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Existence of Equilibria for Generalized Abstract Economies and System of Quasi-Minimax Inequalities[1]

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Abstract. In this paper, we consider a more general form of generalized abstract economy with a fuzzy constraint correspondence studied in [15, 17]. We establish some existence results for an equilibrium of our generalized abstract economy. We also consider a system of quasi-minimax inequalities and note that it is a unified model of several problems. As applications of our equilibrium existence results, we derive some existence results for a solution of system of quasi-minimax inequalities. Several special cases are also considered.

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tem of quasi-minimax inequalities, Upper (lower) semicontinuity, Escaping sequence

1. Introduction

The study of existence of equilibria of various kind of economic models is one of the main problems in mathematical economics. In 1954, Arrow and Debreu [3] first established a result on the existence of Walrasian equilibria of an abstract economy. Since than, this result has been generalized and extended in many directions; See for example [1, 5, 6, 7, 9, 12, 13, 14, 16, 18, 21, 22, 23, 24, 26, 27, 28, 29, 30] and references therein.

Because of the fuzziness of consumer's behaviour or market situations, any preference of a real agent could be unstable. That's why, Kim and Tan [15] introduced a general model of abstract economy with a fuzzy constraint correspondence. They proved two general equilibrium existence theorems for their model by using Himmelberg fixed point theorem and Eilenberg-Montgomery fixed point theorem. These results of Kim and Tan [15] include the previous equilibrium existence results in [5, 21, 27]. This model of abstract economy is further studied by Lin et al [17]. By using their maximal element theorem, they established two equilibrium existence theorems for this model under the assumption of compactly open lower sections of the correspondences involved in it. As we know that the open lower sections property of a correspondence implies the lower semicontinuity but the converse is not true in general; See for example [27].

In this paper, we consider a more general form of generalized abstract economy with a fuzzy constraint correspondence studied in [15, 17]. By using two different maximal element theorems due to Yuan et al [14, 29], we establish some existence results for an equilibrium of our generalized abstract economy in the setting of noncompact set of actions and with lower semicontinuous constraint correspondences, a lower semicontinuous fuzzy constraint correspondence and a L^*-majoried preference correspondence. We also derive some existence results for an equilibrium of abstract economy considered and studied in [1, 5, 6, 9, 12, 13, 14, 16, 18, 21, 27, 29, 30] and references therein. We also consider a system of quasi-minimax inequalities which includes system of generalized implicit quasi-variational inequalities, system of quasi-equilibrium problems and Debreu type equilibrium problem (also known as constrained Nash equilibrium problem) as special cases. As applications of our equilibrium existence results, we establish some existence results for a solution of system of quasi-minimax inequalities. Several special cases are also discussed.

2. Formulations and Preliminaries

Let Ω be a nonempty set. We denote by 2^Ω (respectively, $\mathrm{F}(\Omega)$) the family of all subsets (respectively, finite subsets) of Ω. If Ω is a subset of a vector space, then $\mathrm{co}\Omega$ denotes the convex hull of Ω. If Ω and Δ are subsets of a topological space $\mathcal{X}$ such that $\Omega \subseteq \Delta$, then the closure (respectively, interior) of Ω in Δ is denoted by $\mathrm{cl}_\Delta\Omega$ (respectively, $\mathrm{int}_\Delta\Omega$); In case $\Delta = \mathcal{X}$, we write $\mathrm{cl}\Omega$ and $\mathrm{int}\,\Omega$ instead of $\mathrm{cl}_\mathcal{X}\Omega$ and $\mathrm{int}_\mathcal{X}\Omega$, respectively. A subset Ω of a topological space $\mathcal{X}$ is said to be *compactly open* in $\mathcal{X}$ if for each nonempty compact subset C of $\mathcal{X}$, $\Omega \cap C$ is open in C. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological vector spaces and $\Phi, \Psi : \mathcal{X} \to 2^\mathcal{Y}$ be correspondences. Then $\mathrm{co}\Phi, \Phi \cap \Psi : \mathcal{X} \to 2^\mathcal{Y}$ are defined as $(\mathrm{co}\Phi)(x) = \mathrm{co}\Phi(x)$ and $(\Phi \cap \Psi)(x) = \Phi(x) \cap \Psi(x)$ for all $x \in \mathcal{X}$, respectively. For a nonempty subset V of $\mathcal{Y}$, $\Phi^{-1}(V) = \{x \in \mathcal{Y} : \Phi(x) \cap V \neq \emptyset\}$ and also $x \in \Phi^{-1}(y)$ if and only if $y \in \Phi(x)$. Φ is said to have *open lower sections* (respectively, *compactly open lower sections*) if for each $y \in \mathcal{Y}, \Phi^{-1}(y)$ is open (respectively, compactly open) in $\mathcal{X}$. We also define $\overline{\Phi}, \mathrm{cl}\Phi : \mathcal{X} \to 2^\mathcal{Y}$ by

$$\overline{\Phi}(x) = \{y \in \mathcal{Y} : (x, y) \in \mathrm{cl}_{\mathcal{X}\times\mathcal{Y}} Gr(\Phi)\},$$

where $Gr(\Phi) = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y \in \Phi(x)\}$ denotes the graph of Φ, and

$$\mathrm{cl}\Phi(x) = \mathrm{cl}_\mathcal{Y}\big(\Phi(x)\big) \quad \text{for all } x \in \mathcal{X}, \text{ respectively.}$$

It is easy to see that $\mathrm{cl}\Phi(x) \subseteq \overline{\Phi}(x)$ for all $x \in \mathcal{X}$. Clearly, if Φ has an open graph in $\mathcal{X} \times \mathcal{Y}$, then Φ has open lower section and $\Phi(x)$ is open for all $x \in \mathcal{X}$. However, a lower semicontinuous correspondence may not have open lower section in general; See for example [27]. Furthermore, if Φ has open lower section, then Φ is lower semicontinuous.

