Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 321 (2006) 132–146
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Recession methods for generalized vector equilibrium problems

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Received 29 May 2005

Available online 8 September 2005

Submitted by H. Frankowska

Abstract

By using the recession method, we give some necessary and/or sufficient condition of solutions of generalized vector equilibrium problems.

Keywords: Generalized vector equilibrium problems; Recession cones

1. Introduction and formulations

A mathematical formulation of many problems, for instance, vector variational inequality problem, vector complementarity problem, vector optimization problem, vector saddle point problem, Nash equilibrium problem for vector-valued functions and fixed point problem, arise in mechanics, operations research, nonlinear analysis and game theory may be stated in the following form:

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.059

Given a nonempty set K in a real reflexive Banach space X and a vector-valued function $F : K \times K \to Y$, where Y is a real normed space with an ordered cone C, that is, a proper, closed and convex cone such that $\operatorname{int} C \neq \emptyset$,

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \notin -\operatorname{int} C, \quad \forall y \in K, \tag{1.1}$$

where $\operatorname{int} C$ denotes the interior of C.

Since (1.1) is equivalent to find an equilibrium point of a vector optimization problem, it is known as *vector equilibrium problem* (for short, VEP) and it has been the focus of attention of many researchers in the recent years; see, for example, [2,3,6,9,12,14,15,19] and references therein.

Let 2^Y be the family of all subsets of Y. There are several possible ways to generalize VEP for a given multivalued map $F : K \times K \to 2^Y \setminus \{\emptyset\}$, for instance,

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \not\subseteq -\operatorname{int} C, \quad \forall y \in K, \tag{1.2}$$

and

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in K. \tag{1.3}$$

The latter problem can also be written in the following form:

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \subseteq Y \setminus (-\operatorname{int} C), \quad \forall y \in K.$$

These problems are called *generalized vector equilibrium problems* (for short, GVEP). (1.2) is studied in [1,5,7,17] and references therein. While (1.3) is considered in [4]. Later, it is also studied by Georgiev and Tanaka [13], see, also [20,21]. In most of the papers appeared in the literature, the problem (1.2) is considered since it provides the weak formulation of Stampacchia type generalized vector variational inequality problems (for short, $(\text{SGVVIP})_w$). In the recent past, $(\text{SGVVIP})_w$ is used as a tool to provide the existence of a weak efficient solution[1] of vector optimization problem (for short, VOP); see, for example, [8] and references therein. However the strong formulation of Stampacchia type generalized vector variational inequality problems (for short, $(\text{SGVVIP})_s$) provides the necessary and sufficient condition for a weak efficient solution of VOP; see, for example, [10]. Moreover, (1.3) includes a more general form of generalized vector variational inequality problems (for short, GGVVIP) which contains $(\text{SGVVIP})_w$ and $(\text{SGVVIP})_s$.

It is clear that problem (1.3) is more stronger than problem (1.2) as every solution of (1.3) is also a solution of (1.2).

The following problem which is closely related to GVEP (1.3) can be termed as *dual generalized vector equilibrium problem* (for short, DGVEP):

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(y, \bar{x}) \cap (\operatorname{int} C) = \emptyset, \quad \forall y \in K. \tag{1.4}$$

[1] $x \in K$ is a weak efficient solution of the following vector optimization problem:

$$\min_{x \in K} \varphi(x), \quad \text{where } \varphi : K \to Y$$

if and only if $\varphi(y) - \varphi(x) \notin -\operatorname{int} C$ for all $y \in K$.

We shall denote by E_p and E_d the solution set of GVEP (1.3) and DGVEP (1.4), respectively.

We further consider the following strong formulation of GVEP:

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(\bar{x}, y) \subseteq C, \quad \forall y \in K, \tag{1.5}$$

and its dual form:

$$\text{find} \quad \bar{x} \in K \quad \text{such that} \quad F(y, \bar{x}) \subseteq -C, \quad \forall y \in K. \tag{1.6}$$

Problems (1.5) and (1.6) shall be called the *strong generalized vector equilibrium problem* (for short, SGVEP) and *dual strong generalized vector equilibrium problem* (for short, DSGVEP), respectively.

The absolute solution[2] of VOP can be found by using (1.5) and strong formulation of GGVVIP. The solution set of problem (1.5) and (1.6) are denoted by E_{sp} and E_{sd}, respectively.

In most of the papers appeared in the literature on the existence theory of solutions of VEP and GVEPs, either the set K is compact (in topological vector space setting)/bounded (in reflexive Banach space setting) or some coercivity condition is assumed. In the recent past, Flores-Bazán and Flores-Bazán [12] studied the existence of solutions of VEP under the asymptotic analysis, where neither compactness of K nor any coercivity condition is assumed. They gave some characterizations of nonemptiness of the solution set and also presented several alternative necessary and/or sufficient conditions for the solution set to be nonempty and compact.

In this paper, we extend the ideas of Flores-Bazán and Flores-Bazán [12] for GVEP. By using recession method, we provide several alternative necessary and/or sufficient conditions for the solution set of GVEP to be nonempty and bounded. In Section 2, preliminary definitions, results and notations are given. Section 3 deals with the alternative necessary and/or sufficient conditions for the solution set of GVEP to be nonempty and bounded.

2. Preliminaries

Throughout this paper X will be a real reflexive Banach space. For any given weakly closed set K in X, we define the recession cone of K as the set

$$K^\infty = \{x \in X\colon \exists t_n \downarrow 0,\ \exists x_n \in K,\ t_n x_n \rightharpoonup x\},$$

where "$\rightharpoonup$" means convergence in the weak topology. We set $\emptyset^\infty = \emptyset$.

In case K is also convex, it is known that

$$K^\infty = \{x \in X\colon \exists x_0 \in K,\ x_0 + tx \in K,\ \forall t > 0\}.$$

This cone does not depend on $x_0 \in K$.

We give some basic properties of recession cones in the following result which will be used in the sequel.

