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Fixed point and maximal element theorems with applications to abstract economies and minimax inequalities ☆,☆☆

Lai-Jiu Lin,[a,*] Zenn-Tsuen Yu,[b] Qamrul Hasan Ansari,[c] and Li-Ping Lai [a]

[a] *Department of Mathematics, National Changhua University of Education, Changhua 50058, Taiwan, ROC*
[b] *Department of Electrical Engineering, Nan-Kai College, Nantour 542, Taiwan, ROC*
[c] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*

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Abstract

In this paper, we establish some fixed point theorems for a family of multivalued maps under mild conditions. By using our fixed point theorems, we derive some maximal element theorems for a particular family of multivalued maps, namely the Φ-condensing multivalued maps. As applications of our results, we prove some general equilibrium existence theorems in the generalized abstract economies with preference correspondences. Further applications of our results are also given to minimax inequalities for a family of functions.

Keywords: Fixed point theorems; Maximal element theorems; Φ-condensing multivalued maps; Transfer compactly open valued maps; Generalized abstract economies; Minimax inequalities

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* Corresponding author.
E-mail address: maljlin@math.ncue.edu.tw (L.-J. Lin).

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1. Introduction

In [3], Border not only gave a wide list of applications of fixed point theorems to, say, general equilibrium theorem in economics, but, more interesting, it is also proved that some result appearing in economics about the existence of equilibria is indeed equivalent to some classical fixed point theorem coming from pure mathematics. Thus, in a way, we could say that not only such results in pure mathematics (namely, fixed point theorems) have applications in some other sciences or disciplines (e.g., game theory, optimization theory, and economics), but actually starting from contexts of other disciplines (e.g., economics) we can restate and reobtain classical results in mathematics.

It is well known that the famous Browder fixed point theorem [4] is equivalent to a maximal element theorem (see [28]). In the last decade, many generalized forms of Browder fixed point theorem are used to establish the maximal element theorems for a family of multivalued maps. Such kind of maximal element theorems are useful to establish the existence of a solution of abstract economies or generalized games, system of variational inequalities, etc; see, for example, [6–9,15–19,27,29,30] and references therein. The Browder fixed point theorem has also been generalized for a family of multivalued maps with applications to maximal elements theory, generalized games or abstract economies, system of variational inequalities, etc; see, for example, [1,2,7,8,14,23,29] and references therein.

Motivated by the fact that any preference of a real agent could be unstable by the fuzziness of consumers' behaviour or market situations, Kim and Tan [13] introduced the generalized abstract economies with general preference correspondences and proved the existence of their solution by using the Himmelberg fixed point theorem and the Eilenberg–Montgomery fixed point theorem.

In this paper, we establish some fixed point theorems for a family of multivalued maps under weaker assumptions than in [1,14] and references therein. Our fixed point results can be seen as generalizations of Browder fixed point theorem. By using our fixed point results, we derive some maximal element theorems for a particular family of multivalued maps, namely the Φ-condensing multivalued maps. As applications of our maximal element theorems, we prove some general equilibrium existence theorems in the generalized abstract economies with preference correspondences. Further applications of our results are also given to minimax inequalities for a family of functions. The results of this paper are more general than the ones given in the literature.

2. Preliminaries

Let X and Y be nonempty sets. Let M be a nonempty subset of X and $T : X \to 2^Y$ a multivalued map. Then for all $x \in X$ and $y \in Y$, we have $T(M) = \bigcup\{T(x)\colon x \in M\}$ and $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

For a nonempty set D, we denote by 2^D (respectively, $\langle D\rangle$) the family of all subsets (respectively, family of all nonempty finite subsets) of D. If D is a nonempty subset of a vector space, then $\operatorname{co} D$ denotes the convex hull of D. When D is a nonempty subset of a topological space, $\bar{D}$ denotes the closure of D.

A nonempty subset D of a topological space X is said to be *compactly open* (respectively, compactly closed) if for every nonempty compact subset C of X, $D \cap C$ is open (respectively, compactly closed) in C. The *compact interior* of D [11] is defined by

$$\operatorname{cint} D = \bigcup \{G:\ G \subseteq D \text{ and } G \text{ is compactly open in } X\}.$$

It is easy to see that $\operatorname{cint} D$ is a compactly open set in X and for each nonempty compact subset C of X with $D \cap C \neq \emptyset$, we have $(\operatorname{cint} D) \cap C = \operatorname{int}_C(D \cap C)$, where $\operatorname{int}_C(D \cap C)$ denotes the interior of $D \cap C$ in C. It is clear that a subset D of X is compactly open in X if and only if $\operatorname{cint} D = D$.

Let X and Y be topological spaces and $T : X \to 2^Y$ a multivalued map, then T is said to be *transfer compactly open valued on* X (see [12]) if for every $x \in X$ and for all compact subset D of Y with $T(x) \cap D \neq \emptyset$, $y \in T(x) \cap D$ implies that there exists a point $\hat{x} \in X$ such that $y \in \operatorname{int}_D(T(\hat{x}) \cap D)$. T is said to be *transfer open valued on* X if for every $x \in X$, $y \in T(x)$, there exists a point $\hat{x} \in X$ such that $y \in \operatorname{int} T(\hat{x})$.

Let X be a topological space. A real-valued function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous at* $\bar{x} \in X$ if for every $\varepsilon > 0$, there exists an open neighborhood U of $\bar{x}$ such that $f(x) > f(\bar{x}) - \varepsilon$ for each $x \in U \cap X$.

f is said to be *lower semicontinuous on* X if it is lower semicontinuous at each point of X.

Let X be a convex subset of a vector space. A function $f : X \to \mathbb{R}$ is said to be *quasiconvex* on X if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, we have $f(\lambda x_1 + (1 - \lambda)x_2) \leqslant \max\{f(x_1), f(x_2)\}$.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

By using the argument of Lemma 2.1 in [5], it is easy to derive the following result.