In a real market, any preference of a real agent could be unstable because of the fuzziness of consumers' behaviour or market situations. Therefore, Kim and Tan [15] and Lin et al [17] considered and studied *generalized abstract economy* with a fuzzy constraint correspondence. In this paper, we introduce the following more general form of generalized abstract economy.

Let I be any set of agents (countable or uncountable). For each $i \in I$, let X_i be a nonempty set of actions available to the agent i in a topological vector space E_i and $X = \prod_{i\in I} X_i$. A *generalized abstract economy* (or *generalized game*) $\Gamma = (X_i, A_i, B_i, F_i, P_i)_{i\in I}$ is defined as a family of ordered quintuples $(X_i, A_i, B_i, F_i, P_i)$ where $A_i, B_i : X \to 2^{X_i}$ are constraint correspondences such that $A_i(x)$ and $B_i(x)$ are the states attainable for the agent i at x, $F_i : X \to 2^{X_i}$ is a fuzzy constraint correspondence such that $F_i(x)$ is the unstable state for the agent i, and $P_i : X \times X \to 2^{X_i}$ is a preference correspondence such that $P_i(x)$ is the state preference by the agent i at x. An *equilibrium* for Γ is a point $(\hat{x},\hat{y}) \in X \times X$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $P_i(\hat{x},\hat{y}) \cap A_i(\hat{x}) = \emptyset$.

If for each $i \in I$, $A_i \equiv B_i$ then our generalized abstract economy is the same as considered in [15, 17].

If for each $i \in I$ and for all $x \in X$, $F_i(x) = X_i$ and the preference correspondence P_i is independent of the second variable y, that is, P_i is a correspondence of single variable x, our definitions of a generalized abstract economy and an equilibrium coincide with the usual definitions of an abstract economy and an equilibrium due to Kim and Yuan [14], Lin and Ansari [16], Mehta et al [18], Wu [26] Yuan and Tarafdar [30] and Yuan et al [29]; See also references in these papers.

Furthermore, if $\overline{B_i}(x) = \text{cl}B_i(x)$ for all $x \in X$ and for each $i \in I$ (which is the case when B_i has closed graph in $X \times X_i$; in particular, when $\text{cl}B_i(x)$ is upper semicontinuous with closed values), our definition of an equilibrium point coincides with that of Ansari and Yao [1], Chang [7] and Ding et al [9]; See also references therein.

In addition, if $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition in [5, 27].

Definition 2.1. [14, 29, 30] Let X be a topological space, Y a nonempty subset of a vector space E, $\theta : X \to E$ a single valued map and $\Phi : X \to 2^Y$ be a correspondence.

(i) Φ is said to be of *class* $L_{\theta,C}$ if for every $x \in X$, $\text{co}\Phi(x) \subseteq Y$ and $\theta(x) \notin \text{co}\Phi(x)$ and for all $y \in Y$, $\Phi^{-1}(y)$ is compactly open in X;

(ii) A correspondence $\Phi_x : X \to 2^Y$ is said to be an $L_{\theta,C}$-*majorant of* Φ *at* x if there exists an open neighbourhood N_x of x in X such that

(a) for all $z \in N_x$, $\Phi(z) \subseteq \Phi_x(z)$ and $\theta(z) \notin \text{co}\Phi_x(z)$,
(b) for all $z \in X$, $\text{co}\Phi_x(z) \subseteq Y$,
(c) for all $y \in Y$, $\Phi_x^{-1}(y)$ is compactly open in X;

(iii) Φ is said to be $L_{\theta,C}$-*majorized* if for all $x \in X$ with $\Phi(x) \neq \emptyset$, there exists an $L_{\theta,C}$-majorant of Φ at $x \in X$.

In [14, 29, 30], Yuan et al deal mainly with either case (I) $X = Y$, which is a nonempty convex subset of a topological vector space and $\theta = I_X$ the identity map on X, or case (II) $X = \prod_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a nonempty convex subset of a topological vector space. In both cases (I) and (II), we shall write L in place of $L_{\theta,C}$.

In this paper, we introduce the following more general form of majorized maps.

Definition 2.2. Let X be a topological space, Y a nonempty subset of a vector space E, $\sigma : X \times X \to E$ a single valued map, and $\Psi : X \times X \to 2^Y$ be a correspondence.