[2] $x \in K$ is called absolute solution of VOP if and only if $\varphi(y) - \varphi(x) \in C$ for all $y \in K$. It is clear that every absolute solution is a weak efficient solution, see [16].

Proposition 2.1. *The following holds*:

(a) $K_1 \subseteq K_2$ *implies* $K_1^\infty \subseteq K_2^\infty$;
(b) $(K + x)^\infty = K^\infty$, $\forall x \in X$;
(c) *let* $\{K_i\}_{i\in I}$ *be any family of nonempty sets in* X, *then*

$$\left(\bigcap_{i\in I} K_i\right)^\infty \subset \bigcap_{i\in I} (K_i)^\infty.$$

If, in addition, $\bigcap_{i\in I} K_i \neq \emptyset$ *and each set* K_i *is closed and convex, then we obtain an equality in the previous inclusion.*

Definition 2.1. Let K be a nonempty convex subset of X. For a given closed convex cone P of a real normed space Y, the multivalued map $S : K \to 2^Y \setminus \{\emptyset\}$ is called:

(i) *P-convex* if

$$\alpha S(x) + (1-\alpha)S(y) \subseteq S\big(\alpha x + (1-\alpha)y\big) + P, \quad \forall x, y \in K \text{ and } \forall \alpha \in [0,1];$$

(ii) *properly P-quasiconvex* if, $\forall x, y \in K$ and $\forall \alpha \in]0,1[$,

$$S(x) \subseteq S\big(\alpha x + (1-\alpha)y\big) + P \quad \text{or} \quad S(y) \subseteq S\big(\alpha x + (1-\alpha)y\big) + P;$$

(iii) *explicitly δ-quasiconvex* [17] if, $\forall x, y \in K$, $\forall \alpha \in]0,1]$, we have either

$$S(x) \subseteq S\big(\alpha x + (1-\alpha)y\big) + P \quad \text{or} \quad S(y) \subseteq S\big(\alpha x + (1-\alpha)y\big) + P$$

and if $(S(y) - S(x)) \cap (-\operatorname{int} P) \neq \emptyset$, we have

$$S(x) \subseteq S\big(\alpha x + (1-\alpha)y\big) + \operatorname{int} P, \quad \forall \alpha \in]0,1];$$

(iv) *weakly lower semicontinuous at* $x \in K$ if, for any $y \in S(x)$ and for any sequence $x_n \in K$ converges weakly to x, there exists a sequence $y_n \in S(x_n)$ converges strongly to y.

S is weakly lower semicontinuous on K if, it is weakly lower semicontinuous at each point of K.

3. Existence results for a solution of GVEP

For a nonempty subset A of a vector space, we denote by $\operatorname{co} A$ the convex hull of A.

The next abstract result is a generalization of [12, Theorem 3.1] for multivalued maps.

Theorem 3.1. *Let K be a nonempty closed, convex and bounded set in X and let W be any nonempty subset of Y. Let $F : K \times K \to 2^Y \setminus \{\emptyset\}$ be a multivalued map satisfying the following conditions*:

(A0) $F(x,x) \subseteq W$, $\forall x \in K$;
(A1) *for all* $x, y \in K$, $F(x,y) \subseteq W$ *implies* $F(y,x) \subseteq -W$;

(A2) *for all $x \in K$, the set $\{\xi \in K\colon\ F(x,\xi) \subseteq -W\}$ is (sequentially) weakly closed*;
(A3) *for all $x \in K$, the set $\{\xi \in K\colon\ F(x,\xi) \not\subseteq W\}$ is convex*;
(A4) *for all $x, y \in K$, $F(\xi, x) \subseteq -W$ for all $\xi \in]x, y]$ implies $F(x, y) \subseteq W$, where $]x, y]$ denotes the line segment joining x and y but not containing x.*

Then, the solution set to the problem

$$\textit{find} \quad \bar{x} \in K \quad \textit{such that} \quad F(\bar{x}, y) \subseteq W, \quad \forall y \in K$$

and that of the problem

$$\textit{find} \quad \bar{x} \in K \quad \textit{such that} \quad F(y, \bar{x}) \subseteq -W, \quad \forall y \in K,$$

are nonempty, weakly closed and both coincide.

Proof. Although, it is similar to the proof of [12, Theorem 3.1], we present it for the convenience of the readers.

We first find $\bar{x} \in K$ such that

$$\bar{x} \in \bigcap_{y \in K} \{x \in K\colon\ F(y, x) \subseteq -W\}.$$

To that end, we shall use the famous Ky Fan lemma [11] which is a generalization of KKM lemma (see, [22]). Set

$$G(y) = \{x \in K\colon\ F(y, x) \subseteq -W\}.$$

Assumption (A2) implies that for each $y \in K$, $G(y)$ is weakly closed and bounded, and since K is weakly compact, $G(y)$ is weakly compact. In order to apply the Fan lemma, we need to prove that for any finite subset $\{y_1, \ldots, y_k\}$ of K, $\mathrm{co}\{y_1, \ldots, y_k\} \subseteq \bigcup_{i=1}^k G(y_i)$. If $y = \sum_{i=1}^k \alpha_i y_i \notin \bigcup_{i=1}^k G(y_i)$ for some $\alpha_i \geqslant 0$, $i = 1, \ldots, k$, $\sum_{i=1}^k \alpha_i = 1$, then $y \notin G(y_i)$ for all $i = 1, \ldots, k$. Thus for each $i = 1, \ldots, k$, $F(y_i, y) \not\subseteq -W$ which implies $F(y, y_i) \not\subseteq W$ by assumption (A1). Thus $F(y, y) \not\subseteq W$ because of assumption (A3), which contradicts assumption (A0). This proves that for any finite subset $\{y_1, \ldots, y_k\}$ of K, $\mathrm{co}\{y_1, \ldots, y_k\} \subseteq \bigcup_{i=1}^k G(y_i)$. Hence by Fan–KKM lemma, there exists $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$, i.e., $F(y, \bar{x}) \subseteq -W$ for all $y \in K$, in other words, the second problem has a solution. By applying the assumption (A4), such a solution is also a solution to the first problem (see, [5, the proof of Proposition 3.2]). Since every solution to the first problem is a solution to the second problem by (A1), we deduce that both sets coincide. The weak closedness is a consequence of (A2). □

Remark 3.1. A similar result to that of Theorem 3.1 is derived in [5] for problem (1.2) in the setting of unbounded set K.

