any fixed $x \in X$, $\psi(x, y)$ is a γ-generalized quasiconcave function in y.
(iii) $\phi(x, y) \leq \psi(x, y)$, for all $(x, y) \in X \times X$.
(iv) The set $\{x \in X: \phi(x, y_0) \leq \gamma\}$ is precompact for at least one $y_0 \in X$.

Then, there exists $\bar{x} \in X$ such that

$$\phi(\bar{x}, y) \leq \gamma, \qquad \text{for all } y \in X.$$

In particular, we have

$$\inf_{x \in X} \sup_{y \in X} \phi(x, y) \leq \gamma.$$

Proof. Define two multivalued mappings $T, G: X \to 2^X$ by

$$T(y) = \{x \in X: \psi(x, y) \leq \gamma\} \text{ and } G(y) = \{x \in X: \phi(x, y) \leq \gamma\},$$

for all $y \in X$. Condition (i) implies that G is a transfer closed-valued mapping on X. Indeed, if $x \notin G(y)$, then $\phi(x, y) > \gamma$. Since $\phi(x, y)$ is γ-transfer lower semicontinuous in x, there is a $y' \in X$ and a neighborhood $N(x)$ of x such that

$$\phi(z, y') > \gamma, \qquad \text{for all } z \in N(x).$$

Then, $G(y') \subset X \backslash N(x)$. Hence, $x \notin \overline{G(y')}$. Thus, G is transfer closed-valued. From (ii), T is a generalized KKM mapping. From condition (iii), we have that

$$T(y) \subset G(y), \qquad \text{for all } y \in X,$$

and hence G is also a generalized KKM mapping. So, $\bar{G}$ is also a generalized KKM mapping. Now, condition (iv) implies that $G(y_0)$ is precompact.

Hence, $\overline{G(y_0)}$ is compact. By Theorem 2.1, we have

$$\bigcap_{y\in X} G(y)\neq\varnothing.$$

As a result, there exists $\bar{x}\in X$ such that

$$\phi(\bar{x},y)\leq\gamma, \qquad \text{for all } y\in X. \qquad \square$$

Remark 3.1.

(a) If for every fixed $y\in X$, the function $\phi(x,y)$ is lower semicontinuous in x, then condition (i) of Theorem 3.1 is satisfied immediately.

(b) The following condition (iv)′ implies condition (iv) in Theorem 3.1:

(iv)′ There exist a compact subset L of X and $y_0\in X$ such that $\phi(x,y_0)>\gamma$, for all $x\in X\backslash L$.

As an application of Theorem 3.1, we derive the following existence result for saddle-point problems.

Theorem 3.2. Let E be a Hausdorff topological vector space, and let X be a nonempty closed convex subset of E. Let $\gamma\in(-\infty,+\infty)$ be a given number. Suppose that $\phi: X\times X\to(-\infty,+\infty)$ satisfies the following conditions:

(i) $\phi(x,y)$ is γ-transfer lower semicontinuous in x and γ-generalized quasiconcave in y.

(ii) $\phi(x,y)$ is γ-transfer upper semicontinuous in y and γ-generalized quasiconvex in x.

(iii) There exist $x_1, y_1\in X$ such that the sets

$$\{x\in X: \phi(x,y_1)\leq\gamma\} \text{ and } \{y\in X: \phi(x_1,y)\geq\gamma\}$$

are precompact.