(i) A correspondence $\Psi_{(x,y)} : X \times X \to 2^Y$ is said to be an $L^*_{\sigma,C}$-*majorant of* Ψ *at* (x, y) if there exists an open neighbourhood $N_{(x,y)}$ of (x, y) in $X \times X$ such that

  (a) for all $(z, w) \in N_{(x,y)}$, $\Psi(z, w) \subseteq \Psi_{(x,y)}(z, w)$ and $\sigma(z, w) \notin \mathrm{co}\Psi_{(x,y)}(z, w)$,
  (b) for all $(z, w) \in X \times X$, $\mathrm{co}\Psi_{(x,y)}(z, w) \subseteq Y$,
  (c) for all $r \in Y$, $\Psi^{-1}_{(x,y)}(r)$ is compactly open in $X \times X$;

(ii) Ψ is said to be $L^*_{\sigma,C}$-*majorized* if for all $(x, y) \in X \times X$ with $\Psi(x, y) \neq \emptyset$, there exists an $L^*_{\sigma,C}$-majorant of Ψ at $(x, y) \in X \times X$.

Mainly we deal with $X = \prod_{i\in I} X_i$, $\sigma = \pi_j : X \times X \to X_j$ is the projection of $X \times X$ onto X_j and $Y = X_j$ is a nonempty convex subset of a topological vector space. In this case we shall write L^* in place of $L^*_{\sigma,C}$.

Let us recall the following definitions and results which will be used in the sequel.

Lemma 2.1. [6] *Let X be a nonempty subset of a topological space, Y a nonempty subset of a topological vector space E, and V be a nonempty open subset of E. If $\Phi : X \to 2^Y$ is lower semicontinuous, then the multivalued map $H : X \to 2^Y$ defined by $H(x) = (\Phi(x) + V) \cap Y$ for all $x \in X$, has an open graph in $X \times Y$.*

Lemma 2.2. [29] *Let $\mathcal{X}$ be a topological space, D a nonempty subset of a topological vector space E, $\mathcal{B}$ a base for the neighbourhoods of zero in E and $\Phi : \mathcal{X} \to 2^D$ a multivalued map with nonempty values. For each $V \in \mathcal{B}$, let $\Phi_V : \mathcal{X} \to 2^D$ be defined by $\Phi_V(x) = (\Phi(x) + V) \cap D$ for all $x \in \mathcal{X}$. If $\tilde{x} \in \mathcal{X}$ and $\tilde{y} \in D$ are such that $\tilde{y} \in \bigcap_{V\in\mathcal{B}} \overline{\Phi_V}(\tilde{x})$, then $\tilde{y} \in \overline{\Phi}(\tilde{x})$.*

Theorem 2.1. [14] *For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff topological vector space E_i and let $X = \prod_{i\in I} X_i$. For each $i \in I$, assume that*

(i) the multivalued map $P_i : X \to 2^{X_i}$ is L-majorized,
(ii) the set $\{x \in X : P_i(x) \neq \emptyset\}$ is open and paracompact,
(iii) there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ with the property that for each $\tilde{x} \in X \setminus K$, there exists $j \in I$ such that $P_j(\tilde{x}) \cap C_j \neq \emptyset$.

Then there exists $\hat{x} \in K$ such that $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

Definition 2.3. [4] Let X be a subset of a topological vector space E such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence (in the sense that

$C_n \subseteq C_{n+1}$) of nonempty compact subsets of E. A sequence $\{y_n\}_{n=1}^{\infty}$ in X is said to be *escaping* from X (relative to $\{C_n\}_{n=1}^{\infty}$) if for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $y_k \notin C_n$ for all $k \geq m$.

Theorem 2.2. [29] *For each $i \in I$, let X_i be a subset of a topological vector space (not necessarily Hausdorff) E_i such that $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$, where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of E_i. For each $i \in I$, let $P_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ be an L-majorized multivalued map such that*

(i) $\bigcup_{i\in I}\{x \in X : P_i(x) \neq \emptyset\} = \bigcup_{i\in I} \operatorname{int}_X\{x \in X : P_i(x) \neq \emptyset\}$, *and*

(ii) *for any sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ where $C_n = \prod_{i\in I} C_{i,n}$ for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $x_m \in C_m$ such that $\pi_i(x_m) \in \operatorname{co}P_i(y_m)$ for each $i \in I(y_m)$, where $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$ and $\pi_i(x_m)$ is the projection of x_m onto X_i.*

Then there exists $\hat{x} \in X$ such that $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

3. Existence of Equilibria for Generalized Abstract Economies

In this section, we establish some equilibrium existence theorems for generalized abstract economies by using Theorems 2.1 and 2.2.

Theorem 3.1. *Let $(X_i, A_i, B_i, P_i, F_i)_{i\in I}$ be a generalized abstract economy such that $X = \prod_{i\in I} X_i$ is paracompact and $X \times X$ is perfectly normal and paracompact and for each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i. For each $i \in I$, assume that*

(i) A_i, $B_i : X \to 2^{X_i}$ *are multivalued maps such that A_i is lower semicontinuous and for all $x \in X$, $A_i(x) \neq \emptyset$ and $\operatorname{co}A_i(x) \subseteq B_i(x)$,*

(ii) $F_i : X \to 2^{X_i}$ *is a lower semicontinuous multivalued map with nonempty convex values,*

(iii) the multivalued map $Q_i : X \times X \to 2^{X_i}$ defined as $Q_i(x,y) = A_i(x) \cap P_i(x,y)$, is L^*-majorized,

(iv) $\mathrm{G}_i = \{(x,y) \in X \times X : A_i(x) \cap P_i(x,y) \neq \emptyset\}$ *is an open subset of $X \times X$, and*

(v) there exist nonempty compact subsets K and M of X and nonempty compact convex subsets C_i and D_i of X_i for each $i \in I$ with the property that for each $(\tilde{x}, \tilde{y}) \in X \times X \setminus K \times M$, there exists $j \in I$ such that $A_j(\tilde{x}) \cap P_j(\tilde{x}, \tilde{y}) \cap C_j \neq \emptyset$ and $F_j(\tilde{x}) \cap D_j \neq \emptyset$.

