Journal of Global Optimization **29:** 45–57, 2004.

The System of Vector Quasi-Equilibrium Problems with Applications

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(Received 14 June 2002; accepted in final form: 12 October 2003)

Abstract. In this paper, we consider the system of vector quasi-equilibrium problems with or without involving Φ-condensing maps and prove the existence of its solution. Consequently, we get existence results for a solution to the system of vector quasi-variational-like inequalities. We also prove the equivalence between the system of vector quasi-variational-like inequalities and the Debreu type equilibrium problem for vector-valued functions. As an application, we derive some existence results for a solution to the Debreu type equilibrium problem for vector-valued functions.

Key words: Debreu type equilibrium problem, maximal element theorem, partial Gâteaux derivative, Φ-condensing maps, system of vector quasi-equilibrium problems, system of vector quasi-variational-like inequalities.

1. Introduction and Formulations

In the recent years, the vector equilibrium problem (for short, VEP) has been studied in [1–7] and the references therein which is a unified model of several problems, for instance, vector variational inequality, vector variational-like inequality (also called pre-variational inequality), vector complementarity problems, vector saddle point problems and vector optimization problems. A comprehensive bibliography on vector equilibrium problems, vector variational inequalities, vector variational-like inequalities and their generalizations can be found in a recent volume [3]. Very recently, Ansari and Yao [8] generalized the quasi-equilibrium problem, studied in [9, 10] to the case of vector-valued functions, called *vector quasi-equilibrium problem* (for short, VQEP). They established some existence results for a solution to (VQEP) with or without generalized pseudomonotonicity assumption. As a result, they also derived the existence results for a solution to the vector quasi-optimization problem, vector quasi-saddle point problem, vector quasi-variational inequality and vector quasi-variational-like inequality [11, 12]. The system of vector equilibrium problems (for short, SVEP), that is, a family of equilibrium problems for vector-valued bifunctions defined on a product set, is introduced by Ansari et al. [13] with applications in vector optimization problems and Nash equilibrium problem [14] for vector-valued functions. The (SVEP) contains system of equilibrium problems, systems of vector variational inequalities, system of vector variational-like inequalities, system of optimization problems and the Nash equilibrium problem for vector-valued functions as special cases.

But, by using (SVEP), we cannot establish the existence of a solution to the Debreu type equilibrium problem [15] for vector-valued functions which extends the classical concept of Nash equilibrium problem for a noncooperative game. For this purpose, we introduce the following concept of system of vector quasi-equilibrium problems.

Let I be any index set and for each $i \in I$, let X_i be a topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i. We denote by $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and let C_i: $K \rightrightarrows Y_i$ be a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper (i.e., $C_i(x) \neq Y_i$), closed and convex cone with apex at the origin and int $C_i(x) \neq \emptyset$, where int C denotes the topological interior of C. For each $i \in I$, let f_i: $K \times K_i \to Y_i$ be a bifunction and A_i: $K \rightrightarrows K_i$ be a multivalued map with nonempty values. We consider the following *system of vector quasi-equilibrium problems* (in short, SVQEP), that is, to find a $\bar{x} \in K$ such that for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}): f_i(\bar{x}, y_i) \notin -\text{int } C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

If, $\forall\, i \in I$, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+$ $\forall\, x \in K$, then (SVQEP) is known as a *system of quasi-equilibrium problems*, see [16, 17] and references therein.

When $A_i(x) = K_i$ $\forall\, x \in K$, then (SVQEP) reduces to (SVEP).

If the index set I is singleton, then (SVQEP) becomes the vector quasi-equilibrium problem which contains vector quasi-optimization problem, vector quasi-variational inequality, vector quasi-variational-like inequality and vector quasi-saddle point problem as special cases, see [8].

2. Examples of (SVQEP)

For each $i \in I$, we denote by $L(X_i, Y_i)$ the space of continuous linear operators from X_i into Y_i and let T_i: $K \to L(X_i, Y_i)$ be a map. We denote by $\langle s_i, x_i \rangle$ the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$.

(1) For each $i \in I$, let η_i: $K_i \times K_i \to X_i$ be a map. If, $\forall\, i \in I$,

$$f_i(x, y_i) = \langle T_i(x), \eta_i(y_i, x_i) \rangle$$

then (SVQEP) is equivalent to the following problem of finding $\bar{x} \in K$ such that $\forall\, i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) : \langle T_i(\bar{x}), \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

We call it a *system of vector quasi-variational-like inequalities* (for short, SVQVLI).

When $\eta_i(y_i, x_i) = y_i - x_i$, then (SVQVLI) is called a *system of vector quasi-variational inequalities* (for short, SVQVI). If, $\forall\ i \in I$, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+\ \forall\ x \in K$, (SVQVI) is studied in [16, 17].

If, $\forall\ i \in I$, $A_i(x) = K_i\ \forall\ x \in K$, (SVQVLI) and (SVQVI) reduce to the system of vector variational-like inequalities and system of vector variational inequalities, respectively, which are studied by Ansari et al. [13].