Then, there exists a saddle point of $\phi(x,y)$; that is, there exists $(\bar{x},\bar{y})\in X\times X$ such that

$$\phi(\bar{x},y)\leq\phi(\bar{x},\bar{y})\leq\phi(x,\bar{y}), \qquad \text{for all } x,y\in X.$$

Moreover, we also have

$$\inf_{x\in X}\sup_{y\in X}\phi(x,y)=\sup_{y\in X}\inf_{x\in X}\phi(x,y)=\gamma.$$

Proof. By Theorem 3.1 with $\phi=\psi$, there exists $\bar{x}\in X$ such that

$$\phi(\bar{x},y)\leq\gamma, \qquad \text{for all } y\in X. \tag{1}$$

Let $f: X \times X \to (-\infty, +\infty)$ be defined as $f(y, x) = -\phi(x, y)$. By assumption (ii), $f(x, y)$ is γ-transfer lower semicontinuous in x and $-\gamma$-generalized quasiconcave in y. Therefore, again by Theorem 3.1, there exists $\bar{y} \in X$ such that

$$f(\bar{y}, x) = -\phi(x, \bar{y}) \leq -\gamma, \qquad \text{for all } x \in X,$$

from which it follows that

$$\phi(x, \bar{y}) \geq \gamma, \qquad \text{for all } x \in X. \tag{2}$$

Combining (1) and (2), we have $\phi(\bar{x}, \bar{y}) = \gamma$ and

$$\phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}), \qquad \text{for all } x, y \in X. \tag{3}$$

Finally, again by (1) and (2), we have that

$$\begin{aligned}
&\sup_{y \in X} \inf_{x \in X} \phi(x, y) \\
&\leq \inf_{x \in X} \sup_{y \in X} \phi(x, y) \\
&\leq \sup_{y \in X} \phi(\bar{x}, y) \\
&\leq \phi(\bar{x}, \bar{y}), \text{ by (3)}, \\
&\leq \inf_{x \in X} \phi(x, \bar{y}), \text{ by (3)}, \\
&\leq \sup_{y \in X} \inf_{x \in X} \phi(x, y).
\end{aligned}$$

Consequently,

$$\inf_{x \in X} \sup_{y \in X} \phi(x, y) = \sup_{y \in X} \inf_{x \in X} \phi(x, y) = \gamma,$$

and the proof is completed. □

Remark 3.2. Condition (iii) in Theorem 3.2 can be replaced by the following condition:

(iii)′ There exist a compact subset L of X and $x_1, y_1 \in X$ such that

$$\phi(x, y_1) > \gamma, \qquad \text{for all } x \in X \backslash L,$$

$$\phi(x_1, y) < \gamma, \qquad \text{for all } y \in X \backslash L.$$

For results related to Theorem 3.2, see for example Refs. 12–18.

4. Generalized Variational Inequalities

In this section, we shall employ Theorem 2.1 to derive some existence results for generalized variational inequalities. Let Y and Z be two topological spaces. The multivalued mapping $T: Y \to 2^Z$ is said to be upper semicontinuous at $x_0 \in Y$ (Ref. 19) if, for any open set V in Z containing $T(x_0)$, there is an open neighborhood U of x_0 in Y such that $T(x) \subset V$, for all $x \in U$. We say that T is upper semicontinuous in Y (Ref. 19) if it is upper semicontinuous at each point of Y and if also $T(x)$ is a compact set for each $x \in Y$. For any topological vector space E over real or complex numbers, E^* denotes the vector space of all linear continuous functionals on E and $\langle u, v\rangle$ denotes the paring between $u \in E^*$ and $v \in E$.

Theorem 4.1. Let E be a Hausdorff topological vector space, and let X be a nonempty closed convex subset of E. Let $\gamma \in (-\infty, +\infty)$ be a given number. Suppose that the following conditions are satisfied:

(i) $f: X \times X \to (-\infty, +\infty)$ is lower semicontinuous in the first variable and γ-diagonally concave in the second variable such that $f(x, x) \leq \gamma$, for all $x \in X$.