We now adapt the previous abstract result to our problem and we shall give simpler verifiable conditions on F ensuring the validity of all assumptions imposed in Theorem 3.1.

The basic assumptions on F are listed in hypothesis (H1) below.

Hypothesis (H1). The multivalued map $F : K \times K \to 2^Y \setminus \{\emptyset\}$ is such that

(f_0) for all $x \in K$, $F(x,x) \subseteq l(C) := C \cap (-C)$;
(f_1) for all $x, y \in K$, $F(x,y) \cap (-\operatorname{int} C) = \emptyset$ implies $F(y,x) \cap \operatorname{int} C = \emptyset$;
(f_2) for all $x \in K$, the mapping $F(x,\cdot) : K \to 2^Y \setminus \{\emptyset\}$ is C-convex;
(f_3) for all $x, y \in K$, the set $\{\xi \in [x,y]\colon\ F(\xi,y) \cap (-\operatorname{int} C) = \emptyset\}$ is closed. Here $[x,y]$ stands for the closed line segment joining x and y;
(f_4) for all $x \in K$, $F(x,\cdot)$ is weakly lower semicontinuous.

Remark 3.2.

(a) One can check immediately that the C-convexity of $F(x,\cdot)$ implies that for all $x \in K$, the set
$$\bigl\{\xi \in K\colon\ F(x,\xi) \not\subseteq Y \setminus (-\operatorname{int} C)\bigr\}$$
is convex. Hence condition (A3) (with $W = Y \setminus (-\operatorname{int} C)$) of Theorem 3.1 holds.
(b) It can be easily seen that the weakly lower semicontinuity of $F(x,\cdot)$ asserts the (sequential) weak closedness of
$$\bigl\{\xi \in K\colon\ F(x,\xi) \subseteq Y \setminus (-\operatorname{int} C)\bigr\}$$
for all $x \in K$. Thus condition (A2) (with $W = Y \setminus (-\operatorname{int} C)$) of Theorem 3.1 is satisfied.
(c) Assumptions (f_0), (f_2) and (f_3) in hypothesis (H1) imply that, given any $x \in K$,
$$0 \in F(y,x) + \bigl(Y \setminus (-\operatorname{int} C)\bigr), \quad \forall y \in K, \quad \text{implies}$$
$$F(x,y) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in K. \tag{3.1}$$
Hence condition (A4) (with $W = Y \setminus (-\operatorname{int} C)$) of Theorem 3.1 holds.

Indeed, for every $y \in K$ consider $x_t = x + t(y - x)$ for $t \in]0,1[$. Clearly $x_t \in K$. The C-convexity of $F(x_t,\cdot)$ implies
$$tF(x_t,y) + (1-t)F(x_t,x) \subseteq F(x_t,x_t) + C \subseteq C.$$
Since $0 \in F(x_t,x) + (Y \setminus (-\operatorname{int} C))$, there exists $\xi(x_t,x) \in F(x_t,x)$ such that $\xi(x_t,x) \notin \operatorname{int} C$. From a previous inclusion one has
$$tF(x_t,y) \subseteq -(1-t)\xi(x_t,x) + C \subseteq \bigl(Y \setminus (-\operatorname{int} C)\bigr) + C \subseteq Y \setminus (-\operatorname{int} C).$$
It turns out that $F(x_t,y) \subseteq Y \setminus (-\operatorname{int} C)$ or, equivalently, $F(x_t,y) \cap (-\operatorname{int} C) = \emptyset$. Letting $t \downarrow 0$, we obtain by assumption (f_3), $F(x,y) \cap (-\operatorname{int} C) = \emptyset$. Since y was arbitrary, the desired result is proved. □

The following result shows that (3.1) also holds if we replace the C-convexity of $F(x,\cdot)$ by the explicitly δ-quasiconvexity for each $x \in K$.

Proposition 3.1. *Assume that the multivalued map $F : K \times K \to 2^Y \setminus \{\emptyset\}$ satisfies assumptions (f_0) and (f_3) such that $F(x,\cdot)$ is explicitly δ-quasiconvex for each $x \in K$. Then* (3.1) *holds.*

Proof. For a given $x \in K$, let

$$0 \in F(y,x) + \big(Y \setminus (-\operatorname{int} C)\big), \quad \forall y \in K. \tag{3.2}$$

Suppose there exists $y \in K$ such that $F(x,y) \cap (-\operatorname{int} C) \neq \emptyset$. (f_3) can be written, in an equivalent way, as

(f_3) for all $x, y \in K$, the set $\{\xi \in [x,y]\colon\ F(\xi,y) \cap (-\operatorname{int} C) \neq \emptyset\}$ is open (in $[x,y]$).

Since $x \in \{\xi \in [x,y]\colon\ F(\xi,y) \cap (-\operatorname{int} C) \neq \emptyset\} := M$, there exists $\alpha \in]0,1[$ such that

$$z := x + \alpha(y-x) = \alpha y + (1-\alpha)x \in M,$$

that is,

$$z \in [x,y] \quad \text{and} \quad F(z,y) \cap (-\operatorname{int} C) \neq \emptyset. \tag{3.3}$$

Now by explicitly δ-quasiconvexity of $F(z,\cdot)$, we have

$$F(z,y) \subseteq F(z,z) + C \subseteq C \subseteq W := Y \setminus (-\operatorname{int} C)$$

which implies that

$$F(z,y) \cap (-\operatorname{int} C) = \emptyset,$$

a contradiction of (3.3). Therefore, we can only have

$$F(z,x) \subseteq F(z,z) + C \subseteq C \subseteq W,$$

that is,

$$F(z,x) \cap (-\operatorname{int} C) = \emptyset. \tag{3.4}$$

Relations (3.3) and (3.4) imply

$$\big[F(z,y) - F(z,x)\big] \cap (-\operatorname{int} C) \neq \emptyset.$$

By explicitly δ-quasiconvexity of $F(z,\cdot)$, we have

$$F(z,x) \subseteq F(z,z) + \operatorname{int} C \subseteq C + \operatorname{int} C \subseteq \operatorname{int} C$$

which contradicts (3.2). Hence $F(x,y) \subseteq Y \setminus (-\operatorname{int} C), \forall y \in K$. $\square$

The previous implication is related to a certain maximal pseudomonotonicity condition already discussed by Oettli in [19] and Ansari et al. in [5].

