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Generalized abstract economy and systems of generalized vector quasi-equilibrium problems

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Abstract

The present paper is in two-fold. The first fold is devoted to the existence theory of equilibria for generalized abstract economy with a lower semicontinuous constraint correspondence and a fuzzy constraint correspondence defined on a noncompact/nonparacompact strategy set. In the second fold, we consider systems of generalized vector quasi-equilibrium problems for multivalued maps (for short, SGVQEPs) which contain systems of vector quasi-equilibrium problems, systems of generalized mixed vector quasi-variational inequalities and Debreu-type equilibrium problems for vector valued functions as special cases. By using the results of first fold, we establish some existence results for solutions of SGVQEPs.

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1. Introduction

The notion of an abstract economy (social system) was introduced by Debreu [10]. He proved the existence of an equilibrium point for abstract economy. For the finite number of agents, Shafer and Sonnenschein [19] and Borglin and Keiding [8] extended Debreu's result to abstract economy without order preferences. During the last two decades, many authors studied the existence of equilibrium of an abstract economy with infinite number of agents but under the compactness/paracompactness of strategy set; See for example [24,26] and references therein. In 1990, Tian [23] proved an equilibrium existence theorem for noncompact abstract economy with a countable number of agents. In the recent past, many authors studied the existence of equilibria for abstract economy with infinite number of agents; See for example [11–13,28] and references therein for paracompact/compact strategy set; for noncompact/nonparacompact strategy set we refer to [6,12,13,17] and references therein. Recently, Kim and Tan [15] and Lin et al. [18] considered abstract economy with a fuzzy constraint correspondence, known as generalized abstract economy. They established

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some existence results for an equilibrium point of generalized abstract economy under the assumption of open lower section of correspondences involved. This notion of generalized abstract economy generalizes the concept of abstract economies considered in the references given in this paper and references therein.

System of vector (quasi-) equilibrium problems (for short, SV(Q-)EP) is a unified model of several problems, for instance, system of vector (quasi-) variational inequalities (for short, SV(Q-)VI), system of vector (quasi-) optimization problems and Debreu-type equilibrium problem, also known as noncooperative game, for vector valued functions (for short, Debreu VEP). Recently, Ansari et al. [1] used SVQEP as a tool to study the existence of a solution of (Debreu VEP)(I) (See Section 3.1). They also used SVQVI to prove the existence of a solution of Debreu VEP for nonconvex but differentiable (in some sense) vector valued functions. In [2], system of generalized vector quasi-variational inequalities is used to establish the existence of solution of (Debreu VEP)(I) (See Section 3.1) for nondifferentiable and nonconvex vector valued functions.

The present paper is divided into two folds. The first fold deals with the study of existence of equilibria for generalized abstract economy with a lower semicontinuous constraint correspondence and a fuzzy constraint correspondence defined on noncompact/nonparacompact strategy set. In the second fold, we consider systems of generalized vector quasi-equilibrium problems for multivalued trifunction (for short, SGVQEPs) which contain systems of vector quasi-equilibrium problems for trifunctions, systems of mixed vector quasi-variational inequalities and Debreu-type equilibrium problems for vector valued functions as special cases. As applications of results of first fold, we establish some existence results for solutions of SGVQEPs.

2. Generalized abstract economy

For a subset Ω of a vector space, we denote by coΩ the convex hull of Ω. If Ω and Δ are subsets of a topological space $\mathscr{X}$ such that $\Omega \subseteq \Delta$, then the closure (respectively, interior) of Ω in Δ is denoted by $\mathrm{cl}_\Delta \Omega$ (respectively, $\mathrm{int}_\Delta \Omega$); In case $\Delta = \mathscr{X}$, we write clΩ and int Ω instead of $\mathrm{cl}_{\mathscr{X}}\Omega$ and $\mathrm{int}_{\mathscr{X}}\Omega$, respectively. Let $\mathscr{X}$ and $\mathscr{Y}$ be topological vector spaces and $\Phi, \Psi : \mathscr{X} \to 2^{\mathscr{Y}}$ be correspondences. Then coΦ, $\Phi \cap \Psi : \mathscr{X} \to 2^{\mathscr{Y}}$ are defined as $(\mathrm{co}\Phi)(x) = \mathrm{co}\Phi(x)$ and $(\Phi\cap\Psi)(x)=\Phi(x)\cap\Psi(x)$ for all $x \in \mathscr{X}$, respectively. For a nonempty subset V of $\mathscr{Y}$, $\Phi^{-1}(V)=\{x \in \mathscr{Y} : \Phi(x)\cap V \neq \emptyset\}$ and also $x \in \Phi^{-1}(y)$ if and only if $y \in \Phi(x)$. Φ is said to have an *open lower section* if for each $y \in \mathscr{Y}$, $\Phi^{-1}(y)$ is open in $\mathscr{X}$. We also define $\overline{\Phi}$, cl$\Phi : \mathscr{X} \to 2^{\mathscr{Y}}$ by

$$\overline{\Phi}(x) = \{y \in \mathscr{Y} : (x, y) \in \mathrm{cl}_{\mathscr{X}\times\mathscr{Y}}\, \mathrm{Gr}(\Phi)\},$$

where $\mathrm{Gr}(\Phi) = \{(x, y) \in \mathscr{X} \times \mathscr{Y} : y \in \Phi(x)\}$ denotes the graph of Φ, and

$$\mathrm{cl}\Phi(x) = \mathrm{cl}_{\mathscr{Y}}(\Phi(x)) \quad \text{for all } x \in \mathscr{X}, \text{ respectively.}$$

It is easy to see that $\mathrm{cl}\Phi(x) \subseteq \overline{\Phi}(x)$ for all $x \in \mathscr{X}$.

In a real market, any preference of a real agent could be unstable because of the fuzziness of consumers' behavior or market situations. Thus, Kim and Tan [15] introduced the fuzzy constraint correspondences in defining the following generalized abstract economy.

Let I be any set of agents (countable or uncountable). For each $i \in I$, let X_i be a nonempty set of actions available to the agent i in a topological vector space E_i and $X = \prod_{i\in I} X_i$. A *generalized abstract economy* (or *generalized game*) $\Gamma = (X_i, A_i, F_i, P_i)_{i\in I}$ [15] is defined as a family of ordered quadruples (X_i, A_i, F_i, P_i) where $A_i : X \to 2^{X_i}$ is a constraint correspondence such that $A_i(x)$ is the state attainable for the agent i at x, $F_i : X \to 2^{X_i}$ is a fuzzy constraint correspondence such that $F_i(x)$ is the unstable state for the agent i, and $P_i : X \times X \to 2^{X_i}$ is a preference correspondence such that $P_i(x, y)$ is the state preference by the agent i at (x, y). An *equilibrium* for Γ is a point $(\hat{x}, \hat{y}) \in X \times X$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$.

This problem is further considered and studied in [18] with or without involving Φ-condensing correspondences.

If for each $i \in I$ and for all $x \in X$, $F_i(x) = X_i$ and the preference correspondence P_i is independent of y, that is, $P_i(x, y) = P_i(x)$ for all $x, y \in X$, our definitions of a generalized abstract economy and an equilibrium coincide with the usual definitions of an abstract economy and an equilibrium due to Shafer and Sonnenschein [19]; See also [11,21,27] and references therein.