(ii) $T: X \to 2^{E^*}$ has nonempty values such that, for each fixed $y \in X$,

$$\left\{x \in X: \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + f(x, y) \leq \gamma\right\}$$

$$\subset \left\{x \in X: \sup_{w \in T(y)} \operatorname{Re}\langle w, x-y\rangle + f(x, y) \leq \gamma\right\}.$$

(iii) There exist a nonempty compact subset L of X and $y_0 \in X$ such that

$$\sup_{w \in T(y_0)} \operatorname{Re}\langle w, x-y_0\rangle + f(x, y_0) > \gamma, \qquad \text{for all } x \in X \backslash L.$$

Then, there exists $\bar{x} \in L \subset X$ such that

$$\sup_{w \in T(y)} \operatorname{Re}\langle w, \bar{x}-y\rangle + f(\bar{x}, y) \leq \gamma, \qquad \text{for all } y \in X.$$

Proof. We define two multivalued mappings $G, F: X \to 2^X$ by

$$G(y) = \left\{x \in X: \sup_{w \in T(y)} \operatorname{Re}\langle w, x-y\rangle + f(x, y) \leq \gamma\right\},$$

$$F(y) = \left\{x \in X: \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + f(x, y) \leq \gamma\right\},$$

for all $y \in X$. Then, $G(y)$ is nonempty since $y \in G(y)$, for each $y \in X$. For each fixed $y \in X$, since the mapping

$$x \mapsto \sup_{w \in T(y)} \operatorname{Re}\langle w, x-y\rangle + f(x,y)$$

is lower semicontinuous, $G(y)$ is closed for each $y \in X$. So, $y \mapsto G(y)$ is transfer closed-valued.

Let $\phi: X \times X \to (-\infty, +\infty)$ be defined as follows:

$$\phi(x,y) = \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + f(x,y).$$

For each fixed $x \in X$, the mapping

$$y \mapsto \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle$$

is concave, and hence is 0-diagonally concave. Therefore, $\phi(x,y)$ is γ-diagonally concave in y. In particular, $\phi(x,y)$ is γ-generalized quasiconcave in y. As a result, F is a generalized KKM mapping by Proposition 3.1. Since

$$F(y) \subset G(y), \qquad \text{for all } y \in X,$$

by assumption (ii), G is also a generalized KKM mapping. So, $\bar{G}$ is also a generalized KKM mapping. Also, $\overline{G(y_0)}$ is compact by assumption (iii). Therefore, by Theorem 2.1,

$$\bigcap_{y \in X} G(y) \neq \emptyset.$$

That is, there exists $\bar{x} \in L \cap G(y_0) \subset X$ such that

$$\sup_{w \in T(y)} \operatorname{Re}\langle w, \bar{x}-y\rangle + f(\bar{x},y) \leq \gamma, \qquad \text{for all } y \in X,$$

and the proof is completed. □

By adopting the argument of Shih and Tan (Ref. 20, Lemma 1), we derive the following lemma which will be used to prove the next theorem.

Lemma 4.1. Let B be a reflexive Banach space, and let X be a nonempty convex subset of B. Let $T: X \to 2^{B^*}$ be upper semicontinuous from the line segments in X to the weak topology of B^*, and let $h: X \to (-\infty, +\infty)$ be a convex and lower semicontinuous function. Then, for each fixed $y \in X$, the intersection of the following set:

$$A = \left\{ x \in X: \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + h(x) \leq 0 \right\}$$

with any line segment is closed in X.

Proof. For $x_1, x_2 \in X$, let $[x_1, x_2]$ denote the line segment

$$[x_1, x_2] = \{tx_1 + (1-t)x_2 : t \in [0, 1]\}.$$

Let $\{x_n\}$ be a sequence in $A \cap [x_1, x_2]$ such that $x_n \to x_0 \in [x_1, x_2]$. For each n, since $T(x_n)$ is weakly compact, there exists $u_n \in T(x_n)$ such that

$$\mathrm{Re}\langle u_n, x_n - y\rangle + h(x_n) = \inf_{u \in T(x_n)} \mathrm{Re}\langle u, x_n - y\rangle + h(x_n) \leq 0.$$