(2) For each $i \in I$, let φ_i: $K \to Y_i$ be a vector-valued function and let $K^i = \prod_{j \in I,\ j \neq i} K_j$ and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we write $x = (x^i, x_i)$. If, $\forall\ i \in I$,

$$f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x)$$

then (SVQEP) is equivalent to the following Debreu type equilibrium problem for vector-valued functions (for short, Debreu VEP): Find $\bar{x} \in K$ such that $\forall\ i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) : \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}\ C_i(\bar{x}), \quad \forall\ y_i \in A_i(\bar{x}).$$

Of course, if $\forall\ i \in I$, φ_i is a scalar-valued function, then (Debreu VEP) is the same as the one introduced and studied by Debreu in [15]. In this case, a large number of papers have already been appreared in the literature; see [16, 17] and references therein. To the best of our knowledge, there is no study yet when each φ_i is a vector-valued function. This paper is the first effort in this direction.

The rest of the paper is arranged in the following manner. The next section deals with some preliminary definitions, notations and results which will be used in the sequel. In Section 3, we establish existence results for a solution to (SVQEP) with or without involving Φ-condensing maps by using well-known maximal element theorems for a family of multivalued maps, and consequently, we also get some existence results for a solution to (SVQVLI). In Section 4, we first prove the equivalence between (SVQVLI) and (Debreu VEP) and, as an application of results of Section 3, we then derive some existence results for a solution to (Debreu VEP) for convex or nonconvex functions.

3. Preliminaries

Let E and Z be topological vector spaces and Q: $E \rightrightarrows Z$ be a multivalued map. The *inverse* Q^{-1} of Q is the multivalued map from the range of Q to E defined by

$$x \in Q^{-1}(y) \Leftrightarrow y \in Q(x).$$

DEFINITION 1 [2, 7, 18]. Let M be a nonempty subset of a topological vector space E, and let Z be a topological vector space with a proper, closed and convex cone P with apex at the origin and $\text{int}\,P \neq \emptyset$. A vector-valued function ϕ: $M \to Z$

is said to be *P-lower semicontinuous* (respectively, *P-upper semicontinuous*) *at* $x_0 \in M$ iff, for any neighbourhood V of $\phi(x_0)$ in Z, $\exists$ a neighbourhood U of x_0 in E such that

$$\phi(x) \in V + P, \quad \forall\ x \in U \cap M$$

$$(\text{respectively, } \phi(x) \in V - P, \quad \forall\ x \in U \cap M).$$

Furthermore, ϕ is *P-lower semicontinuous* (respectively, *P-upper semicontinuous*) *on* M iff, it is P-lower semicontinuous (respectively, P-upper semicontinuous) at each $x \in M$.

ϕ is *P-continuous on* M iff, it is both P-lower semicontinuous and P-upper semicontinuous on M.

Remark 1. In [2] it is shown that a function ϕ: $M \to Z$ is P-lower semicontinuous iff, $\forall\ \alpha \in Z$, the set

$$L(\alpha) := \{x \in M : \phi(x) - \alpha \notin \operatorname{int} P\}$$

is closed in M.

Similarly we can show that ϕ is P-upper semicontinuous iff, $\forall\ \alpha \in Z$, the set

$$U(\alpha) := \{x \in M : \phi(x) - \alpha \notin -\operatorname{int} P\}$$

is closed in M.

DEFINITION 2. [1]Let M be a nonempty and convex subset of a topological vector space, and let Z be a topological vector space with a closed and convex cone P with apex at the origin. A vector-valued function ϕ: $M \to Z$ is called

(i) *P-function* iff, $\forall\ x, y \in M$ and $\forall\ \lambda \in [0,1]$,

$$\phi(\lambda x + (1-\lambda)y) \in \lambda\phi(x) + (1-\lambda)\phi(y) - P;$$

(ii) *natural P-quasifunction* iff, $\forall\ x, y \in M$ and $\forall\ \lambda \in [0,1]$,

$$\phi(\lambda x + (1-\lambda)y) \in \text{ conv } \{\phi(x), \phi(y)\} - P,$$

where $\operatorname{conv} B$ denotes the convex hull of B.

(iii) *P-quasifunction* iff, $\forall\ \alpha \in Z$, the set $\{x \in M\colon \phi(x) - \alpha \in -P\}$ is convex.

Remark 2. (a) Every P-function is natural P-quasifunction and every natural P-quasifunction is P-quasifunction, but converse assertions are not true; see, for example, Remark 2.1 in [19].

(b) ϕ is natural P-quasifunction iff, $\forall\ x, y \in M$ and $\forall\ \lambda \in [0,1]$, $\exists\ \mu \in [0,1]$ such that

$$\phi(\lambda x + (1-\lambda)y) \in \mu\phi(x) + (1-\mu)\phi(y) - P.$$

[1]The terms P-convex, natural P-quasiconvex and P-quasiconvex are used in [2, 7, 19] instead of P-function, natural P-quasifunction and P-quasifunction which are suggested by Prof. F. Giannessi.

(c) If ϕ is P-quasifunction, then the set $\{x \in M\colon \phi(x) - \alpha \in -\text{int}\, P\}$ is also convex; see [2].