Lemma 2.1. *Let X and Y be two topological spaces and let $G : X \to 2^Y$ be a multivalued map. Then G is transfer compactly open valued if and only if*

$$\bigcup_{x \in X} G(x) = \bigcup_{x \in X} \operatorname{cint} G(x).$$

By applying Lemma 2.1 and following the argument of Proposition 1 in [15], we have the following lemma.

Lemma 2.2. *Let X and Y be two topological spaces and let $G : X \to 2^Y$ be a multivalued map. Then the following statements are equivalent*:

(i) $G^{-1} : Y \to 2^X$ *is transfer compactly open valued and for all $x \in X$, $G(x)$ is nonempty*;
(ii) $X = \bigcup_{y \in Y} \operatorname{cint} G^{-1}(y)$.

Definition 2.1 [21]. Let E be a topological vector space and let C be a lattice with a minimal element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^E \to C$ is called a *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^E$:

(a) $\Phi(\overline{\operatorname{co}}\, M) = \Phi(M)$, where $\overline{\operatorname{co}}\, M$ denotes the closed convex hull of M;

(b) $\Phi(M) = \mathbf{0}$ if and only if M is precompact;
(c) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 2.2 [21]. Let E be a topological vector space, $X \subseteq E$, and let Φ be a measure of noncompactness on E. A multivalued map (correspondence) $T : X \to 2^E$ is called *Φ-condensing* provided that if $M \subseteq X$ with $\Phi(T(M)) \geqslant \Phi(M)$ then M is relative compact, that is, $\bar{M}$ is compact.

Remark 2.1. Note that every multivalued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If E is locally convex, then a compact multivalued map (i.e., $T(X)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : X \to 2^E$ is Φ-condensing and $T' : X \to 2^E$ satisfies $T'(x) \subseteq T(x)$ for all $x \in X$, then T' is also Φ-condensing.

Lemma 2.3 [18]. *Let X be a nonempty, closed, and convex subset of a topological vector space E. Let Φ be a measure of noncompactness on X. Let $T : X \to 2^X$ be a Φ-condensing multivalued map. Then there exists a nonempty compact convex subset K of X such that $T(K) \subseteq K$.*

Remark 2.2. In [18], E is assumed to be a locally convex topological vector space, but Lemma 2.3 is true for any topological vector space as we can see in the proof.

3. Fixed point theorems for a family of multivalued maps

Throughout the paper, we shall use the following notations.

Let I be any index set. For each $i \in I$, let E_i be a topological vector space and let X_i be a nonempty subset of E_i. Let $X = \prod_{i \in I} X_i$ and $X^i = \prod_{j \in I,\, j \neq i} X_j$, and we write $X = X^i \otimes X_i$.

Theorem 3.1. *For each $i \in I$, let X_i be a nonempty convex set in a topological vector space E_i and let K_i be a nonempty compact subset of X_i. Let $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$. For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(i) *For each $i \in I$ and for all $x \in X$, $\operatorname{co} S_i(x) \subseteq T_i(x)$;*
(ii) *For each $i \in I$, $X = \bigcup\{\operatorname{cint} S_i^{-1}(y_i):\ y_i \in X_i\}$;*
(iii) *For each $i \in I$ and for all $M_i \in \langle X_i \rangle$, there exists a compact convex subset $L_{M_i} \subseteq X_i$ containing M_i such that for all $x \in X \setminus K$ and for each $i \in I$, there exists $\tilde{y}_i \in L_{M_i}$ such that $x \in \operatorname{cint} S_i^{-1}(\tilde{y}_i)$.*

Then there exists $\bar{x} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

Proof. Since for each $i \in I$,

$$X = \bigcup\left\{\operatorname{cint} S_i^{-1}(y_i):\ y_i \in X_i\right\},$$

and K is a nonempty compact subset of X, for each $i \in I$, there exists $M_i \in \langle X_i \rangle$ such that

$$K \subseteq \bigcup \{ \operatorname{cint} S_i^{-1}(y_i):\ y_i \in M_i \}.$$

For M_i, consider the compact convex set $L_{M_i} \subseteq X_i$ as in condition (iii) such that $M_i \subseteq L_{M_i}$ and

$$X \setminus K \subseteq \bigcup \{ \operatorname{cint} S_i^{-1}(y_i):\ y_i \in L_{M_i} \} \quad \text{for each } i \in I. \tag{3.1}$$

Let $M = \prod_{i \in I} M_i$ and $L_M = \prod_{i \in I} L_{M_i}$. Then L_M is a compact and convex subset of X containing M. Since $L_M \setminus K \subseteq X \setminus K$, by (3.1) we have

$$L_M \setminus K \subseteq \bigcup \{ \operatorname{cint} S_i^{-1}(y_i):\ y_i \in L_{M_i} \} \quad \text{for each } i \in I.$$

Since $M_i \subseteq L_{M_i}$, we have

$$L_M \subseteq \bigcup \{ \operatorname{cint} S_i^{-1}(y_i):\ y_i \in L_{M_i} \} \quad \text{for each } i \in I.$$

Since L_M is compact, for each $i \in I$, there exists a finite set $N_i = \{y_i^{(1)}, \dots, y_i^{(n_i+1)}\}$ of L_{M_i} for some $n_i \in \mathbb{N}$ such that

$$L_M \subseteq \bigcup_{j=1}^{n_i+1} \{ \operatorname{cint} S_i^{-1}(y_i^{(j)}) \}.$$

Since L_M is compact, there also exists a continuous partition of unity $\{\beta_i^{(1)}, \dots, \beta_i^{(n_i+1)}\}$ subordinated to the compactly open covering $\{\operatorname{cint} S_i^{-1}(y_i^{(j)})\}_{j=1}^{n_i+1}$, that is, for each $j = 1, \dots, n_i + 1$, $\beta_i^{(j)} : L_M \to [0,1]$ is continuous such that for all $x \in L_M$, $\sum_{j=1}^{n_i+1} \beta_i^{(j)}(x) = 1$ and for each $j = 1, \dots, n_i + 1$, $\beta_i^{(j)}(x) = 0$ for $x \notin \operatorname{cint} S_i^{-1}(y_i^{(j)})$. In other words, $\beta_i^{(j)}(x) \neq 0$ implies $x \in \operatorname{cint} S_i^{-1}(y_i^{(j)}) \subseteq S_i^{-1}(y_i^{(j)})$, that is, $y_i^{(j)} \in S_i(x)$ for all $j = 1, \dots, n_i + 1$ and for each $i \in I$.