Furthermore, if $\overline{A_i}(\hat{x}) = \mathrm{cl}A_i(\hat{x})$, our definition of an equilibrium point coincides with the standard definition in [8,22,25,26].

2.1. Preliminaries

In this section, we collect together some known definitions and results which will be needed in the sequel. Throughout the paper, unless otherwise specified, we assume that I is any (countable or uncountable) index set.

Lemma 2.1.1 (*Tan and Yuan [20]*). *Let $\mathscr{X}$ be a topological space, D a nonempty subset of a topological vector space E, $\mathscr{B}$ a base for neighborhoods of zero in E and $B : \mathscr{X} \to 2^D$ a multivalued map with nonempty values. For each $V \in \mathscr{B}$, let $B_V : \mathscr{X} \to 2^D$ be defined by $B_V(x) = (B(x) + V) \cap D$ for all $x \in \mathscr{X}$. If $\hat{x} \in \mathscr{X}$ and $\hat{y} \in D$ are such that $\hat{y} \in \bigcap_{V \in \mathscr{B}} \overline{B_V}(\hat{x})$, then $\hat{y} \in \overline{B}(\hat{x})$.*

The following results are the main tools to study the existence of equilibria of generalized abstract economy.

Theorem 2.1.1 (*Kim and Yuan [16]*). *For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff topological vector space E_i. For each $i \in I$, let $S_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ be a multivalued map such that*

(i) *for all $x = (x_i)_{i \in I} \in X$, $x_i \notin \mathrm{co}S_i(x)$;*
(ii) *for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X; and*
(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of X_i for each $i \in I$ with the property that for each $x \in X \backslash K$ there exists $j \in I$ such that $S_j(x) \cap D_j \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $S_i(\hat{x}) = \emptyset$ for each $i \in I$.

Definition 2.1.1 (*Border [7]*). Let E be a topological vector space and X be a subset of E such that $X = \bigcup_{n=1}^{\infty} G_n$, where $\{G_n\}_{n=1}^{\infty}$ is an increasing (in the sense that $G_n \subseteq G_{n+1}$) sequence of nonempty compact sets. A sequence $\{y_n\}_{n=1}^{\infty}$ in X is said to be *escaping* from X (relative to $\{G_n\}_{n=1}^{\infty}$) if for each $n \in \mathbb{N}$, there exists $M > 0$ such that $y_k \notin G_n$ for all $k \geqslant M$.

Theorem 2.1.2 (*Yuan et al. [27]*). *For each $i \in I$, let X_i be a subset of a topological vector space (not necessarily Hausdorff) E_i such that $X_i = \bigcup_{j=1}^{\infty} G_{i,j}$, where $\{G_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of E_i. For each $i \in I$, let $S_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ be a multivalued map such that*

(i) *for all $x = (x_i)_{i \in I} \in X$, $x_i \notin \mathrm{co}S_i(x)$;*
(ii) *for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X;*
(iii) *$\{x \in X : S_i(x) \neq \emptyset\} = \mathrm{int}_X\{x \in X : S_i(x) \neq \emptyset\}$; and*
(iv) *for any sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in G_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{G_n\}_{n=1}^{\infty}$ where $G_n = \Pi_{i \in I} G_{i,n}$ for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $x_m \in G_m$ such that $\pi_i(x_m) \in \mathrm{co}S_i(y_m)$ for each $i \in I(y_m)$, where $I(x) = \{i \in I : S_i(x) \neq \emptyset\}$ and $\pi_i(x_m)$ is the projection of x_m onto X_i.*

Then there exists $\hat{x} \in X$ such that $S_i(\hat{x}) = \emptyset$ for each $i \in I$.

2.2. Existence of equilibria for generalized abstract economy

In this section, we establish some existence results for equilibria of generalized abstract economy and abstract economy with lower semicontinuous correspondences.

Theorem 2.2.1. *Let $\Gamma = (X_i, A_i, F_i, P_i)_{i \in I}$ be a generalized abstract economy and $X = \prod_{i \in I} X_i$. For each $i \in I$, assume that the following conditions are satisfied.*

(i) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space E_i;*
(ii) *$A_i : X \to 2^{X_i}$ and $F_i : X \to 2^{X_i}$ are lower semicontinuous multivalued maps with nonempty convex values;*
(iii) *$P_i : X \times X \to 2^{X_i}$ has an open graph and $x_i \notin \mathrm{co}P_i(x, y)$ for all $(x, y) \in X \times X$;*

(iv) *There exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\tilde{D}_i$ and D_i of X_i for each $i \in I$ with the property that for each $(x, y) \in X \times X \backslash K \times M$, there exists $j \in I$ such that $(A_j(x) \cap P_j(x, y)) \cap \tilde{D}_j \neq \emptyset$ and $F_j(x) \cap D_j \neq \emptyset$.*

Then there exists an equilibrium point $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ of generalized abstract economy.

Proof. For each $i \in I$, let $\mathscr{B}_i$ be the collection of all open convex neighborhoods of zero in E_i and $\mathscr{B} = \prod_{i\in I}\mathscr{B}_i$. For any given $V \in \mathscr{B}$, let $V = \prod_{i\in I} V_i$, where $V_i \in \mathscr{B}_i$ for each $i \in I$. For any fixed $i \in I$, define $A_{V_i}, F_{V_i} : X \to 2^{X_i}$ by $A_{V_i}(x) = (A_i(x) + V_i) \cap X_i$ and $F_{V_i}(x) = (F_i(x) + V_i) \cap X_i$ for all $x \in X$. Since A_i and F_i are convex valued and lower semicontinuous, from [9, Lemma 4.1], we obtain that A_{V_i} and F_{V_i} are convex valued and have open graph in $X \times X_i$. By [26, Corollary 4.1], A_{V_i} and F_{V_i} have open lower sections. For each $i \in I$, let $W_{V_i} = \{(x, y) \in X \times X : (x_i, y_i) \in \overline{A_{V_i}}(x) \times \overline{F_{V_i}}(x)\}$, then each W_{V_i} is closed in $X \times X$. Define a multivalued map $Q_{V_i} : X \times X \to 2^{X_i\times X_i}$ by

$$Q_{V_i}(x, y) = \begin{cases} (A_{V_i}(x) \cap \mathrm{co}P_i(x, y)) \times F_{V_i}(x) & \text{if } (x, y) \in W_{V_i} \\ A_{V_i}(x) \times F_{V_i}(x) & \text{if } (x, y) \notin W_{V_i}. \end{cases}$$