DEFINITION 3 [20]. Let E be a Hausdorff topological vector space and L a lattice with least element, denoted by 0. A mapping Φ: $2^E \to L$ is called a *measure of noncompactness* provided that the following conditions hold $\forall\ M, N \in 2^E$:

(i) $\Phi(M) = 0$ iff M is precompact (i.e., it is relatively compact).
(ii) $\Phi(\overline{\text{conv}}\, M) = \Phi(M)$, where $\overline{\text{conv}}\, M$ denotes the convex closure of M.
(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

It follows from (iii) that, if $M \subseteq N$, then $\Phi(M) \leqslant \Phi(N)$.

DEFINITION 4 [20]. Let Φ: $2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A multivalued map Q: $D \rightrightarrows E$ is called Φ-*condensing* provided that, if $M \subseteq D$ with $\Phi(Q(M)) \geqslant \Phi(M)$, then M is relatively compact.

Remark 3. Note that every multivalued map defined on a compact set is Φ-condensing. If E is locally convex, then a compact multivalued map (i.e., $Q(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if Q: $D \rightrightarrows E$ is Φ-condensing and Q': $D \rightrightarrows E$ satisfies $Q'(x) \subseteq Q(x)$ $\forall$ $x \in D$, then Q' is also Φ-condensing.

We shall use the following particular form of a maximal element theorem for a family of multivalued maps due to Deguire et al. (Theorem 7 in [21]).

THEOREM 1. *Let I be any index set. $\forall\ i \in I$, let K_i be a nonempty and convex subset of a Hausdorff topological vector space X_i, and let S_i: $K = \prod_{i\in I} K_i \rightrightarrows K_i$ be a multivalued map. Assume that the following conditions hold*:

(i) *$\forall\ i \in I$ and $\forall\ x \in K$, $S_i(x)$ is convex.*
(ii) *$\forall\ i \in I$ and $\forall\ x \in K$, $x_i \notin S_i(x)$, where x_i is the ith component of x.*
(iii) *$\forall\ i \in I$ and $\forall\ y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K.*
(iv) *There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\ i \in I$, such that $\forall\ x \in K \setminus N$ $\exists\ i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$.*

Then $\exists\ \bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ $\forall\ i \in I$.

Remark 4. If $\forall\ i \in I$, K_i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, then condition (iv) of Theorem 1 can be replaced by the following condition:

(iv)′ The multivalued map S: $K \rightrightarrows K$ defined as $S(x) := \prod_{i\in I} S_i(x)$ $\forall\ x \in K$, is Φ-condensing.

(See, Corollary 4 in [22]).

4. Existence Results

Throughout the paper, unless otherwise specified, we assume that I is any index set and $\forall\ i\in I$, Y_i is a topological vector space, $K=\prod_{i\in I}K_i$, C_i: $K\rightrightarrows Y_i$ is a multivalued map such that $\forall\ x\in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x)\neq\emptyset$ and $P_i=\bigcap_{x\in K}C_i(x)$. We also assume that $\forall i\in I$, A_i: $K\rightrightarrows K_i$ is a multivalued map such that $\forall\ x\in K$, $A_i(x)$ is nonempty and convex, $A_i^{-1}(y_i)$ is open in K $\forall\ y_i\in K_i$ and the set $\mathcal{F}_i$: $=\{x\in K\colon x_i\in A_i(x)\}$ is closed in K, where x_i is the ith component of x.

THEOREM 2. *For each $i\in I$, let K_i be a nonempty and convex subset of a Hausdorff topological vector space X_i and f_i: $K\times K_i\to Y_i$ a bifunction. Assume that the following conditions hold*:

(i) $\forall\ i\in I$ *and* $\forall\ x\in K$, $f_i(x,x_i)\notin-\text{int}C_i(x)$, *where x_i is the ith component of x.*
(ii) $\forall\ i\in I$ *and* $\forall\ x\in K$, *the vector-valued function $y_i\mapsto f_i(x,y_i)$ is natural P_i-quasifunction.*
(iii) $\forall\ i\in I$ *and* $\forall\ y_i\in K_i$, *the set* $\{x\in K\colon f_i(x,y_i)\notin-\text{int}C_i(x)\}$ *is closed in K.*
(iv) *There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\ i\in I$, such that $\forall\ x\in K\setminus N$ $\exists\ i\in I$ and $\exists\ \tilde{y}_i\in B_i$, such that $\tilde{y}_i\in A_i(x)$ and $f_i(x,\tilde{y}_i)\in-\text{int}C_i(x)$.*

Then (SVQEP) *has a solution.*

Proof. For each given $i\in I$, we define a multivalued map Q_i: $K\rightrightarrows K_i$ by

$$Q_i(x)=\{y_i\in K_i\colon f_i(x,y_i)\in-\text{int}\,C_i(x)\},\quad \forall\ x\in K.$$

Then $\forall\ i\in I$ and $\forall\ x\in K$, $Q_i(x)$ is convex.