For each $i \in I$, let $\varphi_i : L_M \to \Delta_{n_i}$ be a map defined by

$$\varphi_i(x) = \sum_{j=1}^{n_i+1} \beta_i^{(j)}(x) e_i^{(j)} \quad \text{for all } x \in L_M,$$

where $e_i^{(j)}$ is the jth unit vector in $\mathbb{R}^{n_i+1}$ and Δ_{n_i} denotes the standard n_i-simplex. Then clearly for each $i \in I$, φ_i is continuous.

For each $i \in I$, define a map $g_i : \Delta_{n_i} \to \operatorname{co} A_i \subseteq L_{M_i}$ by

$$g_i\left(\sum_{j=1}^{n_i+1} \alpha_i^{(j)} e_i^{(j)} \right) = \sum_{j=1}^{n_i+1} \alpha_i^{(j)} y_i^{(j)},$$

where $\alpha_i^{(j)} \geqslant 0$ for each $i \in I$, $1 \leqslant j \leqslant n_i + 1$ and $\sum_{j=1}^{n_i+1} \alpha_i^{(j)} = 1$. Then for each $i \in I$, g_i is also continuous.

Let $J_i(x) = \{ j \in \{1, \dots, n_i + 1\}:\ \beta_i^{(j)}(x) \neq 0\}$. Then for all $x \in L_M$,

$$g_i\varphi_i(x) = g_i\left(\sum_{j=1}^{n_i+1}\beta_i^{(j)}(x)e_i^{(j)}\right) = \sum_{j=1}^{n_i+1}\beta_i^{(j)}(x)y_i^{(j)}$$
$$= \sum_{j\in J_i(x)}\beta_i^{(j)}(x)y_i^{(j)} \in \operatorname{co} S_i(x) \subseteq T_i(x) = T_i|_{L_M}(x).$$

For each $i \in I$, let E_i be the smallest finite-dimensional vector space containing Δ_{n_i} and $C = \prod_{i\in I}\Delta_{n_i}$. Then C is a compact convex subset of a locally convex Hausdorff topological vector space. Define two maps $h : C \to L_M$ and $\Psi : L_M \to C$ by

$$h(z) = \big(g_i(z_i)\big)_{i\in I} \quad \text{for all } z \in C$$

and

$$\Psi(x) = \big(\varphi_i(x)\big)_{i\in I} \quad \text{for all } x \in L_M,$$

where z_i is the ith projection of z. Since for each $i \in I$, g_i and φ_i are continuous, h and Ψ are also continuous.

Let $F = \Psi \circ h$; then $F : C \to C$ is a well-defined and continuous function. By Tychonoff's fixed point theorem [26], there exists $\bar{u} \in C$ such that $\bar{u} = F(\bar{u}) = \Psi \circ h(\bar{u})$. Let $\bar{x} = (\bar{x}_i)_{i\in I} = h(\bar{u})$; then $\bar{u} = \Psi(\bar{x})$, that is, for each $i \in I$, $\bar{u}_i = \varphi_i(\bar{x})$. Therefore, for each $i \in I$, $\bar{x}_i = g_i(\bar{u}_i) = g_i(\varphi_i(\bar{x})) \in T_i(\bar{x})$. □

Remark 3.1. (a) Theorem 3.1 generalizes Theorem 1 in [1] and thus Theorem 2.1 in [14] in several ways.

(b) When I is a singleton set, Theorem 3.1 generalizes Corollary 1 in [1] and thus Fan–Browder fixed point theorem [4], Theorem 2 in [10], Theorem 1 in [24], Corollary 3 in [27] and Theorem 3.2 in [28].

Remark 3.2. In view of Lemma 2.2, condition (ii) of Theorem 3.1 can be replaced by each of the following conditions:

(ii)(a) For each $i \in I$, S_i^{-1} is transfer compactly open valued on X and for all $x \in X$, $S_i(x)$ is nonempty;

(ii)(b) For each $i \in I$ and for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is compactly open in X and for all $x \in X$, $S_i(x)$ is nonempty.

Corollary 3.1. *For each $i \in I$, let X_i be a nonempty convex set in a topological vector space E_i and $X = \prod_{i\in I} X_i$. For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(i) *For each $i \in I$ and for all $x \in X$, $\operatorname{co} S_i(x) \subseteq T_i(x)$;*

(ii) *For each $i \in I$, $X = \bigcup\{\operatorname{cint} S_i^{-1}(y_i):\ y_i \in X_i\}$;*

(iii) *For each $i \in I$, there exist a nonempty compact convex subset $C_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $x \in X \setminus K$ and for each $i \in I$, there exists $\tilde{y}_i \in C_i$ such that $x \in \operatorname{cint} S_i^{-1}(\tilde{y}_i)$.*

Then there exists $\bar{x} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

Proof. For each $i \in I$ and for all $M_i \in \langle X_i \rangle$, let $L_{M_i} = \operatorname{co}(C_i \cup M_i)$. Then L_{M_i} is a compact convex subset of X_i containing M_i. The result follows from Theorem 3.1. $\square$

When I is a singleton set, the following result is a consequence of Corollary 3.1.