Then there exists an equilibrium point $(\hat{x}, \hat{y}) \in K \times M$ *of generalized abstract economy.*

Proof. For each $i \in I$, let $\mathcal{B}_i$ be the collection of all open convex neighbourhoods of zero in E_i and $\mathcal{B} = \prod_{i\in I} \mathcal{B}_i$. For any given $V \in \mathcal{B}$, let $V = \prod_{i\in I} V_i$, where $V_i \in \mathcal{B}_i$ for each $i \in I$. For any fixed $i \in I$ and for all $x \in X$, define A_{V_i}, B_{V_i}, $F_{V_i} : X \to 2^{X_i}$ by $A_{V_i}(x) = (\text{co}A_i(x)+V_i)\cap X_i$, $B_{V_i}(x) = (B_i(x)+V_i)\cap X_i$, and $F_{V_i}(x) = (F_i(x) + V_i) \cap X_i$, respectively. Since for each $i \in I$, A_i is lower semicontinuous, it follows from Proposition 2.6 in [[19], pp. 366] that coA_i is lower semicontinuous. By Lemma 2.1, A_{V_i} and F_{V_i} have open graphs in $X \times X_i$ and these maps are convex valued. For each $i \in I$, let $\text{H}_{V_i} = \{(x, y) \in X\times X : x_i \in \overline{B_{V_i}}(x) \text{ and } y_i \in \overline{F_{V_i}}(x)\}$, then each H_{V_i} is a closed subset of $X\times X$. For each $i \in I$, define multivalued map $Q_{V_i} : X\times X \to 2^{X_i\times X_i}$ by

$$Q_{V_i}(x, y) = \begin{cases} \{A_i(x) \cap P_i(x, y)\} \times F_i(x) & \text{if} \quad (x, y) \in \text{H}_{V_i} \\ A_i(x) \times F_i(x) & \text{if} \quad (x, y) \notin \text{H}_{V_i}. \end{cases}$$

Then the set $\{(x, y) \in X \times X : Q_{V_i}(x, y) \neq \emptyset\}$ is open. Indeed,

$$\{(x, y) \in X\times X : Q_{V_i}(x, y) \neq \emptyset\} = \{\text{H}_{V_i}\cap \text{G}_i\}\cup(X\times X\backslash \text{H}_{V_i}) = (X\times X\backslash \text{H}_{V_i})\cup \text{G}_i.$$

Furthermore, $\{(x, y) \in X \times X : Q_{V_i}(x, y) \neq \emptyset\}$ is paracompact since each open subset of a perfectly normal and paracompact set is paracompact [10].

In order to apply Theorem 2.1, we show that Q_{V_i} is L-majorized for each $i \in I$.

For each $i \in I$, let $(x, y) \in X \times X$ be a point. We consider the following two cases.

Case I. Suppose that $(x, y) \notin \text{H}_{V_i}$. Let $\Phi_{(x,y)}(z_1, z_2) = A_{V_i}(z_1) \times F_{V_i}(z_1)$ for all $(z_1, z_2) \in X \times X$ and $N_{(x,y)} = X \times X \setminus \text{H}_{V_i}$, then $N_{(x,y)}$ is an open neighbourhood of (x, y) in $X \times X$ such that

(a) for all $(z_1, z_2) \in N_{(x,y)}$, $Q_{V_i}(z_1, z_2) \subseteq A_{V_i}(z_1) \times F_{V_i}(z_1) = \Phi_{(x,y)}(z_1, z_2)$ and by (iii) $\pi_i(z_1, z_2) \notin \Phi_{(x,y)}(z_1, z_2) = \text{co}\Phi_{(x,y)}(z_1, z_2)$;

(b) $\text{co}\Phi_{(x,y)}(z_1, z_2) \subseteq X_i \times X_i$ for all $(z_i, z_2) \in X_i \times X_i$ also by (iii); and

(c) $\Phi^{-1}_{(x,y)}(r_i, s_i) = \{A^{-1}_{V_i}(r_i) \cap F^{-1}_{V_i}(s_i)\} \times X$ is open in $X \times X$ since A_{V_i} and F_{V_i} have open graphs.

Therefore, $\Phi_{(x,y)}$ is an L-majorant of Q_{V_i} at (x, y).