As T is upper semicontinuous on $[x_1, x_2]$, which is compact, and since each $T(x_n)$ is weakly compact, the set $\bigcup_{x \in [x_1, x_2]} T(x)$ is also weakly compact (Theorem 3, p. 110, Ref. 19). By the Eberlein–Smulian theorem (Ref. 21), $\bigcup_{x \in [x_1, x_2]} T(x)$ is weakly sequentially compact. Thus, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u_0$ in the weak topology for some $u_0 \in \bigcup_{x \in [x_1, x_2]} T(x)$. By the upper semicontinuity of T, $u_0 \in T(x_0)$. We note that $\mathrm{Re}\langle u_{n_k} - u_0, x_0 - y\rangle \to 0$, since $u_{n_k} \to u_0$ in the weak topology and since $\mathrm{Re}\langle u_{n_k}, x_{n_k} - x_0\rangle \to 0$ as $\{u_{n_k}\}$ is bounded. Consequently,

$$\begin{aligned}&\mathrm{Re}\langle u_{n_k}, x_{n_k} - y\rangle - \mathrm{Re}\langle u_0, x_0 - y\rangle \\ &= \mathrm{Re}\langle u_{n_k} - u_0, x_0 - y\rangle + \mathrm{Re}\langle u_{n_k}, x_{n_k} - x_0\rangle \to 0,\end{aligned}$$

so that

$$\mathrm{Re}\langle u_0, x_0 - y\rangle = \lim_k \mathrm{Re}\langle u_{n_k}, x_{n_k} - y\rangle.$$

Since h is a convex and lower semicontinuous function, it is also lower semicontinuous in the weak topology of B. Hence, we have

$$\begin{aligned}h(x_0) &\leq \liminf_k h(x_{n_k}) \\ &\leq \liminf_k (-\mathrm{Re}\langle u_{n_k}, x_{n_k} - y\rangle) \\ &= -\limsup_k \mathrm{Re}\langle u_{n_k}, x_{n_k} - y\rangle \\ &= -\mathrm{Re}\langle u_0, x_0 - y\rangle.\end{aligned}$$

Therefore,

$$\inf_{u \in T(x_0)} \mathrm{Re}\langle u, x_0 - y\rangle + h(x_0) \leq \mathrm{Re}\langle u_0, x_0 - y\rangle + h(x_0) \leq 0,$$

and hence

$$x_0 \in A \cap [x_1, x_2].$$

As a result, the set $A \cap [x_1, x_2]$ is closed, and the proof is completed. □

Now, we can derive the following result for generalized variational inequalities.

Theorem 4.2. Let B be a reflexive Banach space, and let X be a nonempty closed convex subset of B. Suppose that the following conditions are satisfied:

(i) $f: X \times X \to (-\infty, +\infty)$ is convex and lower semicontinuous in the first variable and concave in the second variable such that $f(x, x) = 0$, for all $x \in X$.

(ii) $T: X \to 2^{B^*}$ is upper semicontinuous from the line segments in X to the weak topology of B^* such that, for each $x \in X$, $T(x)$ is a nonempty subset of B^* and, for each fixed $y \in X$,

$$\left\{x \in X: \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + f(x,y) \leq 0\right\}$$

$$\subset \left\{x \in X: \sup_{w \in T(y)} \operatorname{Re}\langle w, x-y\rangle + f(x,y) \leq 0\right\}.$$

(iii) There exist a nonempty weakly compact subset L of X and $y_0 \in X$ such that

$$\sup_{w \in T(y_0)} \operatorname{Re}\langle w, x-y_0\rangle + f(x, y_0) > 0, \qquad \text{for all } x \in X \backslash L.$$

Then, there exists $\bar{x} \in L \subset X$ such that

$$\inf_{u \in T(\bar{x})} \operatorname{Re}\langle u, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0, \qquad \text{for all } y \in X. \tag{4}$$