For each $i \in I$ and for all $(x, y) \in X \times X$, it is easy to see that $Q_{V_i}(x, y)$ is convex and $(x_i, y_i) \notin Q_{V_i}(x, y)$. Since each P_i has an open graph, again by Corollary 4.1 and Lemma 5.1 both in [26], we have $\mathrm{co}P_i$ has open lower section. We note that for each $i \in I$ and for each $(x'_i, y'_i) \in X_i \times X_i$, the set

$$\begin{aligned} Q^{-1}_{V_i}(x'_i, y'_i) &= \{(x, y) \in X \times X : (x'_i, y'_i) \in Q_{V_i}(x, y)\} \\ &= \{(x, y) \in W_{V_i} : (x'_i, y'_i) \in (A_{V_i}(x) \cap \mathrm{co}P_i(x, y)) \times F_{V_i}(x)\} \\ &\quad \cup \{(x, y) \in X \times X\backslash W_{V_i} : (x'_i, y'_i) \in A_{V_i}(x) \times F_{V_i}(x)\} \\ &= \{(x, y) \in X \times X : (x'_i, y'_i) \in (A_{V_i}(x) \cap \mathrm{co}P_i(x, y)) \times F_{V_i}(x)\} \\ &\quad \cup \{(x, y) \in X \times X\backslash W_{V_i} : (x'_i, y'_i) \in A_{V_i}(x) \times F_{V_i}(x)\} \\ &= [(A^{-1}_{V_i}(x'_i) \times X) \cap (\mathrm{co}P_i)^{-1}(x'_i) \cap (F^{-1}_{V_i}(y'_i) \times X)] \\ &\quad \cup [(X \times X\backslash W_{V_i}) \cap (A^{-1}_{V_i}(x'_i) \times X) \cap (F^{-1}_{V_i}(y'_i) \times X)] \end{aligned}$$

is open in $X \times X$. By (iv), for each $(x, y) \in X \times X\backslash K \times M$ there exists $j \in I$ such that $Q_{V_j}(x, y) \cap (\tilde{D}_j \times D_i) \neq \emptyset$. By Theorem 2.1.1, there exists a point $(x_V, y_V) \in K \times M$ such that $Q_{V_i}(x_V, y_V) = \emptyset$ for each $i \in I$, where $x_V = (x_{V_i})_{i\in I}$ and $y_V = (y_{V_i})_{i\in I}$. Since for each $i \in I$ and for all $x \in X$, $A_i(x)$ and $F_i(x)$ are nonempty, we must have $(x_{V_i}, y_{V_i}) \in \overline{A_{V_i}}(x_V) \times \overline{F_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V, y_V) = \emptyset$.

Since for each $i \in I$, $A_i : X \to 2^{X_i}$ is lower semicontinuous and $P_i : X \times X \to 2^{X_i}$ has open graph, we have that $A_i \cap P_i : X \times X \to 2^{X_i}$ is lower semicontinuous [7, pp. 59–61]. Then it is easy to see that the set $\mathscr{H} = \{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is open.

Indeed, let $(x, y) \in \mathscr{H}$, then $A_i(x) \cap P_i(x, y) \neq \emptyset$. For each fixed $i \in I$, let O be any open set such that $O \cap A_i(x) \cap P_i(x, y) \neq \emptyset$. Since $A_i \cap P_i$ is lower semicontinuous, there exist neighborhoods $U(x)$ of x and $W(y)$ of y such that $O \cap A_i(u) \cap P_i(u, v) \neq \emptyset$ for all $(u, v) \in U(x) \times W(y)$. Therefore, $A_i(u) \cap P_i(u, v) \neq \emptyset$ for all $(u, v) \in U(x) \times W(y)$. Hence $(u, v) \in \mathscr{H}$ and thus $U(x) \times W(y) \subseteq \mathscr{H}$. This shows that $\mathscr{H}$ is open and the set $\{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) = \emptyset\}$ is closed.

Let $Q_V = \{(x, y) \in K \times M : x_i \in \overline{A_{V_i}}(x), y_i \in \overline{F_{V_i}}(y)$ and $A_i(x) \cap P_i(x, y) = \emptyset\}$. Then Q_V is a nonempty closed subset of $K \times M$.

Following the same argument as in [16, Proof of Theorem 4.3] and applying Lemma 2.1.1, we get the conclusion. □

Remark 2.2.1. Conditions (ii) and (iii) of Theorem 2.2.1 can be replaced, respectively, by the following conditions. For each $i \in I$,

(ii)′ $A_i : X \to 2^{X_i}$ is a multivalued map with nonempty convex values and has open lower section, and $F_i : X \to 2^{X_i}$ is a lower semicontinuous multivalued map with nonempty convex values.
(iii)′ $P_i : X \times X \to 2^{X_i}$ has open lower section and $x_i \notin \mathrm{co}P_i(x, y)$ for all $(x, y) \in X \times X$.

If for each $i \in I$ and for all $x \in X$, $F_i(x) = X_i$ and $P_i(x, y)$ is independent of the variable y, then from Theorem 2.2.1, we obtain the following existence result for equilibria of abstract economy.

Corollary 2.2.1. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy and $X = \prod_{i\in I} X_i$. For each $i \in I$, assume that the following conditions are satisfied*:

(i) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space E_i*;
(ii) *$A_i : X \to 2^{X_i}$ is a lower semicontinuous multivalued map with nonempty convex values*;
(iii) *$P_i : X \to 2^{X_i}$ has an open graph and $x_i \notin \mathrm{co}P_i(x)$ for all $x \in X$*;
(iv) *There exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of X_i for each $i \in I$ with the property that for each $x \in X\backslash K$, there exists $j \in I$ such that $(A_j \cap P_j)(x) \cap D_j \neq \emptyset$.*

Then there exists an equilibrium point $\hat{x} = (\hat{x}_i)_{i\in I} \in K$ of abstract economy, that is, for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Remark 2.2.2. Conditions (ii) and (iii) of Corollary 2.2.1 can be replaced, respectively, by the following conditions. For each $i \in I$,

(ii)′ $A_i : X \to 2^{X_i}$ is a multivalued map with nonempty convex values and has open lower section.
(iii)′ $P_i : X \to 2^{X_i}$ has open lower section such that $x_i \notin \mathrm{co}P_i(x)$ for all $x \in X$ and the set $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open.

Remark 2.2.3. (1) If for each $i \in I$, $\mathrm{cl}A_i$ is an upper semicontinuous multivalued map, then Corollary 2.2.1 with Remark 2.2.2 generalizes [23, Theorem 2, 24, Theorem 2.5] and [26, Theorem 6.1].

(2) Corollary 2.2.1 improves Corollaries 4.5 and 4.4 of Kim and Yuan [16] in the following ways:

(a) The set X is neither perfectly normal nor paracompact;
(b) The mapping $A_i \cap P_i$ is L-majorized is replaced by conditions (ii) and (iii).

If for each $i \in I$, E_i is not necessarily Hausdorff, we have the following existence result for equilibria of generalized abstract economy.

Theorem 2.2.2. *For each $i \in I$, let X_i be a nonempty subset of a locally convex topological vector space E_i and $X=\prod_{i\in I} X_i$. Let $\Gamma=(X_i, A_i, F_i, P_i)_{i\in I}$ be a generalized abstract economy such that for each $i \in I$, $X_i \times X_i = \bigcup_{j=1}^{\infty} G_{i,j}$ where $\{G_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subset of a locally convex topological vector space $E_i \times E_i$. For each $i \in I$, assume that the following conditions are satisfied*:

(i) *$A_i : X \to 2^{X_i}$ and $F_i : X \to 2^{X_i}$ are lower semicontinuous multivalued maps with nonempty convex values*;
(ii) *$P_i : X \times X \to 2^{X_i}$ has an open graph and $x_i \notin \mathrm{co}P_i(x, y)$ for each $(x, y) \in X \times X$; and*
(iii) *For each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \prod_{i\in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m) \cap P_i(x_m, y_m)$ and $\pi_i(\tilde{y}_m) \in F_i(x_m)$ for each $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then there exists an equilibrium point $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in X \times X$ of generalized abstract economy.