Corollary 3.2. *Let X be a nonempty convex subset of a topological vector space E. Let $S, T : X \to 2^X$ be multivalued maps satisfying the following conditions*:

(i) *For all $x \in X$,* $\operatorname{co} S(x) \subseteq T(x)$;
(ii) *For all $x \in X$, $x \notin T(x)$ and S^{-1} is transfer compactly open valued on X*;
(iii) *There exist a nonempty compact convex subset $C \subseteq X$ and a nonempty compact subset K of X such that for each $x \in X \setminus K$, there exists $\tilde{y} \in C$ such that $x \in \operatorname{cint} S^{-1}(\tilde{y})$.*

Then there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

4. Maximal element theorems for a family of multivalued maps

If we have a multivalued map (correspondence) S on a set X and there is an element $x \in X$ such that $S(x)$ is empty, then, in a way, such element x is "maximal." This situation is specially interesting when the multivalued map (correspondence) S is associated to a strict ordering $\prec$ defined on X so that $S(x)$ is the "upper contour set" of the element x, i.e., $S(x) = \{z \in X\colon\ x \prec z\}$. This scope is frequently encountered in the literature concerning ordered sets, see, for example, [19,25,28,29] and references therein. The existence of maximal elements for multivalued mappings in topological vector spaces and its important applications to mathematical economies have been studied by many authors in both mathematics and economies, see, for example, [6–10,16,18,19,28–30] and references therein.

In 1983, Yannelis and Prabhakar [28] generalized the previous results in the literature on the existence of maximal elements over compact subsets of Hausdorff topological vector spaces. It is well known that each fixed point theorem has an equivalent version of a maximal element theorem. Recently, Deguire and Yuan [8] and Deguire et al. [9] established several results for the existence of a maximal element for the family of multivalued maps.

In this section, we prove the existence of a maximal element for a particular family of multivalued maps, namely the Φ-condensing multivalued maps defined on a product of noncompact sets.

Theorem 4.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i and $X = \prod_{i\in I} X_i$. For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(i) *For each $i \in I$ and for all $x \in X$,* $\operatorname{co} S_i(x) \subseteq T_i(x)$;
(ii) *For each $i \in I$ and for all $x \in X$, $x_i \notin T_i(x)$ and S_i^{-1} is transfer compactly open valued on X_i*;
(iii) *For all $x \in X$, $I(x) = \{i \in I\colon\ S_i(x) \neq \emptyset\}$ is finite*;

(iv) *For each $i \in I$, there exist a nonempty compact convex subset $C_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $x \in X \setminus K$ and for each $i \in I(x)$, there exists $\tilde{y}_i \in C_i$ such that $x \in \operatorname{cint} S_i^{-1}(\tilde{y}_i)$.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Proof. For all $x \in X$ and for each $i \in I(x)$, define two multivalued maps $A_i, B_i : X \to 2^X$ by

$$A_i(x) = X^i \otimes S_i(x) \quad \text{and} \quad B_i(x) = X^i \otimes T_i(x).$$

Further, we define other two multivalued maps $F, G : X \to 2^X$ by

$$F(x) = \begin{cases} \bigcap_{i \in I(x)} A_i(x) & \text{if } I(x) \neq \emptyset, \\ \emptyset & \text{if } I(x) = \emptyset, \end{cases}$$

and

$$G(x) = \begin{cases} \bigcap_{i \in I(x)} B_i(x) & \text{if } I(x) \neq \emptyset, \\ \emptyset & \text{if } I(x) = \emptyset. \end{cases}$$

To complete the proof, it is sufficient to show that $I(\bar{x}) = \emptyset$ for some $\bar{x} \in X$. Suppose otherwise that $I(x) \neq \emptyset$ for all $x \in X$. Fix an arbitrary $x \in X$; since $I(x) \neq \emptyset$, there exists $i \in I(x)$ such that $S_i(x) \neq \emptyset$. By condition (i), for each $y \in X$, $\operatorname{co} F(y) \subseteq G(y)$. Condition (ii) implies that for each $y \in X$, $y \notin G(y)$ and F^{-1} is transfer compactly open valued on X.

Indeed, for any nonempty compact subset D of X, if $x \in F^{-1}(y) \cap D$, then $y \in A_i(x)$ for all $i \in I(x)$ and $x \in D$. This implies that $y_i \in S_i(x)$ for all $i \in I(x)$ and $x \in D$. Therefore, $x \in S_i^{-1}(y_i) \cap D$ for all $i \in I(x)$.

Since S_i^{-1} is transfer compactly open valued on X_i, there exists $\hat{y}_i \in X_i$ such that

$$x \in \operatorname{int}_D\big(S_i^{-1}(\hat{y}_i) \cap D\big) \quad \text{for all } i \in I(x). \tag{4.1}$$

For each $i \in I(x)$, let $\hat{y} = (y^i, \hat{y}_i)$, where $y^i \in X^i$ is a fixed element. Now

$$\begin{aligned} u \in A_i^{-1}(\hat{y}) \quad &\Leftrightarrow \quad \hat{y} \in A_i(u) = X^i \otimes S_i(u) \\ &\Leftrightarrow \quad \hat{y}_i \in S_i(u) \text{ and } y^i \in X^i \\ &\Leftrightarrow \quad u \in S_i^{-1}(\hat{y}_i) \text{ and } y^i \in X^i. \end{aligned}$$

This shows that $A_i^{-1}(\hat{y}) = S_i^{-1}(\hat{y}_i)$ for each fixed $y^i \in X^i$. Therefore,

$$x \in \operatorname{int}_D\big[(A_i)^{-1}(\hat{y}) \cap D\big] \quad \text{for all } i \in I(x),$$

and thus

$$x \in \bigcap_{i \in I(x)} \operatorname{int}_D\big[A_i^{-1}(\hat{y}) \cap D\big] \subseteq \operatorname{int}_D\big[F^{-1}(\hat{y}) \cap D\big].$$

Hence F^{-1} is transfer compactly open valued on X.