In addition, if $T(\bar{x})$ is convex, then there exists $\bar{u} \in T(\bar{x})$ such that

$$\operatorname{Re}\langle \bar{u}, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0, \qquad \text{for all } y \in X. \tag{5}$$

Proof. By assumption (i), $f(x, y)$ is γ-diagonally concave in y with

$$\gamma = \sup_{x \in X} f(x, x) = 0.$$

Since all the assumptions of Theorem 4.1 are satisfied, there exists $\bar{x} \in L \subset X$ such that

$$\sup_{w \in T(y)} \operatorname{Re}\langle w, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0, \qquad \text{for all } y \in X. \tag{6}$$

Suppose that there exists $\bar{y} \in X$ such that

$$\inf_{u \in T(\bar{x})} \operatorname{Re}\langle u, \bar{x}-\bar{y}\rangle + f(\bar{x}, \bar{y}) > 0. \tag{7}$$

Let

$$y_t = t\bar{y} + (1-t)\bar{x} \in X,\ t \in [0, 1].$$

By (6), we have that, for each $t \in [0, 1]$,

$$\sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - y_t\rangle + f(\bar{x}, y_t) \leq 0,$$

from which together with the facts that $f(x, y)$ is concave in y and $f(\bar{x}, \bar{x}) = 0$, it follows that

$$\begin{aligned}
&\sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - y_t\rangle \\
&= t \sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle \\
&\leq -f(\bar{x}, y_t) \\
&\leq -tf(\bar{x}, \bar{y}) - (1-t)f(\bar{x}, \bar{x}) \\
&= -tf(\bar{x}, \bar{y}).
\end{aligned}$$

Consequently,

$$\sup_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle + f(\bar{x}, \bar{y}) \leq 0, \qquad \text{for all } t \in [0, 1]. \tag{8}$$

Now, by Lemma 4.1 and (7), the set

$$U = \left\{x \in X\colon \inf_{u \in T(x)} \mathrm{Re}\langle u, x - \bar{y}\rangle + f(x, \bar{y}) > 0\right\} \cap [\bar{y}, \bar{x}]$$

is open in $[\bar{y}, \bar{x}]$ and contains $\bar{x}$. Since $y_t \to \bar{x}$ as $t \to 0^+$, there exists $t_0 \in (0, 1]$ such that $y_t \in U$ for all $t \in (0, t_0)$, so that

$$\inf_{w \in T(y_t)} \mathrm{Re}\langle w, y_t - \bar{y}\rangle + f(y_t, \bar{y}) > 0, \qquad \text{for all } t \in (0, t_0].$$

Since $f(x, y)$ is convex in x and

$$y_t - \bar{y} = (1-t)(\bar{x} - \bar{y}),$$

it then follows that, for each $t \in (0, t_0)$,

$$\begin{aligned}
&(1-t) \inf_{w \in T(y_t)} \mathrm{Re}\langle w, \bar{x} - \bar{y}\rangle \\
&> -f(y_t, \bar{y}) \\
&\geq -tf(\bar{y}, \bar{y}) - (1-t)f(\bar{x}, \bar{y}) \\
&= -(1-t)f(\bar{x}, \bar{y}).
\end{aligned}$$

Consequently,

$$\inf_{w\in T(y_t)} \mathrm{Re}\langle w, \bar{x}-\bar{y}\rangle + f(\bar{x},\bar{y}) > 0, \qquad \text{for all } t\in(0, t_0],$$

which contradicts (8). This contradiction proves (4).