Proof. For each $i \in I$, let $\mathscr{B}_i$ be the collection of all open convex neighborhoods of zero in E_i and $\mathscr{B} = \prod_{i\in I} \mathscr{B}_i$. For any given $V \in \mathscr{B}$, let $V = \prod_{i\in I} V_i$, where $V_i \in \mathscr{B}_i$ for each $i \in I$. For any fixed $i \in I$, define A_{V_i}, F_{V_i}, W_{V_i}, and Q_{V_i} as in Proof of Theorem 2.2.1. Following the same argument as in Proof of Theorem 2.2.1, we have that for each $i \in I$ and for all $(x, y) \in X \times X$, $Q_{V_i}(x, y)$ is convex and $(x_i, y_i) \notin Q_{V_i}(x, y)$, and the set $Q_{V_i}^{-1}(x_i', y_i')$ is open in $X \times X$ for all $(x_i', y_i') \in X_i \times X_i$. Since

$$\{(x, y) \in X \times X : Q_{V_i}(x, y) \neq \emptyset\} = \bigcup\{Q_{V_i}^{-1}(x_i', y_i') \subseteq X \times X : (x_i', y_i') \in Q_{V_i}(x, y)\}$$

is open in $X \times X$, we get

$$\{(x, y) \in Q_{V_i}(x, y) \neq \emptyset\} = \mathrm{int}_X\{(x, y) \in Q_{V_i}(x, y) \neq \emptyset\}.$$

Thus, condition (iii) of Theorem 2.1.2 is satisfied. Therefore by Theorem 2.1.2, there exists a point $(x_V, y_V) \in X \times X$ such that $Q_{V_i}(x_V, y_V) = \emptyset$ for each $i \in I$, where $x_V = (x_{V_i})_{i\in I}$ and $y_V = (y_{V_i})_{i\in I}$. Since for each $i \in I$ and each $x \in X$, $A_i(x)$ and $F_i(x)$ are nonempty, we must have $(x_{V_i}, y_{V_i}) \in \overline{A_{V_i}}(x_V) \times \overline{F_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V, y_V) = \emptyset$.

As in Proof of Theorem 2.2.1, the set $\{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is open and the set $Q_V = \{(x, y) \in X \times X : x_i \in \overline{A_{V_i}}(x), y_i \in \overline{F_{V_i}}(y)$ and $A_i(x) \cap P_i(x, y) = \emptyset\}$ is nonempty and closed. Following the same argument as in [27, Proof of Theorem 4.3] and applying Lemma 2.1.1, we get the conclusion. □

If for each $i \in I$ and for all $x \in X$, $F_i(x) = X_i$ and $P_i(x, y)$ is independent of the variable y, then from Theorem 2.2.2, we obtain the following existence result for equilibria of abstract economy.

Corollary 2.2.2 (*Yuan et al.* [27]). *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy such that for each $i \in I$, $X_i = \bigcup_{j=1}^{\infty} G_{i,j}$ where $\{G_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subset of a locally convex topological vector space E_i. For each $i \in I$, assume that the following conditions are satisfied*:

(i) *$A_i : X \to 2^{X_i}$ is a lower semicontinuous multivalued map with nonempty convex values*;
(ii) *$P_i : X \to 2^{X_i}$ has an open graph and $x_i \notin \operatorname{co} P_i(x)$ for each $x \in X$*;
(iii) *For each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in G_n = \Pi_{i\in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $\tilde{x}_m \in G_m$ such that $\pi_i(\tilde{x}_m) \in (A_i \cap P_i)(x_m)$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then there exists $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Remark 2.2.4. If for each $i \in I$, X_i is compact and $\operatorname{cl} A_i$ is an upper semicontinuous multivalued map, Corollary 2.2.2 reduces Theorem 4.1 in [8].

3. Systems of generalized vector quasi-equilibrium problems

This section divided into two subsections. In the first subsection, we introduce four types of systems of generalized vector quasi-equilibrium problems (for short, SGVQEPs) and present systems of vector quasi-equilibrium problems, systems of mixed vector quasi-variational inequalities and Debreu-type equilibrium problems for vector valued functions as special cases of SGVQEPs. Some applications of results of Section 2 to establish some existence results for solutions of SGVQEPs are given in the second subsection.

3.1. Formulations

For each $i \in I$, let X_i be a nonempty subset of a topological vector space E_i, Y_i a topological vector space and let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $A_i : X \to 2^{X_i}$, $C_i : X \to 2^{Y_i}$ and $F_i : X \to 2^{X_i}$ be multivalued maps with nonempty values, and let $f_i : X \times X \times X_i \to 2^{Y_i}$ be a multivalued map with nonempty values. We consider the following *systems of generalized vector quasi-equilibrium problems* (in short, SGVQEPs):

$$\text{(SGVQEP)(I)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \subseteq C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

$$\text{(SGVQEP)(II)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \cap C_i(\hat{x}) \neq \emptyset \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

$$\text{(SGVQEP)(III)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \cap (-\operatorname{int} C_i(\hat{x})) = \emptyset \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

$$\text{(SGVQEP)(IV)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \not\subseteq -\operatorname{int} C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

It is clear that every solution of (SGVQEP)(I) (respectively, (SGVQEP)(III)) is a solution of (SGVQEP)(II) (respectively, (SGVQEP)(IV)) but converse assertion does not hold.

A problem similar to (SGVQEP)(I) and some other types of systems of generalized vector quasi-equilibrium problems are studied in [14] in the setting of locally G-convex uniform spaces.

If for each $i \in I$, $F_i \equiv A_i$ and $f_i(x, y, u_i)$ is independent of the variable y, that is, $f_i(x, y, u_i)$ is a multivalued map of two variables x and u_i, then (SGVQEP)(IV) is introduced and studied by Ansari and Khan [2]. They established the existence results for a solution of such problem. As an application of their problem, they derived existence results for a solution of (Debreu VEP)(II) (see below) for nonconvex and nondifferentiable (in some sense) vector valued functions. Furthermore, if for each $i \in I$ and for all $x \in X$, $F_i(x) = A_i(x) = X$. Then (SGVQEP)(IV) is considered and studied by Ansari et al. [4]. Some existence results for solutions of such problem are established. By using these results, existence of solutions of Nash equilibrium problem for vector valued functions are derived.

Special Cases: (a) If for each $i \in I$, f_i is a single valued map, then (SGVQEP)(I) and (SGVQEP)(II), and (SGVQEP)(III) and (SGVQEP)(IV), respectively, reduce to the following *systems of vector quasi-equilibrium problems* (for short, SVQEPs):

$$\text{(SVQEP)(I)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \in C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

$$\text{(SVQEP)(II)} \quad \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{A_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x}, \hat{y}, u_i) \notin -\operatorname{int} C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

Of course, (SVQEP)(I) is more general than (SVQEP)(II) as every solution of (SVQEP)(I) is a solution of (SVQEP)(II) but converse assertion is not true.