In the same lines of reasoning as in [12], we introduce the following cones in order to deal with the unbounded case, that is, when K is an unbounded set,

$$R_0 := \bigcap_{y \in K} \big\{v \in K^\infty\colon\ 0 \in F(y, z + \lambda v) + W,\ \forall \lambda > 0,\ \forall z \in K \text{ such that } F(y,z) \subseteq -C\big\}$$

and

$$R_1 := \bigcap_{y \in K} \big\{v \in K^\infty\colon\ 0 \in F(y, y + \lambda v) + W,\ \forall \lambda > 0\big\},$$

where $W = Y \setminus (-\operatorname{int} C)$.

We note that the sets R_0 and R_1 are nonempty (because of assumption (f_0)) closed cone but not necessarily convex. Clearly, $R_0 \subset R_1$.

Proposition 3.2. *Let K be a nonempty closed convex subset of X and let* (3.1) *hold. Then*

$$\begin{aligned} R_{11} &:= \bigcap_{y\in K} \big\{v \in K^\infty\colon\ 0 \in F(y, y+\lambda v) + \big(Y \setminus (-\operatorname{int} C)\big),\ \forall \lambda > 0\big\} \\ &\subset \bigcap_{y\in K} \big\{v \in K^\infty\colon\ F(y+\lambda v, y) \cap (-\operatorname{int} C) = \emptyset,\ \forall \lambda > 0\big\}. \end{aligned}$$

Proof. It is straightforward. □

Remark 3.3. If $F : K \times K \to 2^Y \setminus \{\emptyset\}$ is a multivalued map satisfying assumption (f_0) such that for all $x \in K$, $F(x, \cdot) : K \to 2^Y \setminus \{\emptyset\}$ is C-convex, then the conclusion of Proposition 3.2 also holds.

Indeed, let $v \in R_{11}$. Then $v \in K^\infty$, and for all $y \in K$ and all $\lambda > 0$, there exists $\xi(y, y+\lambda v) \in F(y, y+\lambda v)$ such that $\xi(y, y+\lambda v) \in -W$. On the other hand, for any $y \in K$ and $\lambda > 0$, the C-convexity of $F(y+\lambda v, \cdot)$ implies

$$\frac{1}{2}F(y+\lambda v, y+\lambda v+\lambda v) + \frac{1}{2}F(y+\lambda v, y) \subset F(y+\lambda v, y+\lambda v) + C \subset C.$$

Thus

$$\frac{1}{2}F(y+\lambda v, y) \subset -\frac{1}{2}\xi(y+\lambda v, y+2\lambda v) + C \subset W + C \subset W.$$

Hence $F(y+\lambda v, y) \subset W$. Since $y \in K$ and $\lambda > 0$ were arbitrary, we conclude the proof.

Theorem 3.2. *Let K be a nonempty closed convex subset of X. Let $F : K \times K \to 2^Y \setminus \{\emptyset\}$ be a multivalued map satisfying assumptions (f_0), (f_1) and (f_4) such that $F(x, \cdot) : K \to 2^Y \setminus \{\emptyset\}$ is C-convex. Then*

$$\begin{aligned} (E_p)^\infty \subset R_1 &\subset \bigcap_{y\in K} \big\{x \in K\colon\ F(x, y) \cap (-\operatorname{int} C) = \emptyset\big\}^\infty \\ &\subset \bigcap_{y\in K} \big\{x \in K\colon\ F(y, x) \cap (\operatorname{int} C) = \emptyset\big\}^\infty. \end{aligned}$$

If, in addition, there exists $x^* \in K$ such that $F(y, x^*) \subseteq -C$ for all $y \in K$, then $E_p^\infty = R_1$.

Proof. As before, set $W := Y \setminus (-\operatorname{int} C)$. Let us prove the first inclusion. Let $v \in (E_p)^\infty$, then there exist $t_k \downarrow 0$, $u_k \in E_p$ such that $t_k u_k \rightharpoonup v$. For $y \in K$ arbitrary, we have $F(u_k, y) \subseteq W$ for all $k \in \mathbb{N}$. By assumption (f_1), we have $F(y, u_k) \subseteq -W$ for all $k \in \mathbb{N}$. Let us fix any $\lambda > 0$. For k sufficiently large, C-convexity of $F(y, \cdot)$ implies

$$(1-\lambda t_k)F(y, y) + \lambda t_k F(y, u_k) \subseteq F\big(y, (1-\lambda t_k)y + \lambda t_k u_k\big) + C.$$

Hence

$$0 \in F\big(y, (1-\lambda t_k)y + \lambda t_k u_k\big) + W + C \subseteq F\big(y, (1-\lambda t_k)y + \lambda t_k u_k\big) + W.$$

From assumption (f_4), it follows that $0 \in F(y, y+\lambda v) + W$. This proves $v \in R_1$.