By condition (iv), for all $x \in X \setminus K$ and for each $i \in I(x)$, there exists $\tilde{y}_i \in C_i$ such that $x \in \operatorname{cint} S_i^{-1}(\tilde{y}_i) = \operatorname{cint} A_i^{-1}(\tilde{y})$. Therefore,

$$x \in \bigcap_{i \in I(x)} \left\{\operatorname{cint} A_i^{-1}(\tilde{y})\right\} \subseteq \operatorname{cint}\left(\bigcap_{i \in I(x)} A_i^{-1}(\tilde{y})\right) = \operatorname{cint} F^{-1}(\tilde{y}).$$

It follows from Corollary 3.2 that there exists $\hat{z} \in X$ such that $F(\hat{z}) = \emptyset$. Since $I(\hat{z}) \neq \emptyset$, $S_i(\hat{z}) \neq \emptyset$ and $A_i(\hat{z}) \neq \emptyset$ for all $i \in I(\hat{z})$, and therefore $F(\hat{z}) = \bigcap_{i \in I(\hat{z})} A_i(\hat{z}) \neq \emptyset$ which contradicts with $F(\hat{z}) = \emptyset$. Therefore, there exists $\bar{x} \in X$ such that $I(\bar{x}) = \emptyset$ which implies that $S_i(\bar{x}) = \emptyset$ for each $i \in I$. $\square$

Remark 4.1. If for each $i \in I$, X_i is nonempty compact convex subset of a topological vector space then the conclusion of Theorem 4.1 holds without condition (iv).

Theorem 4.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i. Let $X = \prod_{i \in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i \in I} E_i$. For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(i) *For each $i \in I$ and for all $x \in X$, $\operatorname{co} S_i(x) \subseteq T_i(x)$;*
(ii) *For each $i \in I$ and for all $x \in X$, $x_i \notin T_i(x)$ and S_i^{-1} is transfer compactly open valued on X_i;*
(iii) *For all $x \in X$, $I(x) = \{i \in I\colon S_i(x) \neq \emptyset\}$ is finite*;
(iv) *$T := \prod_{i \in I} T_i : X \to 2^X$ defined as $T(x) = \prod_{i \in I} T_i(x)$, is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Proof. Since $T : X \to 2^X$ is Φ-condensing, it follows from Lemma 2.3 that there exists a compact convex subset K of X such that $T(K) \subseteq K$. Then the conclusion follows from Theorem 4.1. $\square$

Theorem 4.3. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i. Let $X = \prod_{i \in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i \in I} E_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a multivalued map satisfying the following conditions*:

(i) *For each $i \in I$ and for all $x \in X$, $x_i \notin \operatorname{co} S_i(x)$;*
(ii) *For each $i \in I$, S_i^{-1} is transfer compactly open valued on X_i;*
(iii) *For all $x \in X$, $I(x) = \{i \in I\colon S_i(x) \neq \emptyset\}$ is finite*;
(iv) *$S := \prod_{i \in I} S_i : X \to 2^X$ defined as $S(x) = \prod_{i \in I} S_i(x)$, is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Proof. For each $i \in I$, define $T_i : X \to 2^{X_i}$ by $T_i(x) = \operatorname{co} S_i(x)$. By condition (iv) and Lemma 2.3, there exists a nonempty compact convex subset K of X such that $S(K) \subseteq K$.

Let K_i be the ith projection of K. Then K_i is a compact convex subset of X_i and $S_i : K \to 2^{K_i}$. Therefore $T_i(K) = \bigcup\{\operatorname{co} S_i(x)\colon x \in K\} \subseteq K_i$ and conclusion follows from Theorem 4.1. $\square$

Remark 4.2. Condition (ii) in Theorem 4.3 can be replaced by the following condition:

(ii′) For each $i \in I$ and for all $x \in X$ such that $S_i(x) \neq \emptyset$, there exists $y_i' \in X_i$ such that $x \in \operatorname{cint} S_i^{-1}(y_i')$.

Indeed, let $x \in X$ and $x \in S_i^{-1}(y_i)$ for some $y_i \in X_i$. Then $y_i \in S_i(x) \neq \emptyset$, and by condition (ii′), there exists $y_i' \in X_i$ such that $x \in \operatorname{cint} S_i^{-1}(y_i')$. This shows that S_i^{-1} is compactly transfer open valued on X_i.

When I is a singleton, we have the following result.

Corollary 4.1. *Let X be a nonempty closed convex subset of a topological vector space E and let Φ be a measure of noncompactness on E. Let $S : X \to 2^X$ be a multivalued map satisfying the following conditions*:

(i) *For all $x \in X$, $x \notin \operatorname{co} S(x)$*;
(ii) *S^{-1} is transfer compactly open valued on X*;
(iii) *S is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

Remark 4.3. (a) Since every transfer compactly open valued map is transfer open map, Theorem 4.3 and Corollary 4.1 give a positive answer to an open question of Mehta [18].

(b) Corollary 4.1 generalizes Corollary 2 in [6], Theorem 2 in [16], Theorem 2.2 in [19], and Theorem 3.1 in [30] in several ways.

5. Generalized abstract economies

Because of the fuzziness of consumers' behaviour or market situations, in a real market, any preference of a real agent would be unstable. Therefore, Kim and Tan [13] introduced the fuzzy constraint correspondences in defining the following generalized abstract economy.

Let I be any set of agents (countable or uncountable). For each $i \in I$, let X_i be a nonempty set of actions available to the agent i in a topological vector space E_i and $X = \prod_{i\in I} X_i$. A *generalized abstract economy* (or *generalized game*) $\Gamma = (X_i, A_i, F_i, P_i)_{i\in I}$ [13] is defined as a family of ordered quadruples (X_i, A_i, F_i, P_i), where $A_i : X \to 2^{X_i}$ is a constraint correspondence such that $A_i(x)$ is the state attainable for the agent i at x, $F_i : X \to 2^{X_i}$ is a fuzzy constraint correspondence such that $F_i(x)$ is the unstable state for the agent i, and $P_i : X \times X \to 2^{X_i}$ is a preference correspondence such that $P_i(x)$ is the state preference by the agent i at x. An *equilibrium* for Γ is a point $(\bar{x}, \bar{y}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$, $\bar{y}_i \in F_i(\bar{x})$, and $P_i(\bar{x}, \bar{y}) \cap A_i(\bar{x}) = \emptyset$.