If for each $i \in I$, $F_i \equiv A_i$ and $f_i(x, y, u_i)$ is independent of the variable y, that is, $f_i(x, y, u_i)$ is a single valued map of two variables x and u_i, then the existence of solutions of (SVQEP)(II) is studied in [1] with application to (Debreu VEP)(II).

Furthermore, if for each $i \in I$ and for all $x \in X$, $F_i(x) = A_i(x) = X$. Then (SVQEP)(II) is studied in [3].

(b) We denote by $L(E_i, Y_i)$ the space of all continuous linear operators from E_i into Y_i. For each $i \in I$, let $H_i : X \to 2^{L(E_i,Y_i)}$ be a multivalued map with nonempty values, and let $g_i : X \times X_i \to Y_i$ and $\eta_i : X_i \times X_i \to X_i$ be vector valued mappings. For each $i \in I$, let $f_i(x, y, u_i) = \langle H_i(x), \eta_i(u_i, x_i)\rangle + g_i(x, u_i)$ for all $(x, y, u_i) \in X \times X \times X_i$ and let $F_i \equiv A_i$. Then SGVQEPs become the following *systems of mixed vector quasi-variational-like inequalities* (in short, SMVQVLIs).

$$\text{(SMVQVLIP)(I)} \quad \begin{cases} \text{Find } \hat{x} = (\hat{x}_i)_{i\in I} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ and} \\ \langle \hat{h}_i, \eta_i(u_i, \hat{x}_i)\rangle + g_i(\hat{x}, u_i) \in C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}) \text{ and } \hat{h}_i \in H_i(\hat{x}). \end{cases}$$

$$\text{(SMVQVLIP)(II)} \quad \begin{cases} \text{Find } \hat{x} = (\hat{x}_i)_{i\in I} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ and} \\ \text{for each } u_i \in A_i(\hat{x}), \text{ there exists } \hat{h}_i \in H_i(\hat{x}) \text{ satisfying} \\ \langle \hat{h}_i, \eta_i(u_i, \hat{x}_i)\rangle + g_i(\hat{x}, u_i) \in C_i(\hat{x}). \end{cases}$$

$$\text{(SMVQVLIP)(III)} \quad \begin{cases} \text{Find } \hat{x} = (\hat{x}_i)_{i\in I} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ satisfying} \\ \langle \hat{h}_i, \eta_i(u_i, \hat{x}_i)\rangle + g_i(\hat{x}, u_i) \notin -\operatorname{int} C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}) \text{ and } \hat{h}_i \in H_i(\hat{x}). \end{cases}$$

$$\text{(SMVQVLIP)(IV)} \quad \begin{cases} \text{Find } \hat{x} = (\hat{x}_i)_{i\in I} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ and} \\ \text{for each } u_i \in A_i(\hat{x}), \text{ there exists } \hat{h}_i \in H_i(\hat{x}) \text{ satisfying} \\ \langle \hat{h}_i, \eta_i(u_i, \hat{x}_i)\rangle + g_i(\hat{x}, u_i) \notin -\operatorname{int} C_i(\hat{x}). \end{cases}$$

(SMVQVLIP)(IV) is considered and studied by Ansari and Khan [2] for the case $g_i \equiv 0$ for each $i \in I$. It is also considered in [4] for the case $A_i(x) = X_i$ for each $i \in I$ and for all $x \in X$. For further detail and other particular cases, see for example [1–4,14] and references therein.

(c) For each $i \in I$, let $\varphi_i : X \to Y_i$ be a vector valued map and let $X^i = \prod_{j\in I, j\neq i} X_j$. We write $X = X^i \times X_i$. For each $x = (x_i)_{i\in I} \in X$, let $x^i = (x_j)_{j\in I, j\neq i}$, we write $x = (x^i, x_i)$. If for each $i \in I$, $\eta_i \equiv 0$ and $g_i(x, u_i) = \varphi_i(x^i, u_i) - \varphi_i(x)$ for all $(x, u_i) \in X \times X_i$, then the above four kinds of SMVQVLIPs reduce to the following two classes of *Debreu-type* *equilibrium problems* [10] for vector valued maps:

$$\text{(Debreu VEP)(I)}\quad \begin{cases} \text{Find } \hat{x} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ and} \\ \varphi_i(\hat{x}^i, u_i) - \varphi_i(\hat{x}) \in C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

$$\text{(Debreu VEP)(II)}\quad \begin{cases} \text{Find } \hat{x} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{A_i}(\hat{x}) \text{ and} \\ \varphi_i(\hat{x}^i, u_i) - \varphi_i(\hat{x}) \notin -\operatorname{int} C_i(\hat{x}) \text{ for all } u_i \in A_i(\hat{x}). \end{cases}$$

From the above special cases, it is clear that our SGVQEPs are more general and unifying models of several problems studied in the literature.

3.2. Existence results for solutions of SGVQEPs

First, we recall the following definitions.

Definition 3.2.1. Let X be a nonempty convex subset of a topological vector space E and Y be a topological vector space. A multivalued map $\Phi : X \to 2^Y$ is said to be *concave* if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $\lambda\Phi(x_1) + (1-\lambda)\Phi(x_2) \subseteq \Phi(\lambda x_1 + (1-\lambda)x_2)$.

Definition 3.2.2. Let X be a nonempty convex subset of a topological vector space E and Y be a topological vector space. Let $\Phi : X \times X \to 2^Y$ and $C : X \to 2^Y$ be multivalued maps such that for each $x \in X$, $\Phi(x) \neq \emptyset$ and $C(x)$ is a closed convex cone with $\operatorname{int} C(x) \neq \emptyset$. Let $x \in X$ be arbitrary, then Φ is called

(i) $C(x)$-*quasi-convex* if for all $y_1, y_2 \in X$ and $\lambda \in [0, 1]$, we have either

$$\Phi(x, y_1) \subseteq \Phi(x, \lambda y_1 + (1-\lambda)y_2) + C(x)$$

or

$$\Phi(x, y_2) \subseteq \Phi(x, \lambda y_1 + (1-\lambda)y_2) + C(x);$$

(ii) $C(x)$-*quasi-convex-like* [5] if for all $y_1, y_2 \in X$ and $\lambda \in [0, 1]$, we have either

$$\Phi(x, \lambda y_1 + (1-\lambda)y_2) \subseteq \Phi(x, y_1) - C(x)$$

or

$$\Phi(x, \lambda y_1 + (1-\lambda)y_2) \subseteq \Phi(x, y_2) - C(x);$$

(iii) *natural* $C(x)$-*quasi-concave* if for any $y_1, y_2 \in X$, $\lambda \in [0, 1]$ and $z_1 \in \Phi(x, y_1)$, $z_2 \in \Phi(x, y_2)$, there exists $z \in \Phi(x, \lambda y_1 + (1-\lambda)y_2)$ such that $z \in \operatorname{co}\{z_1, z_2\} - C(x)$;

(iv) $C(x)$-*convex* on X if for any $y_1, y_2 \in X$ and $\lambda \in [0, 1]$,

$$\Phi(x, \lambda y_1 + (1-\lambda)y_2) \subseteq \lambda\Phi(x, y_1) + (1-\lambda)\Phi(x, y_2) - C(x).$$

From now onward, unless otherwise specified, for each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i and let $X = \prod_{i \in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and $C_i : X \to 2^{Y_i}$ be a multivalued map such that for each $x \in X$, $C_i(x)$ is a proper, closed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$.