Case II. Suppose that $(x, y) \in \text{H}_{V_i}$. Since for each $i \in I$, $Q_{V_i}(x, y) = \{A_i(x) \cap P_i(x, y)\} \times F_i(x) \neq \emptyset$ and $Q_i \equiv P_i \cap A_i$ is L^*-majorized, there exist an open neighborhood $N_{(x,y)}$ of (x, y) in $X \times X$ and a correspondence $\Psi_{(x,y)} : X \times X \to 2^{X_i}$ such that

(1) for all $(z, w) \in N_{(x,y)}$, $Q_i(z, w) \subseteq \Psi_{(x,y)}(z, w)$ and $\pi_i(z, w) = z_i \notin \text{co}\Psi_{(x,y)}(z, w)$;

(2) for all $(z, w) \in X \times X$, $\text{co}\Psi_{(x,y)}(z, w) \subseteq X_i$; and
(3) for all $r_i \in X_i$, $\Psi^{-1}_{(x,y)}(r_i)$ is compactly open in $X \times X$.

Define a multivalued map $\Phi_{(x,y)} : X \times X \to 2^{X_i \times X_i}$ by

$$\Phi_{(x,y)}(z, w) = \begin{cases} \{A_{V_i}(x) \cap \Psi_{(x,y)}(z, w)\} \times F_{V_i}(x), & \text{if} \quad (z, w) \in \mathrm{H}_{V_i} \\ A_{V_i}(x) \times F_{V_i}(x), & \text{if} \quad (z, w) \notin \mathrm{H}_{V_i} \end{cases}$$

Then we observe that

(aa) for all $(z, w) \in N_{(x,y)}$, $Q_{V_i}(z, w) \subseteq \Phi_{(x,y)}(z, w)$ by (1) and by the definitions of Q_{V_i} and $\Phi_{(x,y)}$;
(bb) for all $\pi_i(z, w) \in N_{(x,y)}$, $(z_i, w_i) \notin \text{co}\Phi_{(x,y)}(z, w)$ by (1) and by the definition of H_{V_i};
(cc) for all $(z, w) \in X \times X$, $\text{co}\Phi_{(x,y)}(z, w) \subseteq X_i \times X_i$; and
(dd) for all $(r_i, s_i) \in X_i \times X_i$,

$$\Phi^{-1}_{(x,y)}(r_i, s_i) = [\{A^{-1}_{V_i}(r_i) \cap F^{-1}_{V_i}(s_i)\} \times X] \cap [\{\mathrm{H}_{V_i} \cap \Psi^{-1}_{(x,y)}(r_i)\} \cup \{X \times X \setminus \mathrm{H}_{V_i}\}]$$

$$= [\{A^{-1}_{V_i}(r_i) \cap F^{-1}_{V_i}(s_i)\} \times X] \cap [\{X \times X \setminus \mathrm{H}_{V_i}\} \cap \Psi^{-1}_{(x,y)}(r_i)]$$

is compactly open in $X \times X$ since H_{V_i} is closed in $X \times X$, $\Psi_{(x,y)}$ has compactly open lower sections and A_{V_i} and F_{V_i} have open graphs. Therefore, $\Phi_{(x,y)}$ is an L-majorant of Q_{V_i} at (x, y). Hence, Q_{V_i} is L-majorized.

By (v), for all $(\tilde{x}, \tilde{y}) \in X \times X \setminus K \times M$, there exists $j \in I$ such that $Q_{V_j}(\tilde{x}, \tilde{y}) \cap \{C_j \times D_j\} \neq \emptyset$. By Theorem 2.1, there exists $(x_V, y_V) \in K \times M$ such that for each $i \in I$, $Q_{V_i}(x_V, y_V) = \emptyset$. Since for each $i \in I$ and for all $x \in X$, $A_i(x) \neq \emptyset$ and $F_i(x) \neq \emptyset$, $\pi_i(x_V) \in \overline{B_{V_i}}(x_V)$, $\pi_i(y_V) \in \overline{F_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V, y_V) = \emptyset$. Applying Lemma 2.2 and following the same argument as in the proof of Theorem 4.3 in [14], we obtain the conclusion. □

From Theorem 3.1, we derive the following equilibrium existence result due to Kim and Yuan [14] for an abstract economy.

Corollary 3.1. [14] *Let $(X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that $X = \prod_{i\in I} X_i$ is paracompact and perfectly normal and for each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i. For each $i \in I$, assume that*

(i) A_i, $B_i : X \to 2^{X_i}$ *are multivalued maps such that* A_i *is lower semicontinuous on* X *and for all* $x \in X$, $A_i(x) \neq \emptyset$ *and* $\text{co}A_i(x) \subseteq B_i(x)$,
(ii) the multivalued map $Q_i : X \to 2^{X_i}$ *defined as* $Q_i(x) = A_i(x) \cap P_i(x)$ *is* L*-majorized,*
(iii) $\mathrm{G}_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ *is an open subset of* X*, and*

(iv) there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ with the property that for each $\tilde{x} \in X \setminus K$, there exists $j \in I$ such that $A_j(\tilde{x}) \cap P_j(\tilde{x}) \cap C_j \neq \emptyset$.