If for each $i \in I$ and each $x \in X$, $F_i(x) = X_i$ and the preference correspondence P_i satisfies $P_i(x, y) = P_i(x, y')$ for each $x, y, y' \in X$, our definitions of a generalized abstract economy and an equilibrium coincide with the usual definitions of an abstract economy and an equilibrium due to Shafer and Sonnenschein [22].

As applications of Theorem 4.1, we derive the following general equilibrium existence results for the generalized abstract economies with infinitely many commodities and infinitely many agents, and with general preference correspondences.

Theorem 5.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i, $X = \prod_{i \in I} X_i$, $A_i : X \to 2^{X_i}$ a multivalued map, $P_i : X \times X \to 2^{X_i}$ a preference correspondence, and $F_i : X \to 2^{X_i}$ a fuzzy constraint correspondence. For each $i \in I$, assume that the following conditions hold*:

- (i) *For all $x \in X$, $A_i(x)$ and $F_i(x)$ are nonempty and convex*;
- (ii) *For all $y_i \in X_i$, $A_i^{-1}(y_i)$, $F_i^{-1}(y_i)$, and $P_i^{-1}(y_i)$ are compactly open sets*;
- (iii) *For all $(x, y) \in X \times X$, $x_i \notin \operatorname{co} P_i(x, y)$*;
- (iv) *The set $W_i = \{(x, y) \in X \times X\colon\ x_i \in A_i(x)$ and $y_i \in F_i(x)\}$ is compactly closed in $X \times X$*;
- (v) *For $(x, y) \in X \times X$, $I(x, y) = \{i \in I\colon\ A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is finite*;
- (vi) *There exist nonempty compact convex sets C_i, $D_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $(x, y) \in X \times X \setminus K \times K$ and for each $i \in I(x, y)$, there exist $\tilde{u}_i \in C_i$ and $\tilde{v}_i \in D_i$ satisfying $\tilde{u}_i \in P_i(x, y) \cap A_i(x)$ and $\tilde{v}_i \in F_i(x)$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$, $\bar{y}_i \in F_i(\bar{x})$, and $A_i(\bar{x}) \cap P_i(\bar{x}, \bar{y}) = \emptyset$.

Proof. For each $i \in I$, define $S_i, T_i : X \times X \to 2^{X_i \times X_i}$ by

$$S_i(x, y) = \begin{cases} [P_i(x, y) \cap A_i(x)] \times F_i(x) & \text{if } (x, y) \in W_i, \\ A_i(x) \times F_i(x) & \text{if } (x, y) \notin W_i, \end{cases}$$

and

$$T_i(x, y) = \begin{cases} [\operatorname{co} P_i(x, y) \cap A_i(x)] \times F_i(x) & \text{if } (x, y) \in W_i, \\ A_i(x) \times F_i(x) & \text{if } (x, y) \notin W_i. \end{cases}$$

From conditions (i) and (iii), we have $\operatorname{co} S_i(x, y) \subseteq T_i(x, y)$ and $(x_i, y_i) \notin T_i(x, y)$ for each $i \in I$ and for all $(x, y) \in X \times X$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times X_i$, we have

$$\begin{aligned} S_i^{-1}(u_i, v_i) = {} & \bigl[P_i^{-1}(u_i) \cap \bigl(A_i^{-1}(u_i) \times X\bigr) \cap \bigl(F_i^{-1}(v_i) \times X\bigr)\bigr] \\ & \cup \bigl[(X \times X \setminus W_i) \cap \bigl(A_i^{-1}(u_i) \times X\bigr) \cap \bigl(F_i^{-1}(v_i) \times X\bigr)\bigr]. \end{aligned}$$

By conditions (ii) and (iv), $S_i^{-1}(u_i, v_i)$ is compactly open and therefore S_i^{-1} is transfer compactly open valued on $X_i \times X_i$. From condition (iv), we obtain

$$\begin{aligned} X \times X \setminus K \times K & \subseteq \bigcup \bigl\{S_i^{-1}(u_i, v_i)\colon\ u_i, v_i \in C_i\bigr\} \\ & \subseteq \bigcup \bigl\{\operatorname{cint} S_i^{-1}(u_i, v_i)\colon\ u_i, v_i \in C_i\bigr\}. \end{aligned}$$

Theorem 4.1 implies that there exists $(\bar{x}, \bar{y}) \in X \times X$ such that $S_i(\bar{x}, \bar{y}) = \emptyset$ for each $i \in I$. If $(\bar{x}, \bar{y}) \notin W_j$ for some $j \in I$, then either $A_i(\bar{x}) = \emptyset$ or $F_i(\bar{x}) = \emptyset$ which contradicts the fact of $A_i(x)$ and $F_i(x)$ being nonempty for all $x \in X$. Therefore $(\bar{x}, \bar{y}) \in W_i$ for each $i \in I$, and thus $\bar{x}_i \in A_i(\bar{x})$, $\bar{y}_i \in F_i(\bar{x})$, and $A_i(\bar{x}) \cap P_i(\bar{x}, \bar{y}) = \emptyset$. □

Theorem 5.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $A_i : X \to 2^{X_i}$ a multivalued map, $P_i : X \times X \to 2^{X_i}$ a preference correspondence, and $F_i : X \to 2^{X_i}$ a fuzzy constraint correspondence. Let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, assume that the following conditions hold*:

(i) *For all $x \in X$, $A_i(x)$ and $F_i(x)$ are nonempty and convex*;
(ii) *For all $y_i \in X_i$, $A_i^{-1}(y_i)$, $F_i^{-1}(y_i)$, and $P_i^{-1}(y_i)$ are compactly open sets;*
(iii) *For all $(x, y) \in X \times X$, $x_i \notin \operatorname{co} P_i(x, y)$;*
(iv) *The set $W_i = \{(x, y) \in X \times X\colon\ x_i \in A_i(x)$ and $y_i \in F_i(x)\}$ is compactly closed in $X \times X$;*
(v) *For $(x, y) \in X \times X$, $I(x, y) = \{i \in I\colon\ A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is finite*;
(vi) *The multivalued map $(A \times F) = (\prod_{i\in I} A_i \times \prod_{i\in I} F_i) : X \times X \to 2^{X\times X}$ defined as $(A \times F)(x, y) = \prod_{i\in I} A_i(x) \times \prod_{i\in I} F_i(y)$, is Φ-condensing.*

Then there exists $(\bar{x}, \bar{y}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$, $\bar{y}_i \in F_i(\bar{x})$, and $A_i(\bar{x}) \cap P_i(\bar{x}, \bar{y}) = \emptyset$.