Theorem 3.2.1. *For each* $i \in I$, *let* $f_i : X \times X \times X_i \to 2^{Y_i}$ *be a lower semicontinuous multivalued map with nonempty values. For each* $i \in I$, *assume that*

(i) $F_i, A_i : X \to 2^{X_i}$ *are lower semicontinuous multivalued maps with nonempty convex values*;
(ii) $C_i : X \to 2^{Y_i}$ *is an upper semicontinuous multivalued map*;
(iii) *for all* $x = (x_i)_{i \in I}$, $y \in X$, $f_i(x, y, x_i) \subseteq C_i(x)$;

(iv) *for all* $x, y \in X$, *the multivalued map* $u_i \mapsto f_i(x, y, u_i)$ *is* $C_i(x)$*-quasi-convex*; *and*
(v) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets* $\tilde{D}_i$ *and* D_i *of* X_i *for each* $i \in I$ *with the property that for each* $(x, y) \in X \times X \backslash K \times M$, *there exist* $j \in I$ *and* $\tilde{u}_j \in \tilde{D}_j$, $\tilde{v}_j \in D_j$ *such that* $\tilde{u}_j \in A_j(x)$, $\tilde{v}_j \in F_j(x)$ *and* $f_j(x, y, \tilde{u}_j) \not\subseteq C_j(x)$.

Then there exists a solution $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *of* (*SGVQEP*)(*I*).

Proof. For each $i \in I$, let $P_i : X \times X \to 2^{X_i}$ be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \not\subseteq C_i(x)\} \quad \text{for all } (x, y) \in X \times X.$$

By condition (iv), for each $i \in I$ and for all $(x, y) \in X \times X$, $P_i(x, y)$ is convex.

Indeed, let $u_{i_1}, u_{i_2} \in P_i(x, y)$ and $\lambda \in [0, 1]$, then $f_i(x, y, u_{i_j}) \not\subseteq C_i(x)$ for $j = 1, 2$. Let $u_\lambda = \lambda u_{i_1} + (1 - \lambda)u_{i_2}$, then we have to show that $f_i(x, y, u_\lambda) \not\subseteq C_i(x)$. Suppose on the contrary that there exists a $\lambda_0 \in [0, 1]$ such that $f_i(x, y, u_{\lambda_0}) \subseteq C_i(x)$. Since the multivalued map $u_i \mapsto f_i(x, y, u_i)$ is $C_i(x)$-quasi-convex, we have either $f_i(x, y, u_{i_1}) \subseteq f_i(x, y, u_\lambda) + C_i(x)$ or $f_i(x, y, u_{i_2}) \subseteq f_i(x, y, u_\lambda) + C_i(x)$ for all $\lambda \in [0, 1]$. Therefore, in particular, we have either $f_i(x, y, u_{i_1}) \subseteq f_i(x, y, u_{\lambda_0}) + C_i(x)$ or $f_i(x, y, u_{i_2}) \subseteq f_i(x, y, u_{\lambda_0}) + C_i(x)$. Since $f_i(x, y, u_{\lambda_0}) \subseteq C_i(x)$, we have $f_i(x, y, u_{i_j}) \subseteq C_i(x) + C_i(x) = C_i(x)$, $i = 1, 2$ which contradicts to our assumption that $u_{i_j} \in P_i(x, y)$. Hence, $P_i(x, y)$ is convex.

Since for all $x = (x_i)_{i\in I}, y \in X$, $f_i(x, y, x_i) \subseteq C_i(x)$ we have $x_i \notin P_i(x, y) = \mathrm{co}P_i(x, y)$.

For each $i \in I$, P_i has an open graph in $X \times X_i$. Indeed, let $(x, y, u_i) \in \mathrm{cl}[G_r P_i]^c$, then there exists a net $\{(x^\alpha, y^\alpha, u_i^\alpha)\}_{\alpha\in\Lambda}$ in $[G_r P_i]^c$ such that $(x^\alpha, y^\alpha, u_i^\alpha) \to (x, y, u_i) \in X \times X \times X_i$, where Λ is an index set. Then $f_i(x^\alpha, y^\alpha, u_i^\alpha) \subseteq C_i(x^\alpha)$ for all $\alpha \in \Lambda$. For any $z \in f_i(x, y, u_i)$, the lower semicontinuity of f_i on $X \times X \times X_i$ implies that there exists a net $\{z^\alpha\}$ in X_i such that $z^\alpha \to z$ and $z^\alpha \in f_i(x^\alpha, y^\alpha, u_i^\alpha)$ for all $\alpha \in \Lambda$. Hence $z^\alpha \in C_i(x^\alpha)$ for all $\alpha \in \Lambda$and $(x^\alpha, z^\alpha) \to (x, z)$. Since C_i is upper semicontinuous with closed values, it follows that C_i is closed and hence $z \in C_i(x)$. This shows that $f_i(x, y, u_i) \subseteq C_i(x)$. Therefore, $(x, y, u_i) \in [G_r P_i]^c$ and hence $[G_r P_i]^c$ the complement of $G_r P_i$ in $X \times X \times X_i$ is closed.

Condition (v) implies for each $(x, y) \in X \times X \subseteq K \times M$, there exists $j \in I$ such that $(A_j(x) \cap P_j(x, y)) \cap \tilde{D}_j \neq \emptyset$ and $F_j(x) \cap D_j \neq \emptyset$. Then by Theorem 2.2.1 there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$, that is, $f_i(\hat{x}, \hat{y}, u_i) \subseteq C_i(\hat{x})$ for all $u_i \in A_i(\hat{x})$. □

Theorem 3.2.2. *For each* $i \in I$, *let* $f_i : X \times X \times X_i \to 2^{Y_i}$ *be an upper semicontinuous multivalued map with nonempty compact values. For each* $i \in I$, *assume that*

(i) $F_i, A_i : X \to 2^{X_i}$ *are lower semicontinuous multivalued maps with nonempty convex values*;
(ii) $C_i : X \to 2^{Y_i}$ *is an upper semicontinuous multivalued map*;
(iii) *for all* $x = (x_i)_{i\in I}, y \in X$, $f_i(x, y, x_i) \cap C_i(x) \neq \emptyset$;
(iv) *for all* $(x, y) \in X \times X$, *the multivalued map* $u_i \mapsto f_i(x, y, u_i)$ *is* $C_i(x)$*-quasi-convex-like*; *and*
(v) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets* $\tilde{D}_i$ *and* D_i *of* X_i *for each* $i \in I$ *with the property that for each* $(x, y) \in X \times X \backslash K \times M$, *there exist* $j \in I$ *and* $\tilde{u}_j \in \tilde{D}_j$, $\tilde{v}_j \in D_j$ *such that* $\tilde{u}_j \in A_j(x)$,$\tilde{v}_j \in F_j(x)$ *and* $f_j(x, y, \tilde{u}_j) \cap C_j(x) = \emptyset$.