To prove it, let us fix arbitrary $i\in I$ and $x\in K$. Let $y_{i_1},y_{i_2}\in Q_i(x)$ and $\lambda\in[0,1]$, then we have

$$f_i(x,y_{i_j})\in-\text{int}C_i(x),\quad \text{for}\ \ j=1,2. \tag{1}$$

Since $f_i(x,\cdot)$ is natural P_i-quasifunction, $\exists\ \mu\in[0,1]$ such that

$$f_i(x,\lambda y_{i_1}+(1-\lambda)y_{i_2})\in\mu f_i(x,y_{i_1})+(1-\mu)f_i(x,y_{i_2})-P_i. \tag{2}$$

From (1) and (2), we get

$$f_i(x,\lambda y_{i_1}+(1-\lambda)y_{i_2})\in-\text{int}C_i(x)-\text{int}C_i(x)-P_i\subseteq-\text{int}C_i(x).$$

Hence $\lambda y_{i_1}+(1-\lambda)y_{i_2}\in Q_i(x)$ and therefore $Q_i(x)$ is convex. Since $i\in I$ and $x\in K$ are arbitrary, $Q_i(x)$ is convex $\forall\ x\in K$ and $\forall\ i\in I$. By condition (iii), $\forall\ i\in I$ and $\forall\ y_i\in K_i$, the complement of $Q_i^{-1}(y_i)$ in K

$$[Q_i^{-1}(y_i)]^c=\{x\in K: f_i(x,y_i)\notin-\text{int}C_i(x)\}$$

is closed in K.

$\forall\ i\in I$ and $\forall\ x\in K$, we define another multivalued map S_i: $K \rightrightarrows K_i$ by

$$S_i(x)=\begin{cases} A_i(x)\cap Q_i(x), & \text{if } x\in\mathcal{F}_i \\ A_i(x), & \text{if } x\in K\setminus\mathcal{F}_i. \end{cases}$$

Then, it is clear that $\forall\ i\in I$ and $\forall\ x\in K$, $S_i(x)$ is convex, and by condition (i), $x_i\notin S_i(x)$. Since $\forall\ i\in I$ and $\forall\ y_i\in K_i$,

$$S_i^{-1}(y_i)=(A_i^{-1}(y_i)\cap Q_i^{-1}(y_i))\cup((K\setminus\mathcal{F}_i)\cap A_i^{-1}(y_i))$$

(see, for example, the proof of Lemma 2.3 in [9]), and $A_i^{-1}(y_i)$, $Q_i^{-1}(y_i)$ and $K\setminus\mathcal{F}_i$ are open in K, we have $S_i^{-1}(y_i)$ is open in K.

Condition (iv) of Theorem 1 is followed from condition (iv). Hence by Theorem 1, $\exists\ \bar{x}\in K$ such that $S_i(\bar{x})=\emptyset\ \ \forall\ i\in I$. Since $\forall\ i\in I$ and $\forall\ x\in K$, $A_i(x)$ is nonempty, we have $A_i(\bar{x})\cap Q_i(\bar{x})=\emptyset\ \ \forall\ i\in I$. Therefore, $\forall\ i\in I$,

$$\bar{x}_i\in A_i(\bar{x}) \ \text{ and } \ f_i(\bar{x},y_i)\notin -\text{int } C_i(\bar{x}), \quad \forall\ y_i\in A_i(\bar{x}).$$

Remark 5. (1) The condition (iii) of Theorem 2 is satisfied if the following conditions hold $\forall\ i\in I$:

(a) The multivalued map W_i: $K\rightrightarrows Y_i$ defined by $W_i(x)=Y_i\setminus\{-\text{int } C_i(x)\}\ \forall\, x\in K$, is closed in $K\times K_i$.
(b) $\forall\ y_i\in K_i$, $f_i(\cdot,y_i)$: $K\to Y_i$ is continuous (in the usual sense) on K.

(2) If $\forall\ i\in I$ and $\forall\ x\in K$, $C_i(x)=C_i$, a (fixed) proper, closed and convex cone in Y_i, then conditions (ii) and (iii) of Theorem 3 can be replaced, respectively, by the following conditions:

(c) $\forall\ i\in I$ and $\forall\ x\in K$, the vector-valued function $y_i\mapsto f_i(x,y_i)$ is C_i-quasifunction.
(d) $\forall\ i\in I$ and $\forall\ y_i\in K_i$ the vector-valued function $x\mapsto f_i(x,y_i)$ is C_i-upper semicontinuous on K.

(3) Theorem 3 extends and generalizes Theorem 6 in [16], Theorem 2.1 in [13] and Corollary 3.1 in [8] in several ways.

(4) If $\forall\ i\in I$, K_i is a nonempty, compact and convex subset of a Hausdorff topological vector space X_i, then the conclusion of Theorem 3 holds without condition (iv).

Now we establish an existence result for a solution to (SVQEP) involving Φ-condensing maps.