In addition, if the set $T(\bar{x})$ is convex, by the Kneser minimax theorem (Ref. 22), we have

$$\begin{aligned} &\inf_{u\in T(\bar{x})} \sup_{y\in X} \mathrm{Re}\langle u, \bar{x}-y\rangle + f(\bar{x},y) \\ &= \sup_{y\in X} \inf_{u\in T(\bar{x})} \mathrm{Re}\langle u, \bar{x}-y\rangle + f(\bar{x},y) \le 0. \end{aligned} \tag{9}$$

Note that the function

$$u \mapsto \sup_{y\in X} \mathrm{Re}\langle u, \bar{x}-y\rangle + f(\bar{x},y)$$

is convex and weakly lower semicontinuous on B^*. Since $T(\bar{x})$ is weakly compact, it follows from (9) that there exists $\bar{u}\in T(\bar{x})$ such that

$$\mathrm{Re}\langle \bar{u}, \bar{x}-y\rangle + f(\bar{x},y) \le 0, \qquad \text{for all } y\in X,$$

and hence Inequality (5) is proved. □

A multivalued mapping $T: X\to 2^{B^*}$ is said to be monotone if

$$\mathrm{Re}\langle u-v, x-y\rangle \ge 0, \qquad \text{for all } x, y\in X \text{ and for all } u\in T(x), v\in T(y).$$

Remark 4.1. Condition (ii) of Theorem 4.2 can be replaced by the following condition:

(ii)′ $T: X\to 2^{B^*}$ is monotone and upper semicontinuous from the line segments in X to the weak topology of B^* such that, for each $x\in X$, $T(x)$ is a nonempty subset of B^*.

5. Generalized Quasi-Variational Inequalities

In this section, we shall derive some existence results for generalized quasi-variational inequalities.

Theorem 5.1. Let E be a locally convex Hausdorff topological vector space, and let X be a nonempty closed convex subset of E. Let $\gamma\in(-\infty, +\infty)$ be a given number. Suppose that the following conditions are satisfied:

(i) $F: X \to 2^X$ is upper semicontinuous with nonempty convex values such that $F(X) = \bigcup_{x \in X} F(x)$ is a precompact subset of X.
(ii) $f: X \times X \to (-\infty, +\infty)$ is convex and lower semicontinuous in the first variable and γ-diagonally concave in the second variable such that $f(x, x) \leq \gamma$, for all $x \in X$.
(iii) $T: X \to 2^{E^*}$ has nonempty values such that, for each fixed $y \in X$.

$$\left\{x \in X: \inf_{u \in T(x)} \mathrm{Re}\langle u, x - y\rangle + f(x, y) \leq \gamma\right\}$$

$$\subset \left\{x \in X: \sup_{w \in T(y)} \mathrm{Re}\langle w, x - y\rangle + f(x, y) \leq \gamma\right\}.$$

(iv) There exists a set $\{y(z): z \in X\}$ such that, for each $z \in X, y(z) \in F(z)$, we have

$$\sup_{w \in T(y(z))} \mathrm{Re}\langle w, x - y(z)\rangle + f(x, y(z)) > \gamma, \qquad \text{for all } x \in X \backslash F(z).$$

Then, there exists $\bar{x} \in X$ such that:

(a) $\bar{x} \in F(\bar{x})$,
(b) $\sup_{w \in T(y)} \mathrm{Re}\langle w, \bar{x} - y\rangle + f(\bar{x}, y) \leq \gamma, \qquad$ for all $y \in X$.

Proof. Define a multivalued mapping $G: X \to 2^X$ by

$$G(z) = \left\{z^* \in F(z): \sup_{y \in X} \sup_{w \in T(y)} \mathrm{Re}\langle w, z^* - y\rangle + f(z^*, y) \leq \gamma\right\},$$

for each $z \in X$. We note that $G(z)$ is nonempty for each $z \in X$, by Theorem 4.1 with $L = F(z)$ and $y_0 = y(z)$. Since $f(x, y)$ is convex and lower semicontinuous in x by (ii), the set $G(z)$ is closed and convex for each $z \in X$. Since F is upper semicontinuous and the mapping

$$z^* \mapsto \sup_{y \in X} \sup_{w \in T(y)} \mathrm{Re}\langle w, z^* - y\rangle + f(z^*, y)$$

is lower semicontinuous in X, the graph of G is closed in $X \times X$. Since $F(X)$ is precompact, there is a compact set C containing $F(X)$, so $G(z) \subset C$, for each $z \in X$. Therefore, by Theorem 7, page 112, Ref. 19, G is also upper semicontinuous in X. It follows from the Himmelberg fixed-point theorem (Theorem 2, Ref. 23) that G has a fixed point. Hence, there exists $\bar{x} \in X$ such that