Then there exists an equilibrium point $\hat{x} \in K$ for abstract economy, that is, for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

Theorem 3.2. *Let $(X_i, A_i, B_i, P_i, F_i)_{i\in I}$ be a generalized abstract economy such that $X = \prod_{i\in I} X_i$ is paracompact and $X \times X$ is perfectly normal and paracompact and for each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i. For each $i \in I$, assume that*

(i) A_i, $B_i : X \to 2^{X_i}$ are multivalued maps such that A_i has open lower sections and for all $x \in X$ $A_i(x) \neq \emptyset$ and $\mathrm{co}A_i(x) \subseteq B_i(x)$,

(ii) $F_i : X \to 2^{X_i}$ is a lower semicontinuous multivalued map with nonempty convex values,

(iii) $P_i : X \times X \to 2^{X_i}$ is a multivalued map with convex values and has open lower sections such that $x_i \notin P_i(x, y)$ for all $(x, y) \in X \times X$, and

(iv) there exist two nonempty compact subsets K and M of X and two nonempty compact convex subsets C_i and D_i of X_i for each $i \in I$ with the property that for each $(\tilde{x}, \tilde{y}) \in X \times X \setminus K \times M$, there exists $j \in I$ such that $A_j(\tilde{x}) \cap P_j(\tilde{x}, \tilde{y}) \cap C_j \neq \emptyset$ and $F_j(\tilde{x}) \cap D_j \neq \emptyset$.

Then there exists an equilibrium point $(\hat{x}, \hat{y}) \in K \times M$ of generalized abstract economy.

Proof. Since for each $i \in I$, A_i has open lower sections, we have A_i is lower semicontinuous. Clearly, for each $i \in I$ and for all $(x, y) \in X \times X$, we have (a) $A_i(x) \cap P_i(x, y) \subseteq P_i(x, y)$; (b) $x_i \notin \mathrm{co}P_i(x, y) = P_i(x, y)$; (c) $\mathrm{co}P_i(x, y) \subseteq X_i$; (d) P_i has open lower sections. Therefore, $A_i \cap P_i$ is L^*-majorized. Furthermore, $(A_i \cap P_i)$ have open lower sections since A_i and P_i have open lower sections. Therefore, $A_i \cap P_i$ is lower semicontinuous and so that $\{(x, y) \in X \times X : A_i(x) \cap P_i(x, y) \neq \emptyset\}$ is an open subset of $X \times X$. Then all the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists an equilibrium point $(\hat{x}, \hat{y}) \in K \times M$ of generalized abstract economy. □

Corollary 3.2. *Let $(X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that $X = \prod_{i\in I} X_i$ is paracompact and perfectly normal and for each $i \in I$, X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i. For each $i \in I$, assume that*

(i) A_i, $B_i : X \to 2^{X_i}$ are multivalued maps such that A_i has open lower sections and for all $x \in X$, $A_i(x) \neq \emptyset$ and $coA_i(x) \subseteq B_i(x)$,

(ii) $P_i : X \to 2^{X_i}$ is a multivalued map with convex values and has open lower sections such that for all $x \in X$, $x_i \notin P_i(x)$, and

(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ with the property that for each $\tilde{x} \in X \setminus K$, there exists $j \in I$ such that $A_j(\tilde{x}) \cap P_j(\tilde{x}) \cap C_j \neq \emptyset$.*

Then there exists an equilibrium point $\hat{x} \in K$ for abstract economy, that is, for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

Proof. By (i) and (ii), $A_i \cap P_i$ is L-majorized and $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is an open subset in X. By Corollary 3.1, there exists $\hat{x} \in K$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. □

If for each $i \in I$, E_i is not necessarily Hausdorff, then we have the following results.

Theorem 3.3. *Let $(X_i, A_i, B_i, P_i, F_i)_{i \in I}$ be a generalized abstract economy such that for each $i \in I$, $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space E_i and $X = \prod_{i \in I} X_i$. For each $i \in I$, assume that the conditions (i) - (iv) of Theorem 3.1 and the following condition hold.*

(v)′ *For each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ which is escaping from $X \times X$ relative to $\{C_n \times C_n\}_{n=1}^{\infty}$ where $C_n = \prod_{i \in I} C_{i,n}$ for each $n \in \mathbb{N}$, there exist $m \in N$ and $(u_m, v_m) \in C_m \times C_m$ such that for each $i \in I$, $\pi_i(u_m, v_m) \in \{A_i(x_m) \cap P_i(x_m, y_m)\} \times F_i(x_m)$.*

Then there exists an equilibrium point $(\hat{x}, \hat{y}) \in X \times X$ of generalized abstract economy.

Proof. Let V, X_i, A_{V_i}, B_{V_i}, F_{V_i}, H_{V_i} and Q_{V_i} be the same as defined in the proof of Theorem 3.1. Then H_{V_i} is closed. Following the same argument as in Theorem 3.1, it is easy to see that Q_{V_i} is L-majorized and $\{(x,y) \in X \times X : Q_{V_i}(x,y) \neq \emptyset\}$ is open. By (v)′, for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ which is escaping from $X \times X$ relative to $\{C_n \times C_n\}_{n=1}^{\infty}$ where $C_n = \prod_{i \in I} C_{i,n}$ for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $(u_m, v_m) \in C_m$ such that for each $i \in I$, $\pi_i(x_m, y_m) \in Q_{V_i}(x_m, y_m)$. By Theorem 2.2, there exists $(x_V, y_V) \in X \times X$ such that for each $i \in I$, $Q_{V_i}(x_V, y_V) = \emptyset$. Applying Lemma 2.2 and following the same argument as in the last part of Theorem 4.3 in [29], we have the conclusion. □

From above theorem, we derive the following equilibrium existence result for abstract economy.