THEOREM 3. *For each $i\in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, f_i: $K\times K_i\to Y_i$ a bifunction and let the multivalued map $A=\prod_{i\in I}A_i$: $K\rightrightarrows K$ defined as $A(x)=\prod_{i\in I}A_i(x)\ \ \forall\ x\in K$, be Φ-condensing. Assume that the conditions* (i), (ii) *and* (iii) *of Theorem* 2 *hold. Then* (SVQEP) *has a solution.*

Proof. In view of Remark 4, it is sufficient to show that the multivalued map S: $K \rightrightarrows K$ defined as $S(x)=\prod_{i\in I} S_i(x)\ \forall\ x\in K$, is ϕ-condensing, where S_i's are the same as defined in the proof of Theorem 2. By the definition of S_i, $S_i(x)\subseteq A_i(x)\ \forall\ i\in I$ and $\forall\ x\in K$ and therefore $S(x)\subseteq A(x)\ \forall\ x\in K$. Since A is Φ-condensing, by Remark 3, we have S is also Φ-condensing.

Let E and Z be Hausdorff topological vector spaces and σ be the family of bounded subsets of E whose union is total in E, that is, the linear hull of $\bigcup\{U\colon U\in\sigma\}$ is dense in E. Let $\mathcal{B}$ be a neighbourhood base of 0 in Z. When U runs through σ, V through $\mathcal{B}$, the family

$$M(U,V)=\left\{\xi\in L(E,Z):\bigcup_{x\in U}\langle\xi,x\rangle\subseteq V\right\}$$

is a neighbourhood base of 0 in $L(E,Z)$ for a unique translation-invariant topology, called the *topology of uniform convergence* on the sets $U\in\sigma$, or, briefly, the σ-topology (see [23], pp. 79–80).

In order to derive existence results for a solution to (SVQVI) from Theorems 2 and 3, we need the following result of Ding and Tarafdar [24].

LEMMA 1. *Let E and Z be Hausdorff topological vector spaces and $L(E,Z)$ be the topological vector space under the σ-topology. Then, the bilinear mapping $\langle .,.\rangle$: $L(E,Z)\times E\to Z$ is continuous on $L(E,Z)\times E$.*

In addition to the assumptions on C_i: $K\rightrightarrows Y_i$, in the following Corollaries 1 and 2, we further assume that $C_i(x)$ is pointed, $\forall\ i\in I$ and $\forall\ x\in K$. Then the following results can be easily derived, respectively, from Theorems 2 and 3 by setting

$$f_i(x,y_i)=\langle T_i(x),\eta_i(y_i,x_i)\rangle.$$

COROLLARY 1. *For each $i\in I$, let K_i, X_i and W_i be the same as in Theorem* 2 *and Remark* 5, *respectively, and let $L(X_i,Y_i)$ be equipped with the σ-topology. $\forall\ i\in I$, let η_i: $K_i\times K_i\to X_i$ be continuous in the second variable such that $\eta_i(x_i,x_i)=0\ \forall\ x_i\in K_i$, and let T_i: $K\to L(X_i,Y_i)$ be continuous on K such that the map $y_i\mapsto\langle T_i(x),\eta_i(y_i,x_i)\rangle$ is natural P_i-quasifunction, $\forall\ x\in K$. Assume that there exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of $K_i\ \forall\ i\in I$, such that $\forall\ x\in K\setminus N\ \exists\ i\in I$ and $\exists\ \tilde{y}_i\in B_i$ such that $\tilde{y}_i\in A_i(x)$ and $\langle T_i(x),\eta_i(\tilde{y}_i,x_i)\rangle\in-\mathrm{int}\ C_i(x)$. Then* (SVQVLI) *has a solution.*

COROLLARY 2. *For each $i\in I$, let K_i, X_i and η_i, T_i, W_i, $L(X_i,Y_i)$ be the same as in Theorem* 3 *and Corollary* 1, *respectively. Let the multivalued map $A=\prod_{i\in I}A_i$: $K\rightrightarrows K$ defined as $A(x)=\prod_{i\in I}A_i(x)\ \ \forall\ x\in K$, be Φ-condensing. Then* (SVQVLI) *has a solution.*

Remark 6. To the best of our knowledge, there is only one paper [25] appeared in the literature on the scalar quasi-variational-like inequality problems involving Φ-condensing maps. Since the approach in this paper is different from the one adopted in [25], Corollary 2 is a new result in the literature, not only for vector-valued case but also for scalar-valued case.

5. Applications

Let $I=\{1,2,\dots,n\}$ be a finite index set and $\forall\ i\in I$, let X_i be a normed space. Let Z be a normed space. We recall the following definition.

DEFINITION 5 [26]. The function $\phi: X=\prod_{i\in I}X_i\to Z$ is said to be *partial Gâteaux differentiable at* $x=(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n)\in X$ w. r. t. the jth variable x_j if,

$$\langle D_{x_j}\phi(x),h_j\rangle=\lim_{t\to 0}\frac{\phi(x_1,\dots,x_{j-1},x_j+th_j,x_{j+1},\dots,x_n)-\phi(x)}{t}\quad\text{exists,}$$

$\forall\ h_j\in X_j$. $D_{x_j}\phi(x)\in L(X_j,Z)$ is called *partial Gâteaux derivative of* ϕ *at* $x\in X$ w.r.t. the jth variable x_j.