Proof. In view of Theorem 4.2, it is sufficient to show that the multivalued map $T := \prod_{i\in I} T_i : X \times X \to 2^{X\times X}$ defined as $T(x, y) = \prod_{i\in I} T_i(x, y)$, is Φ-condensing, where T_i's are the same as defined in the proof of Theorem 5.1. By the definition of T_i, $T_i(x, y) \subseteq A_i(x) \times F_i(x)$ for each $i \in I$ and for all $(x, y) \in X \times X$ and therefore $T(x, y) \subseteq A(x) \times F(x)$. Since $A \times F$ is Φ-condensing, by Remark 2.1, we have T is also Φ-condensing. □

From Theorems 5.1 and 5.2, we can easily derive the following equilibrium existence results for the abstract economies with or without involving Φ-condensing maps.

Corollary 5.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $A_i : X \to 2^{X_i}$ a multivalued map with nonempty values, and $P_i : X \to 2^{X_i}$ a preference correspondence. For each $i \in I$, assume that the following conditions hold*:

(i) *For all $x \in X$, $A_i(x)$ is convex*;
(ii) *For all $y_i \in X_i$, $A_i^{-1}(y_i)$ and $P_i^{-1}(y_i)$ are compactly open*;
(iii) *For all $x \in X$, $x_i \notin P_i(x)$;*
(iv) *The set $W_i = \{x \in X\colon\ x_i \in A_i(x)\}$ is compactly closed in X;*
(v) *For all $x \in X$, $I(x) = \{i \in I\colon\ A_i(x) \cap P_i(x) \neq \emptyset\}$ is finite*;

(vi) *There exist a nonempty compact convex subset $C_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $x \in X \setminus K$ and for each $i \in I(x)$, there exists $\tilde{y}_i \in C_i$ satisfying*

$$\tilde{y}_i \in P_i(x) \cap A_i(x).$$

Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

Proof. The result follows from Theorem 5.1 by taking $F_i(x) = X_i$ and $P_i(x, y) = P_i(x)$ for all $x, y \in X$ and for each $i \in I$. □

Corollary 5.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $A_i : X \to 2^{X_i}$ a multivalued map, and $P_i : X \to 2^{X_i}$ a preference correspondence. Let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, assume that the following conditions hold*:

(i) *For all $x \in X$, $A_i(x)$ is nonempty and convex*;
(ii) *For all $y_i \in X_i$, $A_i^{-1}(y_i)$ and $P_i^{-1}(y_i)$ are compactly open sets*;.
(iii) *For all $x \in X$, $x_i \notin \operatorname{co} P_i(x)$*;
(iv) *The set $W_i = \{x \in X\colon x_i \in A_i(x)\}$ is compactly closed in X*;
(v) *For all $x \in X$, $I(x) = \{i \in I\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is finite*;
(vi) *The multivalued map $A := (\prod_{i\in I} A_i) : X \to 2^X$ defined as $A(x) = \prod_{i\in I} A_i(x)$, is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

6. Minimax inequalities

From now on, unless otherwise specified, we shall assume that the set $\mathcal{F}_i = \{x \in X\colon x_i \in B_i(x)\}$ is compactly closed.

Theorem 6.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $f_i\colon X^i \times X_i \to \mathbb{R}$ a real function, $B_i : X \to 2^{X_i}$ a multivalued map, and $A_i : X \to 2^{X_i}$ a multivalued map with nonempty values such that for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X. For each $i \in I$, assume that the following conditions hold*:

(i) *For all $x \in X$, $\operatorname{co} A_i(x) \subseteq B_i(x)$*;
(ii) *For all $x^i \in X^i$, $y_i \mapsto f_i(x^i, y_i)$ is quasiconcave*;
(iii) *For all $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is lower semicontinuous on X_i*;
(iv) *For all $y_i \in X_i$, $x^i \mapsto f_i(x^i, y_i)$ is continuous on X^i*;
(v) *For all $x \in X$, $I(x) = \{i \in I\colon A_i(x) \neq \emptyset\}$ is finite*;
(vi) *There exist a nonempty compact convex subset $C_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $x \in X \setminus K$ and each $i \in I$, there exists $\tilde{y}_i \in C_i$ such that $x \in A_i^{-1}(\tilde{y}_i)$ and $f_i(x^i, \tilde{y}_i) > f_i(x^i, x_i)$.*

Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in B_i(\bar{x})$ and $f_i(\bar{x}^i, y_i) \leqslant f_i(\bar{x}^i, \bar{x}_i)$ for all $y_i \in A_i(\bar{x})$.

Proof. For each $i \in I$ and for all $x \in X$, define $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i\colon\ f_i(x^i, y_i) > f_i(x^i, x_i)\}.$$

By condition (ii), for each $x \in X$ and for each $i \in I$, $P_i(x)$ is convex and $x_i \notin P_i(x)$. Condition (iii) implies that $P_i^{-1}(y_i)$ is open in X for all $y_i \in X_i$ and for each $i \in I$.