The proof of the second inclusion is as follows. Let $v \in K^\infty$ such that $0 \in F(y, y+\lambda v) + W$ for all $\lambda > 0$ and all $y \in K$. By the previous proposition $F(y+\lambda v, y) \subseteq W$ for all $\lambda > 0$ and all $y \in K$. For any fixed $y \in K$, set $x_k := y + kv \in K$ for all $k \in \mathbb{N}$. Then $F(x_k, y) \subseteq W$ for all $k \in \mathbb{N}$. By choosing $t_k = \frac{1}{k}$, we have $t_k x_k = \frac{y}{k} + v \to v$ as $k \to +\infty$, that is, $v \in \{x \in K\colon\ F(x, y) \subseteq W\}^\infty$. Since y was arbitrary, the proof of the second inclusion is complete.

The last inclusion is a consequence of assumption (f_1).

Let us prove the last part of the theorem. By our hypothesis, there exists $x^* \in K$ such that $F(y, x^*) \subseteq -C$ for all $y \in K$. Let $v \in R_1$. Then for all $y \in K$ and for all $\lambda > 0$, $0 \in F(y, x^* + \lambda v) + W$. By previous proposition, $F(x^* + \lambda v, y) \subseteq W$. Thus for all $\lambda > 0$, $x^* + \lambda v \in E_p$. Hence $v \in (E_p)^\infty$ and thus $R_1 \subseteq (E_p)^\infty$. Consequently, $R_1 = (E_p)^\infty$. □

Theorem 3.3. *Let K be a nonempty closed convex set in X and let $F : K \times K \to 2^Y \setminus \{\emptyset\}$ be a multivalued map satisfying hypothesis* (H1). *If,*

(∗) *for every sequence $\{x_k\}$ in K, $\|x_k\| \to +\infty$ such that $\frac{x_k}{\|x_k\|} \rightharpoonup v$ with $v \in R_1$ and for all $y \in K$ it exists k_y such that $F(x_k, y) \subseteq Y \setminus (-\operatorname{int} C)$ for all $k \geqslant k_y$, there exists $u \in K$ such that $\|u\| < \|x_k\|$ and $F(x_k, u) \subseteq -C$ for $k \in \mathbb{N}$ sufficiently large,*

then problem (1.3) *admits a solution. Indeed, E_p is a nonempty weakly closed set.*

Proof. For every $k \in \mathbb{N}$, set $K_k := \{x \in K\colon\ \|x\| \leqslant k\}$. We may assume, without loss of generality, that $K_k \neq \emptyset$ for all $k \in \mathbb{N}$. Let us consider the problem

$$\text{find} \quad \bar{x} \in K_k \quad \text{such that} \quad F(\bar{x}, y) \subseteq Y \setminus (-\operatorname{int} C), \quad \forall y \in K_k. \tag{3.5}$$

Taking into account Remark 3.2, we apply Theorem 3.1 (with $W = Y \setminus (-\operatorname{int} C)$) to conclude that problem (3.5) admits a solution, say $x_k \in K_k$ for all $k \in \mathbb{N}$. If $\|x_k\| < k$ for some $k \in \mathbb{N}$, then, we claim that x_k is also a solution to problem (1.3). Suppose to the contrary that x_k is not a solution to problem (1.3). Then there exists $y \in K$ with $\|y\| > k$ such that $F(x_k, y) \not\subseteq Y \setminus (-\operatorname{int} C)$. We choose $z \in K$ with $z \in\,]x_k, y[$ and $\|z\| < k$. Writing $z = \alpha x_k + (1-\alpha)y$ for some $\alpha \in\,]0, 1[$ then by C-convexity of $F(x_k, \cdot)$, we have

$$\alpha F(x_k, x_k) + (1-\alpha)F(x_k, y) \subseteq F(x_k, z) + C.$$

This implies

$$(1-\alpha)F(x_k, y) \subseteq W + C \subseteq W.$$

It follows that $F(x_k, y) \subseteq W = Y \setminus (-\operatorname{int} C)$, which contradicts to our supposition. Hence x_k is a solution to problem (1.3).

We consider now the case $\|x_k\| = k$ for all $k \in \mathbb{N}$. We may assume, up to a subsequence, that $\frac{x_k}{\|x_k\|} \rightharpoonup v$ for $v \in K$. Then $v \neq 0$ and $v \in K^\infty$. For any fixed $y \in K$ and $\lambda > 0$, we have $F(x_k, y) \subseteq W$, $\forall k \in \mathbb{N}$ sufficiently large ($k > \|y\|$). By (f_1), $F(y, x_k) \subseteq Y \setminus (\operatorname{int} C)$ for

all k sufficiently large. For every $\lambda > 0$ and all k sufficiently large, C-convexity of $F(y,\cdot)$ implies

$$\left(1-\frac{\lambda}{\|x_k\|}\right)F(y,y)+\frac{\lambda}{\|x_k\|}F(y,x_k)\subseteq F\left(y,\left(1-\frac{\lambda}{\|x_k\|}\right)y+\frac{\lambda}{\|x_k\|}x_k\right)+C.$$

Hence

$$0\in F\left(y,\left(1-\frac{\lambda}{\|x_k\|}\right)y+\frac{\lambda}{\|x_k\|}x_k\right)+\big(Y\setminus(\operatorname{int}C)\big)+C.$$

Thus, by assumption (f_4) of hypothesis (H1), $0\in F(y,y+\lambda v)+W$. This proves $v\in R_1$. By assumption, there exist $u\in K$ such that $\|u\|<\|x_k\|$ and $F(x_k,u)\subseteq -C$ for k sufficiently large. We claim that x_k is also a solution to problem (1.3). Suppose contrary that x_k is not a solution of problem (1.3). Then there exists $y\in K$, $\|y\|>k$ such that $F(x_k,y)\not\subseteq Y\setminus(-\operatorname{int}C)=W$. Since $\|u\|<\|x_k\|$ we can find $z\in\,]u,y[$ such that $\|z\|<k$. Thus for some $\alpha\in\,]0,1[$, we have by C-convexity of $F(x_k,\cdot)$,

$$\alpha F(x_k,u)+(1-\alpha)F(x_k,y)\subseteq F(x_k,z)+C\subseteq W+C.$$

This implies

$$(1-\alpha)F(x_k,y)\subseteq W,$$

a contradiction to our supposition is proving that x_k is a solution to (1.3). □

Remark 3.4. During the preparation of this paper, authors came to know that Lee and Bu [18] also considered GVEP (1.3) in the setting of finite-dimensional Euclidean space $\mathbb{R}^n$ but for a moving cone.