Then there exists a solution $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *of* (*SGVQEP*)(*II*).

Proof. For each $i \in I$, let $P_i : X \times X \to 2^{X_i}$ be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \cap C_i(x) = \emptyset\} \quad \text{for all } (x, y) \in X \times X.$$

Following the approach of Theorem 3.2.1 and by using condition (iv), we have that $P_i(x, y)$ is convex for each $i \in I$ and for all $(x, y) \in X \times X$.

Condition (iii) implies that $x_i \notin P_i(x, y) = \mathrm{co}P_i(x, y)$ for all $x = (x_i)_{i\in I}, y \in X$.

Since each f_i is upper semicontinuous with nonempty compact values and each C_i is also upper semicontinuous, by using the same argument as in [5], we obtain that $G_r P_i$ is open.

By Theorem 2.2.1, there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$, that is, $f_i(\hat{x}, \hat{y}, u_i) \cap C_i(\hat{x}) \neq \emptyset$ for all $u_i \in A_i(\hat{x})$. □

Theorem 3.2.3. *For each $i \in I$, let $f_i : X \times X \times X_i \to 2^{Y_i}$ be a lower semicontinuous multivalued map with nonempty values. For each $i \in I$, assume that*

(i) *$F_i, A_i : X \to 2^{X_i}$ are lower semicontinuous multivalued maps with nonempty convex values*;
(ii) *$W_i : X \to 2^{Y_i}$ is an upper semicontinuous multivalued map defined as $W_i(x) = Y_i \backslash (-\operatorname{int} C_i(x))$ for all $x \in X$*;
(iii) *for all $x = (x_i)_{i\in I}$, $y \in X$, $f_i(x, y, x_i) \cap (-\operatorname{int} C_i(x)) = \emptyset$*;
(iv) *for all $(x, y) \in X \times X$, the multivalued map $u_i \mapsto f_i(x, y, u_i)$ is natural $C_i(x)$-quasi-concave*;
(v) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\tilde{D}_i$ and D_i of X_i for each $i \in I$ with the property that for each $(x, y) \in X \times X \backslash K \times M$, there exist $j \in I$ and $\tilde{u}_j \in \tilde{D}_j$, $\tilde{v}_j \in D_j$ such that $\tilde{u}_j \in A_j(x)$, $\tilde{v}_j \in F_j(x)$ and $f_j(x, y, \tilde{u}_j) \cap (-\operatorname{int} C_j(x)) \neq \emptyset$.*

Then there exists a solution $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ of (SGVQEP)(III).

Proof. For each $i \in I$, let $P_i : X \times X \to 2^{X_i}$ be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \cap (-\operatorname{int} C_i(x)) \neq \emptyset\} \quad \text{for all } (x, y) \in X \times X.$$

By condition (iv), for each $i \in I$ and for all $(x, y) \in X \times X$, $P_i(x, y)$ is convex. Indeed, let $u_{i_1}, u_{i_2} \in P_i(x, y)$ and $\lambda \in [0, 1]$, then $f_i(x, y, u_{i_j}) \cap (-\operatorname{int} C_i(x)) \neq \emptyset$ for $j = 1, 2$. Let $z_{i_j} \in f_i(x, y, u_{i_j}) \cap (-\operatorname{int} C_i(x))$ for $j = 1, 2$, thus $z_{i_j} \in f_i(x, y, u_{i_j})$ and $z_{i_j} \in -\operatorname{int} C_i(x)$ for $j = 1, 2$. Since $(-\operatorname{int} C_i(x))$ is convex, so that $\operatorname{co}\{z_{i_1}, z_{i_2}\} \subseteq -\operatorname{int} C_i(x)$. Since for each fixed $(x, y) \in X \times X$, $u_i \mapsto f_i(x, y, u_i)$ is natural $C_i(x)$-quasi-concave, it is easy to see that $\lambda u_{i_1} + (1-\lambda) u_{i_2} \in P_i(x, y)$ and hence $P_i(x, y)$ is convex.

Condition (iii) implies that $x_i \notin P_i(x, y) = \operatorname{co} P_i(x, y)$ for all $x = (x_i)_{i\in I}$, $y \in X$.

Since f_i is lower semicontinuous on $X \times X \times X_i$ and W_i is upper semicontinuous on X and by following the same argument as in Proof of Theorem 3.2.1, we have that $G_r P_i$ is open. By Theorem 2.2.1, there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$, that is, $f_i(\hat{x}, \hat{y}, u_i) \cap (-\operatorname{int} C_i(\hat{x})) = \emptyset$ for all $u_i \in A_i(\hat{x})$. □

Remark 3.2.1. The condition (iv) of Theorem 3.2.3 can be replaced by the following assumption.

(iv)′ for each $i \in I$ and for each $(x, y) \in X \times X$, the multivalued map $u_i \mapsto f_i(x, y, u_i)$ is concave.

Indeed, we only remain to show that for each $i \in I$ and for all $(x, y) \in X \times X$, $P_i(x, y)$ is convex.

Let $u_{i_1}, u_{i_2} \in P_i(x, y)$ and $\lambda \in [0, 1]$, then $f_i(x, y, u_{i_j}) \cap (-\operatorname{int} C_i(x)) \neq \emptyset$ for $j = 1, 2$. Let $z_{i_j} \in f_i(x, y, u_{i_j}) \cap (-\operatorname{int} C_i(x))$ for $j = 1, 2$. Then $z_{i_j} \in f_i(x, y, u_{i_j})$ and $z_{i_j} \in -\operatorname{int} C_i(x)$ for $j = 1, 2$. Since $-\operatorname{int} C_i(x)$ is convex, $\lambda z_{i_1} + (1-\lambda) z_{i_2} \in -\operatorname{int} C_i(x)$. Since the multivalued map $u_i \mapsto f_i(x, y, u_i)$ is concave, $\lambda z_{i_1} + (1-\lambda) z_{i_2} \in \lambda f_i(x, y, u_{i_1}) + (1-\lambda) f_i(x, y, u_{i_2}) \subseteq f_i(x, y, \lambda u_{i_1} + (1-\lambda) u_{i_2})$. Thus, $f_i(x, y, \lambda u_{i_1} + (1-\lambda) u_{i_2}) \cap (-\operatorname{int} C_i(x)) \neq \emptyset$, it follows that $\lambda u_{i_1} + (1-\lambda) u_{i_2} \in P_i(x, y)$ and hence $P_i(x, y)$ is convex.