Corollary 3.3 [29] *Let $(X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$, $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space E_i and $X = \prod_{i \in I} X_i$. For each $i \in I$, assume that the conditions (i) - (iii) of Corollary 3.1 and the following condition hold.*

(iv)′ *For each sequence* $\{x_n\}_{n=1}^{\infty}$ *in* X *which is escaping from* X *relative to* $\{C_n\}_{n=1}^{\infty}$ *where* $C_n = \prod_{i\in I} C_{i,n}$ *for each* $n \in \mathbb{N}$*, there exist* $m \in \mathbb{N}$ *and* $u_m \in C_m$ *such that for each* $i \in I$*,* $\pi_i(u_m) \in A_i(x_m) \cap P_i(x_m)$*.*

Then there exists an equilibrium point $\hat{x} \in X$ *for abstract economy, that is, for each* $i \in I$*,* $\hat{x}_i \in \overline{B_i}(\hat{x})$ *and* $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$*.*

Theorem 3.4. *Let* $(X_i, A_i, B_i, P_i, F_i)_{i\in I}$ *be a generalized abstract economy such that for each* $i \in I$*,* $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ *where* $\{C_{i,j}\}_{j=1}^{\infty}$ *is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space* E_i *and* $X = \prod_{i\in I} X_i$*. For each* $i \in I$*, assume that the conditions (i) - (iii) of Theorem 3.2 and the following condition hold.*

(iv)′ *For each sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *in* $X \times X$ *which is escaping from* $X \times X$ *relative to* $\{C_n \times C_n\}_{n=1}^{\infty}$ *where* $C_n = \prod_{i\in I} C_{i,n}$ *for each* $n \in \mathbb{N}$*, there exist* $m \in \mathbb{N}$ *and* $(u_m, v_m) \in C_m$ *such that for each* $i \in I$*,* $\pi_i(u_m, v_m) \in \{A_i(x_m) \cap P_i(x_m, y_m)\} \times F_i(x_m)$*.*

Then there exists an equilibrium point $(\hat{x}, \hat{y}) \in X \times X$ *of generalized abstract economy.*

Proof. Applying Theorem 3.3 and following the same argument as in the proof of Theorem 3.2, we obtain the conclusion. □

Corollary 3.4. *For each* $i \in I$*, let* X_i *be a nonempty compact convex subset of a locally convex topological vector space* E_i *and* $X = \prod_{j\in I} X_j$*. For each* $i \in I$*, let* A_i*,* $B_i : X \to 2^{X_i}$ *be multivalued maps such that* A_i *is lower semicontinuous and for all* $x \in X$*,* $A_i(x) \neq \emptyset$ *and* $\mathrm{co}A_i(x) \subseteq B_i(x)$*. Then there exists* $\hat{x} \in X$ *such that* $\hat{x}_i \in \overline{B_i}(\hat{x})$ *for each* $i \in I$*.*

Proof. For each $i \in I$, let $P_i : X \times X \to 2^{X_i}$, $F_i : X \to 2^{X_i}$ be defined by $P_i(x, y) = \emptyset$ and $F_i(x) = X_i$ for each $(x, y) \in X \times X$. Then the conclusion followed from Theorem 3.3. □

Remark 3.1. Theorems 3.1, 3.2 and Theorems 3.3, 3.4 are extensions of Theorem 4.3 in [14] and Theorem 4.3 in [29], respectively, for generalized abstract economy. Therefore, the results of this section improve and generalize the results of [5, 9, 12, 13, 21]; See also references in [14, 29].

4. System of Quasi-Minimax Inequalities

For each $i \in I$, let X_i be a nonempty subset of a topological vector space E_i and $X = \prod_{i\in I} X_i$. For each $i \in I$, let A_i, $B_i, F_i : X \to 2^{X_i}$ be multivalued maps with nonempty values and $f_i : X \times X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ be a function.

We consider the following *system of quasi-minimax inequalities.*

$$\text{(SQMI)}\quad \begin{cases} \text{Find } (\hat{x},\hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{B_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x},\hat{y},z_i) \leq f_i(\hat{x},\hat{y},\hat{x}_i) \text{ for all } z_i \in A_i(\hat{x}). \end{cases}$$

In the case where for each $i \in I$ and for all $x \in X$, $\overline{B_i}(x) = B_i(x)$, $F_i(x) = X_i$ and f_i is a function of two variables x and z_i, (SQMI) is considered and studied by Lin et al [17].

Of course, if for each $i \in I$ and for all $x = (x_i)_{i\in I} \in X$, $y \in X$, $f_i(x,y,x_i) = 0$, then (SQMI) reduces to the following *system of generalized implicit quasi-variational inequalities.*

$$\text{(SGIQVI)}\quad \begin{cases} \text{Find } (\hat{x},\hat{y}) \in X \times X \text{ such that for each } i \in I, \\ \hat{x}_i \in \overline{B_i}(\hat{x}),\ \hat{y}_i \in \overline{F_i}(\hat{x}) \text{ and } f_i(\hat{x},\hat{y},z_i) \leq 0 \text{ for all } z_i \in A_i(\hat{x}). \end{cases}$$

For each $i \in I$ and for all $x \in X$, $A_i(x) = B_i(x) = X_i$, the weak formulation of (SGIQVI) is considered and studied by Ansari and Yao [2] with applications to Nash equilibrium problem [20].