Example 3.1. The function $F(x,y)=(y-x,x-y)$, $K=\mathbb{R}$, $C\equiv\mathbb{R}^2_+$, does not satisfy condition $(*)$ while $E_p=\mathbb{R}$. Notice also that none of the results appearing in [12,18] is applicable.

We now establish a couple of necessary and sufficient conditions for the nonemptiness of E_p for a class of multivalued map F defined on $K\subseteq\mathbb{R}$, which apply to the previous example. This characterization is new.

(†) for every sequence (x_n) in K with $|x_n|\to+\infty$, $\frac{x_n}{|x_n|}\to v$, $v\in R_1$, and for all $y\in K$ it exists n_y such that $F(x_n,y)\subset Y\setminus(-\operatorname{int}C)$ for all $n\geqslant n_y$, there exist $u\in K$ and $\bar{n}$ such that $|u|<|x_{\bar{n}}|$ and $F(x_{\bar{n}},u)\subseteq Y\setminus(\operatorname{int}C)$.

(††) for every sequence (x_n) in K with $|x_n|\to+\infty$, there exist n_0, $u\in K$, such that $F(x_n,u)\subset Y\setminus\operatorname{int}C$ for all $n\geqslant n_0$.

Theorem 3.4. *Let $K\subseteq\mathbb{R}$ be a closed convex set and let $F:K\to 2^Y\setminus\{\emptyset\}$ be a multivalued map satisfying hypothesis* (H1). *Then E_p is closed convex set, and the following three assertions are equivalent*:

(a) *E_p is nonempty*;
(b) (†) *is satisfied*;
(c) (††) *is satisfied.*

Proof. The closedness of E_p is obtained as before. We reason as follows to prove the convexity: take $x_1, x_2 \in E_p$, $x_1 < x_2$, and $x \in]x_1, x_2[$. Then if $y \in K$, $y > x$, we write $x = \alpha x_1 + (1-\alpha)y$ and use the C-convexity of $F(x, \cdot)$ to obtain $F(x, y) \subset Y \setminus (-\operatorname{int} C)$. In case $y \in K$, $y < x$, we write $x = \alpha x_2 + (1-\alpha)y$ and proceed as before to conclude again $F(x, y) \subset Y \setminus (-\operatorname{int} C)$. Thus $x \in E_p$, proving the convexity of E_p.

We now prove the equivalences.

(c) $\Rightarrow$ (b). It is obvious.

(a) $\Rightarrow$ (c). It follows by taking as u any element in E_p.

(b) $\Rightarrow$ (a). We proceed as in the proof of Theorem 3.3 until being in the case when the sequence $x_n \in K$, satisfies $n = |x_n| \to +\infty$, $\frac{x_n}{|x_n|} \to v \in R_0$ and for all $y \in K$, n_y exists such that $F(x_n, y) \subset Y \setminus (-\operatorname{int} C)$ for all $n \geqslant n_y$. By assumption (†), there exist $u \in K$ and $\bar{n}$ such that $|u| < |x_{\bar{n}}|$ and $F(x_{\bar{n}}, u) \subseteq Y \setminus (\operatorname{int} C)$. We also have $F(x_{\bar{n}}, u) \subseteq Y \setminus (-\operatorname{int} C)$ because of the choice of $x_{\bar{n}}$. We claim that such $x_{\bar{n}}$ is a solution to (1.3). It only remains to check that $F(x_{\bar{n}}, y) \subseteq Y \setminus (-\operatorname{int} C)$ for all $y \in K$ with $|y| > \bar{n}$. In the case when $x_{\bar{n}} \in [u, y]$ or $x_{\bar{n}} \in [y, u]$, the C-convexity of $F(x_{\bar{n}}, \cdot)$ implies, for some $\alpha \in [0, 1]$,

$$\alpha F(x_{\bar{n}}, u) + (1-\alpha) F(x_{\bar{n}}, y) \subseteq F(x_{\bar{n}}, x_{\bar{n}}) + C.$$

Then

$$(1-\alpha) F(x_{\bar{n}}, y) \subseteq C - \alpha F(x_{\bar{n}}, u) \subset Y \setminus (-\operatorname{int} C),$$

proving the claim. If on the contrary $u \in [y, x_{\bar{n}}]$ or $u \in [x_{\bar{n}}, y]$, for some $\alpha \in [0, 1]$, we have as before

$$\alpha F(x_{\bar{n}}, y) + (1-\alpha) F(x_{\bar{n}}, x_{\bar{n}}) \subseteq F(x_{\bar{n}}, u) + C.$$

It follows that

$$(1-\alpha) F(x_{\bar{n}}, y) \subseteq Y \setminus (-\operatorname{int} C).$$

This completes the proof of the claim and therefore $E_p \neq \emptyset$. $\square$

4. Existence of a solution of strong GVEP

For the existence of a solution of strong GVEP, the basic assumptions on F are listed in hypothesis (H2) below.

Hypothesis (H2). The multivalued map $F : K \times K \to 2^Y \setminus \{\emptyset\}$ is such that

(f_0) for all $x \in K$, $F(x, x) \subseteq l(C) := C \cap (-C)$;
(f_1') for all $x, y \in K$, $F(x, y) \subseteq C$ implies $F(y, x) \subseteq -C$;
(f_2') for all $x \in K$, the mapping $F(x, \cdot) : K \to 2^Y \setminus \{\emptyset\}$ is properly C-quasiconvex;

(f_3') for all $x, y \in K$, the set $\{\xi \in [x, y]\colon F(y, \xi) \subseteq C\}$ is closed. Here $[x, y]$ stands for the closed line segment joining x and y;
(f_4') for all $x \in K$, $F(x, \cdot)$ is weakly lower semicontinuous on K.