ϕ is called *partial Gâteaux differentiable on* X if, it is partial Gâteaux differentiable at each point of K w.r.t. each variable.

DEFINITION 6 [27]. Let E be a normed space, Z a normed space with a closed and convex cone P with apex at the origin, M a nonempty subset of E, η: $M\times M\to E$ a function. A Gâteaux differentiable function ϕ: $M\to Z$ is said to be P-invex w.r.t. η iff, $\forall\ x,y\in M$,

$$\phi(y)-\phi(x)-\langle D_x\phi(x),\eta(y,x)\rangle\in P,$$

where $D_x\phi(x)$ denotes the Gâteaux derivative of ϕ at x.

DEFINITION 7 [28]. A subset M of a vector space E is said to be *invex w.r.t.* η: $M\times M\to E$ iff, $\forall\ x,y\in M$ and $\forall\ \lambda\in[0,1]$, $x+\lambda\eta(y,x)\in M$.

DEFINITION 8 [27]. Let M be an invex set in a normed space E w.r.t. η: $M\times M\to E$. A vector-valued function ϕ: $M\to Z$ is said to be *P-preinvex* iff, $\forall\ x,y\in M$ and $\forall\ \lambda\in[0,1]$,

$$\lambda\phi(y)+(1-\lambda)\phi(x)-\phi(x+\lambda\eta(y,x))\in P.$$

Remark 7. It can be easily seen that if M is an invex set of E w.r.t. η: $M\times M\to E$ and ϕ: $M\to Z$ is Gâteaux differentiable on M and P-preinvex, then ϕ is P-invex w.r.t. η. But the converse assertion is, in general, not true.

Now we establish the following sufficient condition for a solution to (Debreu VEP) which will be useful in deriving some existence results for a solution to (Debreu VEP) for nonconvex functions.

PROPOSITION 1. *Let I be a finite index set. $\forall\ i \in I$, let X_i and Y_i be normed spaces, K_i a nonempty, open and convex subset of X_i, A_i: $K \rightrightarrows K_i$ nonempty and convex-valued multivalued map, η_i: $K_i \times K_i \to X_i$, and φ_i: $K = \prod_{i \in I} K_i \to Y_i$ partial Gâteaux differentiable on K and P_i-invex w.r.t. η_i in each argument. Then every solution of* (SVQVLI) *with $T_i(x) = D_{x_i} \varphi_i(x)$ is also a solution of* (*Debreu* VEP).

Proof. Assume that $\bar{x} \in K$ is a solution of (SVQVLI) with $T_i(x) = D_{x_i} \varphi_i(x)$. Then $\forall\ i \in I$,

$$\bar{x}_i \in A_i(\bar{x})\ :\ \langle D_{x_i} \varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}), \quad \forall\ y_i \in A_i(\bar{x}). \tag{3}$$

Since $\forall\ i \in I$, φ_i is P_i-invex w.r.t. η_i in each argument, we have $\forall\ \lambda \in [0, 1]$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) - \langle D_{x_i} \varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i) \rangle \in P_i \subseteq C_i(\bar{x}). \tag{4}$$

Since $a - b \in P$ and $b \notin -\text{int} P \Rightarrow a \notin -\text{int} P$, it follows from (3) and (4) that

$$\bar{x}_i \in A_i(\bar{x})\ :\ \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}), \quad \forall\ y_i \in A_i(\bar{x}).$$

Hence $\bar{x} \in K$ is a solution of (Debreu VEP).

The next result provides the equivalence between (SVQVLI) and (Debreu VEP).

PROPOSITION 2. *Let I be a finite index set. $\forall\ i \in I$, let X_i and Y_i be normed spaces, $K_i \subseteq X_i$ nonempty, open and invex w.r.t. η_i: $K_i \times K_i \to X_i$, A_i: $K \rightrightarrows K_i$ nonempty and invex-valued multivalued map and φ_i: $K = \prod_{i \in I} K_i \to Y_i$ partial Gâteaux differentiable on K and P_i-preinvex in each argument. Then $\bar{x} \in K$ is a solution of* (SVQVLI) *with $T_i(x) = D_{x_i} \varphi_i(x)$ iff it is a solution of* (*Debreu* VEP).

Proof. Assume that $\bar{x} \in K$ is a solution of (SVQVLI). Then by Proposition 1, $\bar{x} \in K$ is a solution of (Debreu VEP).

Conversely, let $\bar{x} \in K$ be a solution of (Debreu VEP). Then $\forall\ i \in I$,

$$\bar{x}_i \in A_i(\bar{x})\ :\ \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}), \quad \forall\ y_i \in A_i(\bar{x}). \tag{5}$$

Since $\bar{x}_i, y_i \in A_i(\bar{x})$ and each $A_i(\bar{x})$ is invex, we have $\bar{x}_i + \lambda \eta_i(y_i, \bar{x}_i) \in A_i(\bar{x}) \forall\ \lambda \in [0, 1]$. Therefore from (5), we get

$$\varphi_i(\bar{x}^i, \bar{x}_i + \lambda \eta_i(y_i, \bar{x}_i)) - \varphi_i(\bar{x}) \in W_i(\bar{x}) = Y_i \setminus \{-\text{int } C_i(\bar{x})\}.$$