Theorem 3.2.4. *Foreach $i \in I$, let $f_i : X \times X \times X_i \to 2^{Y_i}$ be an upper semicontinuous multivalued map with nonempty compact values. For each $i \in I$, assume that*

(i) *$F_i, A_i : X \to 2^{X_i}$ are lower semicontinuous multivalued maps with nonempty convex values*;
(ii) *$W_i : X \to 2^{Y_i}$ is an upper semicontinuous multivalued map defined as $W_i(x) = Y_i \backslash (-\operatorname{int} C_i(x))$ for all $x \in X$*;
(iii) *for all $x = (x_i)_{i\in I}$, $y \in X$, $f_i(x, y, x_i) \nsubseteq -\operatorname{int} C_i(x)$*;
(iv) *for all $(x, y) \in X \times X$, the multivalued map $u_i \mapsto f_i(x, y, u_i)$* is *$C_i(x)$-quasi-convex-like*;
(v) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\tilde{D}_i$ and D_i of X_i for each $i \in I$ with the property that for each $(x, y) \in X \times X \backslash K \times M$, there exist $j \in I$ and $\tilde{u}_j \in \tilde{D}_j$, $\tilde{v}_j \in D_j$ such that $\tilde{u}_j \in A_j(x)$, $\tilde{v}_j \in F_j(x)$ and $f_j(x, y, \tilde{u}_j) \subseteq -\operatorname{int} C_j(x)$.*

Then there exists a solution $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ of (SGVQEP)(IV).

Proof. For each $i \in I$, let $P_i : X \times X \to 2^{X_i}$ be defined by

$$P_i(x, y) = \{u_i \in X_i : f_i(x, y, u_i) \subseteq -\text{int}\, C_i(x)\} \quad \text{for all } (x, y) \in X \times X.$$

By condition (iv), for each $i \in I$ and for all $(x, y) \in X \times X$, $P_i(x, y)$ is convex (see, for example [5, Proof of Theorem 2.1]).

Condition (iii) implies that $x_i \notin P_i(x, y) = \text{co}_i\, P(x, y)$ for all $x = (x_i)_{i \in I}$, $y \in X$.

Since W_i is upper semicontinuous on X and f_i is upper semicontinuous with nonempty compact values, by using the same argument as in Proof of Theorem 3.2.2, we have that $G_r P_i$ is open. By Theorem 2.2.1, there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i \in I}, (\hat{y}_i)_{i \in I}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{A_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$, that is, $f_i(\hat{x}, \hat{y}, u_i) \nsubseteq -\text{int}\, C_i(\hat{x})$ for all $u_i \in A_i(\hat{x})$. □

Remark 3.2.2. Condition (iv) of Theorem 3.2.4 can be replaced by the following condition:

(iv)′ for each $i \in I$ and for all $(x, y) \in X \times X$, the multivalued map $u_i \mapsto f_i(x, y, u_i)$ is $C_i(x)$-convex.

Remark 3.2.3. Theorem 3.2.4 generalizes Theorem 2.1 in [3] by several ways.

Theorem 3.2.5. *For each $i \in I$, let $X_i = \bigcup_{j=1}^{\infty} G_{i,j}$, where $\{G_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subset of E_i, and $f_i : X \times X \times X_i \to 2^{Y_i}$ be a l.s.c. multivalued map with nonempty values. Assume that the conditions* (i)–(iv) *of Theorem* 3.2.1 *and the following condition hold*:

(v)′ *for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \prod_{i \in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m)$,$\pi_i(\tilde{y}_m) \in F_i(x_m)$ and $f_i(x_m, y_m, \pi_i(\tilde{x}_m)) \nsubseteq C_i(x_m)$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then (*SGVQEP*)(*I*) *has a solution.*

Proof. For each $i \in I$, let P_i be the same as defined in Proof of Theorem 3.2.1. By using Theorem 2.2.2, assumption (v)′ and following the same argument as in Proof of Theorem 3.2.1, we get the conclusion. □

Theorem 3.2.6. *For each $i \in I$, let X_i, E_i, $\{G_{i,j}\}_{j=1}^{\infty}$ be the same as in Theorem* 3.2.5 *and f_i be an u.s.c. multivalued map with noncompact values. Assume that conditions* (i)–(iv) *of Theorem* 3.2.2 *and the following condition hold*:

(v)′ *for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \prod_{i \in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m)$, $\pi_i(\tilde{y}_m) \in F_i(x_m)$ and $f_i(x_m, y_m, \pi_i(\tilde{x}_m)) \cap C_i(x_m) = \emptyset$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then (*SGVQEP*)(*II*) *has a solution.*

Proof. For each $i \in I$, let P_i be the same as defined in Proof of Theorem 3.2.2. By using Theorem 2.2.2, assumption (v)′ and following the same argument as in Theorem 3.2.2, we get the conclusion. □

Theorem 3.2.7. *For each $i \in I$, let X_i, E_i, $\{G_{i,j}\}_{j=1}^{\infty}$ and f_i be the same as in Theorem* 3.2.5. *Assume that conditions* (i)–(iv) *of Theorem* 3.2.3 *and the following condition hold*:

(v)′ *for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \Pi_{i \in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m)$, $\pi_i(\tilde{y}_m) \in F_i(x_m)$ and $f_i(x_m, y_m, \pi_i(\tilde{x}_m)) \cap (-\text{int}\, C_i(x_m)) \neq \emptyset$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then (*SGVQEP*)(*III*) *has a solution.*

Proof. For each $i \in I$, let P_i be the same as defined in Proof of Theorem 3.2.3. By using Theorem 2.2.2, assumption (v)′ and following the same argument as in Theorem 3.2.3, we obtain the conclusion. □

Theorem 3.2.8. *For each $i \in I$, let X_i, E_i, $\{G_{i,j}\}_{j=1}^{\infty}$ and f_i be the same as in Theorem* 3.2.5. *Assume that conditions* (i)–(iv) *of Theorem* 3.2.4 *and following condition hold*:

(v)′ *for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \Pi_{i\in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exist $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m)$, $\pi_i(\tilde{y}_m) \in F_i(x_m)$ and $f_i(x_m, y_m, \pi_i(\tilde{x}_m)) \subseteq -\text{int}\, C_i(x_m)$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then (*SGVQEP*)(*IV*) *has a solution*.

Proof. For each $i \in I$, let P_i be the same as defined in Proof of Theorem 3.2.4. By using Theorem 2.2.2, assumption (v)′ and following the same argument as in Proof of Theorem 3.2.4, we get the conclusion. □

4. Conclusions

In this paper, we first considered generalized abstract economy which generalizes the concept of an abstract economy studied in the literature. By using a known maximal element theorem for a family of multivalued maps, we proved the existence of an equilibrium for generalized abstract economy with lower semicontinuous constraint correspondence and a fuzzy constraint correspondence defined on a noncompact/nonparacompact strategy set. As a particular case, we also derived the existence results for an equilibrium of abstract economy. Secondly, we considered systems of generalized vector quasi-equilibrium problems (for short, SGVQEPs) which contain system of vector quasi-equilibrium problems, system of generalized mixed vector quasi-variational inequalities and Debreu-type equilibrium problems for vector valued functions as special cases. As applications of results of Section 2.2, we established some existence results for solutions of SGVQEPs. The existence results for solutions of the problems which are special cases of our SGVQEPs can be easily derived from the results of Section 3.2. By using the technique of Ansari et al. [1–4], it is easy to establish the existence of solutions of (Debreu VEP)(I) and (Debreu VEP)(II) for nondifferentiable (in some sense) and nonconvex vector valued functions. To best of our knowledge, no study has been done on the existence of solutions of (Debreu VEP)(I). This paper can be seen as an effort in this direction.

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