(a) $\bar{x} \in F(\bar{x})$,
(b) $\sup_{w \in T(y)} \mathrm{Re}\langle w, \bar{x} - y\rangle + f(\bar{x}, y) \leq \gamma, \qquad$ for all $y \in X$,

and the proof is completed. □

Finally, we have the following result for generalized quasivariational inequalities.

Theorem 5.2. Let B be a reflexive Banach space, and let X be a nonempty closed convex subset of B. Suppose that the following conditions are satisfied:

(i) $F: X \to 2^X$ is upper semicontinuous with nonempty convex values such that $F(X)$ is a precompact subset of X.
(ii) $f: X \times X \to (-\infty, +\infty)$ is convex and lower semicontinuous in the first variable and concave in the second variable such that $f(x,x)=0$, for all $x \in X$.
(iii) $T: X \to 2^{B^*}$ is upper semicontinuous from the line segments in X to the weak topology of B^* such that, for each $x \in X$, $T(x)$ is a nonempty subset of B^* and, for each fixed $y \in X$,

$$\left\{x \in X: \inf_{u \in T(x)} \operatorname{Re}\langle u, x-y\rangle + f(x,y) \leq 0\right\}$$

$$\subset \left\{x \in X: \sup_{w \in T(y)} \operatorname{Re}\langle w, x-y\rangle + f(x,y) \leq 0\right\}.$$

(iv) There exists a set $\{y(z): z \in X\}$ such that, for each $z \in X, y(z) \in F(z)$, we have

$$\sup_{w \in T(y(z))} \operatorname{Re}\langle w, x-y(z)\rangle + f(x, y(z)) > 0, \qquad \text{for all } x \in X \backslash F(z).$$

Then, there exists $\bar{x} \in X$ such that:

(a) $\bar{x} \in F(\bar{x})$,
(b) $\inf_{u \in T(\bar{x})} \operatorname{Re}\langle u, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0$, for all $y \in F(\bar{x})$.

In addition, if $T(\bar{x})$ is convex, then there exists $\bar{u} \in T(\bar{x})$ such that:

(c) $\bar{x} \in F(\bar{x})$,
(d) $\operatorname{Re}\langle \bar{u}, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0$, for all $y \in F(\bar{x})$.

Proof. By Theorem 5.1, there exists $\bar{x} \in X$ such that $\bar{x} \in F(\bar{x})$ and

$$\sup_{w \in T(y)} \operatorname{Re}\langle w, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0, \qquad \text{for all } y \in F(\bar{x}).$$

By the same argument as that in the proof of Theorem 4.2, we have that:

(a) $\bar{x} \in F(\bar{x})$,
(b) $\inf_{u \in T(\bar{x})} \operatorname{Re}\langle u, \bar{x}-y\rangle + f(\bar{x}, y) \leq 0, \qquad$ for all $y \in F(\bar{x})$.

In addition, if $T(\bar{x})$ is convex, then by the Kneser minimax theorem (Ref. 22), there exists $\bar{u} \in T(\bar{x})$ such that:

(c) $\bar{x} \in F(\bar{x})$,

(d) $\mathrm{Re}\langle \bar{u}, \bar{x} - y \rangle + f(\bar{x}, y) \leq 0$, for all $y \in F(\bar{x})$,

and the proof is completed. □

For other results related to Theorems 5.1–5.2, see for example Theorem 3.3 in Ref. 11, Theorems 5–6 in Ref. 24, and Theorems 1–4 in Ref. 25, where the lower semicontinuity of the multivalued mapping T is assumed.

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