Since, $\forall\ i \in I$, $W_i(\bar{x})$ is a closed cone, we have

$$\lim_{\lambda \to 0} \frac{\varphi_i(\bar{x}^i, \bar{x}_i + \lambda \eta_i(y_i, \bar{x}_i)) - \varphi_i(\bar{x})}{\lambda} \in W_i(\bar{x}).$$

From the partial Gâteaux differentiability of each φ_i, we get, $\forall\ i \in I$

$$\bar{x}_i \in A_i(\bar{x})\ :\ \langle D_{x_i} \varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}), \quad \forall\ y_i \in A_i(\bar{x}).$$

Hence $\bar{x} \in K$ is a solution of (SVQVLI) with $T_i(\bar{x}) = D_{x_i} \varphi_i(\bar{x})\ \forall\ i \in I$.

Remark 8. If $\forall\ i \in I$ and $\forall\ x \in K$, $\eta_i(y_i, x_i) = y_i - x_i$, $A_i(x) = K_i$, $C_i(x) = \mathbb{R}_+$ and $Y_i = \mathbb{R}$, then Proposition 2 reduces to Proposition 4 in ([29], pp. 269). Hence Proposition 2 extends Proposition 4 in [29] in several ways.

By using Proposition 1 and Corollary 1, we can easily derive the following existence result for a solution to (Debreu VEP).

THEOREM 4. *Let I be a finite index set and $\forall\ i \in I$, let X_i and Y_i be normed spaces, K_i be a nonempty, open and convex subset of X_i and W_i be the same as in Remark* 5. *$\forall\ i \in I$, let η_i: $K_i \times K_i \to X_i$ be continuous in the second argument such that $\eta_i(x_i, x_i) = 0\ \ \forall\ x_i \in K_i$, and φ_i: $K \to Y_i$ partial Gâteaux differentiable on K and P_i-invex in each variable such that the function $y_i \mapsto \langle D_{x_i}\varphi_i(x), \eta_i(y_i, x_i)\rangle$ is natural P_i-quasifunction, $\forall\ x \in K$. Assume that there exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of $K_i\ \ \forall\ i \in I$, such that $\forall\ x \in K \setminus N\ \exists\ i \in I$ and $\exists\ \tilde{y}_i \in B_i$, such that $\tilde{y}_i \in A_i(x)$ and $\langle D_{x_i}\varphi_i(x), \eta_i(\tilde{y}_i, x_i)\rangle \in -\mathrm{int}\, C_i x$. Then* (*Debreu* VEP) *has a solution.*

If the index set I need not be finite and $\forall\ i \in I$, φ_i need not be partial Gâteaux differentiable, then we can also easily derive the following existence results for a solution to (Debreu VEP) from Theorems 2 and 3 by setting, $\forall\ i \in I$,

$$f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x).$$

THEOREM 5. *For each $i \in I$, let K_i, X_i and W_i be the same as in Theorem* 2 *and Remark* 5, *respectively, and let φ_i: $K \to Y_i$ be a vector-valued function. Assume that the following conditions hold*:

(i) *$\forall\ i \in I$, φ_i is natural P_i-quasifunction in the ith argument.*
(ii) *$\forall\ i \in I$, φ_i is continuous on K.*
(iii) *There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\ i \in I$, such that $\forall\ x \in K \setminus N\ \exists\ i \in I$ and $\exists\ \tilde{y}_i \in B_i$, such that $\tilde{y}_i \in A_i(x)$ and $\varphi_i(x^i, \tilde{y}_i) - \varphi_i(x) \in -\mathrm{int}\ C_i(x)$.*

Then (*Debreu* EP) *has a solution.*

THEOREM 6. *For each $i \in I$, let K_i, X_i and W_i be the same as in Theorem* 3 *and Theorem* 5, *respectively. Let the multivalued map $A = \prod_{i\in I} A_i$: $K \rightrightarrows K$ defined as $A(x) = \prod_{i\in I} A_i(x)\ \forall\ x \in K$, be Φ-condensing. Assume that the conditions* (i) *and* (ii) *of Theorem* 5 *hold. Then* (*Debreu* EP) *has a solution.*

Remark 9. (1) If $\forall\ i \in I$ and $\forall\ x \in K$, $C_i(x) = C_i$, a (fixed) proper, closed and convex cone in Y_i, then conditions (i) and (ii) in Theorem 5, and subsequently, in Theorem 6 can be replaced, respectively, by the following conditions:

(i)′ $\forall\ i \in I$ and $\forall\ x \in K$, φ_i is C_i-quasifunction in the *i*th argument.

(ii)′ $\forall\, i \in I$, φ_i is C_i-upper semicontinuous on K.

(2) Theorem 6 provides the existence of a solution to the Debreu type equilibrium problem for vector-valued functions involving Φ-condensing map and, consequently, for scalar-valued functions. Therefore, Theorem 6 is a new result in the literature in this direction.