Remark 4.1.

(a) One can check that the proper C-quasiconvexity of $F(x, \cdot)$ implies that the set

$$\{\xi \in K\colon F(x, \xi) \not\subseteq C\}$$

is convex for all $x \in K$. Hence condition (A3) (with $W = C$) of Theorem 3.1 is satisfied.

(b) It can be easily seen that the weakly lower semicontinuity of $F(x, \cdot)$ asserts the weak closedness of

$$\{\xi \in K\colon F(x, \xi) \subseteq -C\}$$

for all $x \in K$. Thus condition (A3) (with $W = C$) of Theorem 3.1 is satisfied.

(c) Similar to part (c) of Remark 3.2, one can prove, under assumptions (f_0), (f_3') in hypothesis (H2) and C-convexity of $F(x, \cdot)$, that given any $x \in K$,

$$0 \in F(y, x) + C, \quad \forall y \in K, \quad \text{implies} \quad F(x, y) \cap (Y \setminus C) = \emptyset, \quad \forall y \in K. \tag{4.1}$$

Hence condition (A4) (with $W = C$) of Theorem 3.1 holds.

The following result is a particular case of [19, Corollary 4], but it is obtained from Theorem 3.1 by specializing $W = C$.

Lemma 4.1. *Let $K \subseteq X$ be nonempty weakly compact convex set and let $F : K \times K \to 2^Y \setminus \{\emptyset\}$ be a multivalued map satisfying hypothesis* (H2) *such that for all $x \in K$, $F(x, \cdot)$ is C-convex. Then, E_{sp} is nonempty and weakly compact.*

Proof. The weak compactness is obtained as usual. To prove the nonemptiness of E_{sp} we will show that the assumption of Theorem 3.1 are satisfied when specialized to $W = C$. This was proved in Remark 4.1. □

In the present situation the cones to be considered are the following:

$$R_0' := \bigcap_{y \in K} \{v \in K^\infty\colon 0 \in F(y, y + \lambda v) + C,\ \forall \lambda > 0, \\ \forall z \in K \text{ such that } F(y, z) \subseteq -C\}$$

and

$$R_1' := \bigcap_{y \in K} \{v \in K^\infty\colon 0 \in F(y, y + \lambda v) + C,\ \forall \lambda > 0\}.$$

Theorem 4.1. *Let $K \subseteq X$ be a nonempty closed convex set. Assume that F satisfies hypothesis* (H2) *such that $F(x, \cdot)$ is C-convex for all $x \in K$. Then, E_{sp} is a nonempty and weakly closed set if and only if the following condition holds*:

$(*)'$ *for every sequence* $\{x_n\}$ *in* K, $\|x_n\| \to +\infty$, $\frac{x_n}{\|x_n\|} \rightharpoonup v$ *with* $v \in R_1'$ *and for all* $y \in K$ *it exists* n_y *such that* $F(x_n, y) \subseteq C$ *for all* $n \geqslant n_y$, *there exist* $u \in K$ *and* $\bar{n}$, *such that* $\|u\| < \|x_{\bar{n}}\|$ *and* $F(x_{\bar{n}}, u) \subseteq -C$.

Proof. To prove $(*)'$ is sufficient for $E_{sp} \neq \emptyset$, a reasoning similar to the proof of Theorem 3.3 is applied. Instead of considering (3.5), we consider the problem

$$\text{find} \quad \bar{x} \in K_n \quad \text{such that} \quad F(\bar{x}, y) \subseteq C, \quad \forall y \in K_n. \tag{4.2}$$

Such a problem admits a solution by Lemma 4.1, say $x_n \in K_n$, for all $n \in \mathbb{N}$. If $\|x_n\| < n$ for some $n \in \mathbb{N}$, we will show that x_n is also a solution to problem (1.5). In fact, for any fixed $y \in K$ with $\|y\| > n$, we take $z \in K$ with $z \in]x_n, y[$ and $\|z\| < n$. Writing $z = \alpha x_n + (1-\alpha)y$ for some $\alpha \in]0, 1[$, we have

$$\alpha F(x_n, x_n) + (1-\alpha)F(x_n, y) \subseteq F(x_n, z) + C.$$

This implies

$$(1-\alpha)F(x_n, y) \subseteq C,$$

proving the desired result.

We consider now the case $\|x_n\| = n$ for all $n \in \mathbb{N}$. We may assume, without loss of generality, that $\frac{x_n}{\|x_n\|} \rightharpoonup v$. Then $v \in K^\infty$. For any fixed $y \in K$, $F(x_n, y) \subseteq C$ for all $n \in \mathbb{N}$ sufficiently large ($n > \|y\|$). For every $\lambda > 0$ and all n sufficiently large, C-convexity of $F(y, \cdot)$ implies

$$\left(1 - \frac{\lambda}{\|x_n\|}\right)F(y, y) + \frac{\lambda}{\|x_n\|}F(y, x_n) \subseteq F\left(y, \left(1 - \frac{\lambda}{\|x_n\|}\right)y + \frac{\lambda}{\|x_k\|}x_n\right) + C.$$

Hence

$$0 \in F\left(y, \left(1 - \frac{\lambda}{\|x_n\|}\right)y + \frac{\lambda}{\|x_n\|}x_n\right) + C.$$

Thus, by (f_4'), $0 \in F(y, z + \lambda v) + C$. This proves $v \in R_1'$. Now, we can use assumption $(*)'$ to ensure the existence of $u \in K$ and $\bar{n}$ such that $\|u\| < \|x_{\bar{n}}\|$ and $F(x_{\bar{n}}, u) \subseteq -C$. Similarly as in the proof of Theorem 3.3, one can check that $x_{\bar{n}}$ is also a solution to problem (1.5). The weakly closedness of E_{sp} follows as usual. The "necessity" of $(*)'$ is shown by taking element in E_{sp} as the point u required in condition $(*)'$. □

For algorithmic purposes it is desirable to know a priori when the solution set is bounded, in this case arises the next condition (f_5') giving rise to the characterization expressed in Theorem 4.2:

(f_5') Any sequence $x_n \in K$ with $\|x_n\| \to +\infty$ such that for all $y \in K$, n_y exists such that

$$F(x_n, y) \subseteq C \quad \text{when } n \geqslant n_y,$$

admits a subsequence $\{x_{n_k}\}$ such that $\{\frac{x_{n_k}}{\|x_{n_k}\|}\}$ converges strongly.