For each $i \in I$, we define other multivalued maps $T_i, S_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} B_i(x) \cap P_i(x) & \text{if } x \in \mathcal{F}_i, \\ B_i(x) & \text{if } x \in X \setminus \mathcal{F}_i, \end{cases}$$

and

$$S_i(x) = \begin{cases} A_i(x) \cap P_i(x) & \text{if } x \in \mathcal{F}_i, \\ A_i(x) & \text{if } x \in X \setminus \mathcal{F}_i. \end{cases}$$

Since for each $i \in I$ and for all $x \in X$, $P_i(x)$ is convex, it follows from condition (i) that $\operatorname{co} S_i(x) \subseteq T_i(x)$. Since for each $i \in I$, $\mathcal{F}_i$ is compactly closed in X, it is easy to see that

$$S_i^{-1}(y_i) = \big(A_i^{-1}(y_i) \cap P_i^{-1}(y_i)\big) \cup \big((X \setminus \mathcal{F}_i) \cap A_i^{-1}(y_i)\big)$$

is compactly open in X for every $y_i \in X_i$ and also for each $i \in I$ and for all $x \in X$, $x_i \notin T_i(x)$. Theorem 4.1 implies that there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$. If $\bar{x} \in X \setminus \mathcal{F}_j$, for some $j \in I$, then $A_j(\bar{x}) = S_j(\bar{x}) = \emptyset$ which contradicts with $A_i(x)$ is nonempty for all $i \in I$ and $x \in X$. Therefore $\bar{x} \in \mathcal{F}_i$ for all $i \in I$. Hence $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$ and thus $\bar{x}_i \in B_i(\bar{x})$, $f_i(\bar{x}^i, y_i) \leqslant f_i(\bar{x}^i, \bar{x}_i)$ for all $y_i \in A_i(\bar{x})$ and all $i \in I$. □

Theorem 6.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i, $X = \prod_{i \in I} X_i$, $f_i : X^i \times X_i \to \mathbb{R}$ a real function, $B_i : X \to 2^{X_i}$ a multivalued map, and $A_i : X \to 2^{X_i}$ a multivalued map with nonempty values such that for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X. Let Φ be a measure of noncompactness on $E = \prod_{i \in I} E_i$. For each $i \in I$, assume that conditions* (i)–(v) *of Theorem* 6.1 *hold. Further assume that the multivalued map $A = \prod_{i \in I} A_i : X \to 2^X$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in X$, is Φ-condensing. Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in B_i(\bar{x})$ and $f_i(\bar{x}^i, y_i) \leqslant f_i(\bar{x}^i, \bar{x}_i)$ for all $y_i \in A_i(\bar{x})$.*

Proof. In view of Theorem 4.2, it is sufficient to show that the multivalued map $S : X \to 2^X$ defined as $S(x) = \prod_{i \in I} S_i(x)$ for all $x \in X$, is Φ-condensing, where S_i's are the same as defined in the proof of Theorem 6.1. By the definition of S_i, $S_i(x) \subseteq A_i(x)$ for all $i \in I$ and for all $x \in K$ and therefore $S(x) \subseteq A(x)$ for all $x \in X$. Since A is Φ-condensing, by Remark 2.1, we have that S is also Φ-condensing. □

Remark 6.1. (a) If for each $i \in I$, X_i is a compact convex subset of topological vector space and for all $x \in X$, $A_i(x) = B_i(x) = X_i$, then Theorems 6.1 and 6.2 reduce to the theorem of equilibria of Nash [20].

(b) Theorems 6.1 and 6.2 generalize Theorem 4.2 in [2] for noncompact setting.

Theorem 6.3. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $f_i : X \times X_i \to \mathbb{R}$ a real function, a_i a real number, $B_i : X \to 2^{X_i}$ a multivalued map, and $A_i : X \to 2^{X_i}$ a multivalued map such that for each $y_i \in X$, $A_i^{-1}(y_i)$ is compactly open in X. For each $i \in I$, assume that the following conditions hold*:

- (i) *For all $x \in X$, $\operatorname{co} A_i(x) \subseteq B_i(x)$*;
- (ii) *For all $x \in X$, $y_i \mapsto f_i(x, y_i)$ is quasiconcave*;
- (iii) *For all $y_i \in X_i$, $x \mapsto f_i(x, y_i)$ is lower semicontinuous on X*;
- (iv) *For all $x \in X$, $f_i(x, x_i) \leqslant a_i$*;
- (v) *For all $x \in X$, $I(x) = \{i \in I\colon\ A_i(x) \neq \emptyset\}$ is finite*;
- (vi) *There exist a nonempty compact convex subset $C_i \subseteq X_i$ and a nonempty compact subset K of X such that for all $x \in X \setminus K$ and each $i \in I(x)$, there exists $\tilde{y}_i \in C_i$ such that $x \in A_i^{-1}(\tilde{y}_i)$ and $f_i(x, \tilde{y}_i) > a_i$.*

Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in B_i(\bar{x})$ and

$$f_i(\bar{x}, y_i) \leqslant a_i \quad \text{for all } y_i \in A_i(\bar{x}).$$

Proof. For each $i \in I$ and for all $x \in X$, define $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \bigl\{y_i \in X_i\colon\ f_i(x, y_i) > a_i\bigr\}.$$

Following the argument of Theorem 6.1, it is easy to show that there exists $\bar{x} \in X$ such that $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$. Therefore $f_i(\bar{x}, y_i) \leqslant a_i$ for all $y_i \in A_i(\bar{x})$ and each $i \in I$. □

Theorem 6.4. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i, $X = \prod_{i\in I} X_i$, $f_i : X \times X_i \to \mathbb{R}$ a real function, a_i a real number, $B_i : X \to 2^{X_i}$ a multivalued map, and $A_i : X \to 2^{X_i}$ a multivalued map such that for each $y_i \in X$, $A_i^{-1}(y_i)$ is compactly open in X. Let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, assume that conditions* (i)–(v) *of Theorem* 6.3 *hold. Further assume that the multivalued map $A = \prod_{i\in I} A_i : X \to 2^X$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in X$ is Φ-condensing. Then there exists $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in B_i(\bar{x})$ and*

$$f_i(\bar{x}, y_i) \leqslant a_i \quad \text{for all } y_i \in A_i(\bar{x}).$$

Proof. Following the argument of Theorems 6.2 and 6.3, we get the conclusion. □

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