6. Acknowledgements

This work was partially done during the visit of first author to the Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong. Authors are thankful to Prof. F. Giannessi and referees for their valuable suggestions and comments to improve the early version of this paper. The first author is also grateful to the Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia for providing excellent research facilities to carry out this work.

References

1. Ansari, Q.H., Yang, X.Q. and Yao, J.C. (2001), Existence and duality of implicit vector variational problems, *Numerical Functional Analysis and Optimization*, 22(7 & 8), 815–829.
2. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone bifunctions, *Journal of Optimization Theory and Applications*, 92, 527–542.
3. Giannessi, F. (Ed.) (2000), *Vector Variational Inequalities and Vector Equilibria. Mathematical Theories*, Kluwer Academic Publishers, Dordrecht/Boston/London.
4. Gong, X.H. (2001), Efficiency and henig efficiency for vector equilibrium problems, *Journal of Optimization Theory and Applications*, 108, 139–154.
5. Hadjisavvas, N. and Schaible, S. (1998), From scalar to vector equilibrium problems in the quasimonotone case, *Journal of Optimization Theory and Applications*, 96, 297–309.
6. Oettli, W. (1997), A remark on vector-valued equilibria and generalized monotonicity, *Acta Mathematica Vietnamica*, 22, 213–221.
7. Tan, N.X. and Tinh, P.N. (1998), On the existence of equilibrium points of vector functions, *Numerical Functional Analysis and Optimization*, 19, 141–156.
8. Ansari, Q.H. and Yao, J.C. (2002), On vector quasi-equilibrium problems, In: Daniele, P., Giannessi, F. and Maugeri, A. (eds), *Equilibrium Problems and Variational Models*, Kluwer Academic Publishers, Dordrecht/Boston/London.
9. Ding, X.P. (2000), Existence of solutions for quasi-equilibrium problems in noncompact topological spaces, *Computers and Mathematics with Applications*, 39, 13–21.
10. Lin, L.-J. and Park, S. (1998), On some generalized quasi-equilibrium problems, *Journal of Mathematical Analysis and Applications*, 224, 167–181.
11. Kim, W.K. and Tan, K.-K. (1999), On generalized vector quasi-variational inequalities, *Optimization*, 46, 185–198.
12. Lee, G.M., Kim, D.S. and Lee, B.S. (1996), On vector quasivariational-like inequality, *Bulletin of the Korean Mathematical Society*, 33, 45–55.
13. Ansari, Q.H., Schaible, S. and Yao, J.C. (2000), The system of vector equilibrium problems and its applications, *Journal of Optimization Theory and Applications*, 107(3), 547–557.
14. Nash, J. (1951), Non-cooperative games, *Annals of Mathematics*, 54, 286–295.
15. Debreu, G. (1952), A social equilibrium existence theorem, *Proceedings of the National Academic Science USA*, 38, 886–893.

16. Ansari, Q.H., Idzik, A. and Yao, J.C. (2000), Coincidence and fixed point theorems with applications, *Topological Methods in Nonlinear Analysis*, 15(1), 191–202.
17. Yuan, G.X.-Z. (1999), *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, Inc., New York, Basel.
18. Luc, D.T. (1989), *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems Vol. 319 Springer-Verlag, Berlin.
19. Tanaka, T. (1994), Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions, *Journal of Optimization Theory and Applications*, 81, 355–377.
20. Petryshyn, W.V. and Fitzpatrick, P.M. (1974), Fixed point theorems of multivalued noncompact acyclic mappings, *Pacific Journal of Mathematics*, 54, 17–23.
21. Deguire, P., Tan, K.K. and Yuan, G.X.-Z. (1999), The study of maximal elements, fixed points for L_S-majorized mappings and their applications to minimax and variational inequalities in the product topological spaces, *Nonlinear Analysis, Theory Methods and Applications*, 37, 933–951.
22. Chebbi, S. and Florenzano, M. (1999), Maximal elements and equilibria for condensing correspondences, *Nonlinear Analysis, Theory Methods and Applications*, 38, 995–1002.
23. Schaefer, H.H. (1971), *Topological Vector Spaces*, Springer-Verlag, New York.
24. Ding, X.P. and Tarafdar, E. (2000), Generalized vector variational-like inequalities without monotonicity, In: *Vector Variational Inequalities and Vector Equilibria. Mathematical Theories*, pp. 113–124, Kluwer Academic Publishers, Dordrecht/Boston/London.
25. Dien, N.H. (1992), Some remarks on variational-like and quasivariational-like inequalities, *Bulletin of the Australian Mathematical Society*, 46, 335–342.
26. Zeidler, E. (1986), *Nonlinear Functional Analysis and Its Applications, I: Fixed Point Theorems*, Springer-Verlag, New York, Berlin.
27. Weir, T. and Jeyakumar, V. (1988), A class of nonconvex functions and mathematical programming, *Journal of the Australian Mathematical Society*, 38, 177–189.
28. Mohan, S.R. and Neogy, S.K. (1995), On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications*, 189, 901–908.
29. Aubin, J.P. (1979), *Mathematical Methods of Game and Economic Theory*, North-Holland Publishing Company, Amsterdam/New York/Oxford.