Remark 4.2. When Y is a finite-dimensional space, a condition implying condition $(*')$ (with $v \in R_1'$) described in the preceding theorem is $R_1' \subset -R_1'$ (in particular, if $R_1' = \{0\}$), since in this case, for all $v \in R_1'$,

$$0 \in F(y, y + \lambda v) + C, \quad \forall \lambda \in \mathbb{R},\ \forall y \in K.$$

Indeed, $(*)'$ is satisfied by taking $u = x_n - \|x_n\| v$. Notice that $\frac{x_n}{\|x_n\|} \to v$ implies $\|u\| < \|x_n\|$ for all n sufficiently large.

Theorem 4.2. *Let $K \subset X$ be a nonempty closed and convex. Assume that F satisfies hypothesis* (H2) *and assumption* (f_5') *such that $F(x, \cdot)$ is C-convex. Then, the following assertions are equivalent*:

(a) *E_{sp} is nonempty and weakly compact*;
(b) *$\exists r > 0$, $\forall x \in K \setminus K_r$, $\exists y \in K_r$: $F(x, y) \not\subseteq C$, where $K_r = \{x \in K\colon \|x\| \leqslant r\} \neq \emptyset$.*

Proof. (a) $\Rightarrow$ (b). It follows from the previous corollary since E_{sp} is also bounded.

(b) $\Rightarrow$ (a). This implication is a consequence of Theorem 4.1 and Corollary 4.1 by taking into account (f_5'), since in this case $(*)'$ holds vacuously. □

References

[1] Q.H. Ansari, I.V. Konnov, J.C. Yao, On generalized vector equilibrium problems, Nonlinear Anal. 47 (2001) 543–554.
[2] Q.H. Ansari, I.V. Konnov, J.C. Yao, Existence of a solution and variational principles for vector equilibrium problems, J. Optim. Theory Appl. 110 (2001) 481–492.
[3] Q.H. Ansari, I.V. Konnov, J.C. Yao, Characterizations of solutions for vector equilibrium problems, J. Optim. Theory Appl. 113 (2002) 435–447.
[4] Q.H. Ansari, W. Oettli, D. Schläger, A generalization of vectorial equilibria, Math. Methods Oper. Res. 46 (1997) 147–152.
[5] Q.H. Ansari, A.H. Siddiqi, S.Y. Wu, Existence and duality of generalized vector equilibrium problems, J. Math. Anal. Appl. 259 (2001) 115–126.
[6] Q.H. Ansari, X.Q. Yang, J.C. Yao, Existence and duality of implicit vector variational problems, Numer. Funct. Anal. Optim. 22 (2001) 815–829.
[7] Q.H. Ansari, J.C. Yao, An existence result for the generalized vector equilibrium problem, Appl. Math. Lett. 12 (1999) 53–56.
[8] Q.H. Ansari, J.C. Yao, On nondifferentiable and nonconvex vector optimization problems, J. Optim. Theory Appl. 106 (2000) 487–500.
[9] M. Bianchi, N. Hadjisavvas, S. Schaible, Vector equilibrium problems with generalized monotone bifunctions, J. Optim. Theory Appl. 92 (1997) 527–542.
[10] G.Y. Chen, B.D. Craven, A vector variational inequality and optimization over an efficient set, ZOR-Methods Models Oper. Res. 3 (1990) 1–12.
[11] K. Fan, A generalization of Tychonoff's fixed point theorem, Math. Ann. 142 (1961) 305–310.
[12] F. Flores-Bazán, F. Flores-Bazán, Vector equilibrium problems under asymptotic analysis, J. Global Optim. 26 (2003) 141–166.
[13] P.G. Georgiev, T. Tanaka, Vector-valued set-valued variants of Ky Fan's inequality, J. Nonlinear Convex Anal. 1 (2000) 245–254.
[14] F. Giannessi (Ed.), Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, Kluwer Academic, Dordrecht, 2000.

[15] N. Hadjisavvas, S. Schaible, From scalar to vector equilibrium problems in the quasimonotone case, J. Optim. Theory Appl. 96 (1998) 297–309.
[16] S. Komlósi, On the Stampacchia and minty variational inequalities, in: G. Giorgi, F. Rossi (Eds.), Generalized Convexity and Optimization for Economic and Financial Decisions, Pitagora Editrica, Bologna, Italy, 1999, pp. 231–260.
[17] I.V. Konnov, J.C. Yao, Existence of solutions for generalized vector equilibrium problems, J. Math. Anal. Appl. 233 (1999) 328–335.
[18] G.M. Lee, I.J. Bu, On vector equilibrium problems with multifunctions, Taiwanese J. Math. (2005), in press.
[19] W. Oettli, A remark on vector-valued equilibria and generalized monotonicity, Acta Math. Vietnam. 22 (1997) 213–221.
[20] W. Oettli, D. Schläger, Generalized vectorial equilibria and generalized monotonicity, in: M. Brokate, A.H. Siddiqi (Eds.), Functional Analysis with Current Applications in Science, Technology and Industry, in: Pitman Res. Notes in Math. Ser., vol. 377, Longman, Essex, 1998, pp. 145–154.
[21] W. Oettli, D. Schläger, Existence of equilibria for monotone multivalued mappings, Math. Methods Oper. Res. 48 (1998) 219–228.
[22] G.X.-Z. Yuan, KKM Theory and Applications in Nonlinear Analysis, Dekker, New York, 1999.