If for each $i \in I$ and for all $x \in X$, $F_i(x) = X_i$ and f_i is independent of the second variable y, that is, f_i is a function of two variables x and z_i, then (SGIQVI) becomes the following *system of quasi-equilibrium problems.*

$$\text{(SQEP)}\quad \begin{cases} \text{Find } \hat{x} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{B_i}(\hat{x}) \text{ and} \\ f_i(\hat{x},z_i) \leq 0 \text{ for all } z_i \in A_i(\hat{x}). \end{cases}$$

For each $i \in I$, let $\varphi_i : X \to \mathbb{R}$ be a function and let $X^i = \prod_{j\in I, j\neq i} X_j$. We write $X = X^i \times X_i$. For each $x = (x_i)_{i\in I} \in X$, let $x^i = (x_j)_{j\in I, j\neq i}$, we write $x = (x^i, x_i)$. If for each $i \in I$, $f_i(x,z_i) = \varphi_i(x^i,x_i) - \varphi_i(x^i,z_i)$ for all $(x,z_i) \in X \times X_i$, then (SQEP) reduces to the following *Debreu type equilibrium problem* [8] (also known as constrained Nash equilibrium problem).

$$\text{(Debreu EP)}\quad \begin{cases} \text{Find } \hat{x} \in X \text{ such that for each } i \in I, \hat{x}_i \in \overline{B_i}(\hat{x}) \text{ and} \\ \varphi_i(\hat{x}^i,\hat{x}_i) - \varphi_i(\hat{x}^i,z_i) \leq 0 \ \text{ for all } z_i \in A_i(\hat{x}). \end{cases}$$

It is easy to see that the Ky Fan minimax inequality [11] (see also [25, 22, 28]) is also a particular case of our (SQMI).

From the above special cases, it is clear that our (SQMI) is more general and unifying model of several problems.

In this section, we apply the results of previous section to establish the existence of solutions of (SQMI). It is easy to derive the existence results for solutions of above particular cases of (SQMI).

Theorem 4.1. *For each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i. Let $X = \prod_{i\in I} X_i$ be paracompact and $X \times X$ be perfectly normal and paracompact. For each $i \in I$, let $f_i : X \times X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that for each fixed $z_i \in X_i$, $(x,y) \mapsto f_i(x,y,z_i)$ is continuous on $X \times X$. For each $i \in I$, assume that* *(i)* $A_i,\ B_i : X \to 2^{X_i}$ *are multivalued maps such that* A_i *has open lower sections and for all* $x \in X$, $A_i(x) \neq \emptyset$ *and* $\mathrm{co}A_i(x) \subseteq B_i(x)$,

(ii) $F_i : X \to 2^{X_i}$ *is a lower semicontinuous multivalued maps with nonempty convex values,*

(iii) for each $N \in \mathrm{F}(X)$ *and for all* $y \in X$, $x = (x_i)_{i\in I} \in \mathrm{co}N$, $\min\{f_i(x,y,z_i) - f_i(x,y,x_i) : z_i \in N_i\} \leq 0$, *where* N_i *is the projection of* N *on* X_i*; and*

(iv) there exist nonempty compact subsets K *and* M *of* X *and nonempty compact convex subsets* C_i *and* D_i *of* X_i *for each* $i \in I$ *with the property that for each* $(x,y) \in X \times X \setminus K \times M$, *there exist* $j \in I$, $u_j \in C_j$ *and* $v_j \in D_j$ *such that* $u_j \in A_j(x)$, $v_j \in F_j(x)$ *and* $f_j(x,y,u_j) > f_j(x,y,x_j)$.

Then there exists a solution $(\hat{x},\hat{y}) \in K \times M$ *of (SQMI).*

Proof. For each $i \in I$, let $P_i,\ P_i' : X \times X \to 2^{X_i}$ be defined by $P_i(x,y) = \{z_i \in X_i : f_i(x,y,z_i) > f_i(x,y,x_i)\}$ and $P_i'(x,y) = \mathrm{co}P_i(x,y)$. Clearly, P_i' has convex values. Furthermore, for all $z_i \in X_i$ we have

$$P_i^{-1}(z_i) = \{(x,y) \in X \times X : f_i(x,y,z_i) > f_i(x,y,x_i)\} \text{ is open in } X \times X$$

since the complement of $P_i^{-1}(z_i)$ in $X \times X$,

$$[P_i^{-1}(z_i)]^c = \{(x,y) \in X \times X : f_i(x,y,z_i) - f_i(x,y,x_i) \leq 0\} \text{ is closed in } X \times X$$

in view of the condition that $(x,y) \mapsto f_i(x,y,z_i)$ is continuous on $X \times X$. Then by Lemma 5.1 in [27], P_i' has open lower sections.

Suppose that there exist $j \in I$ and $(x,y) \in X \times X$ such that $x_j \in \mathrm{co}P_j(x,y) = P_j'(x,y)$. Then there exists $\{z_{j_1}, z_{j_2}, \dots, z_{j_m}\} \subseteq P_j(x,y)$ such that $x_j \in \mathrm{co}\{z_{j_1}, z_{j_2}, \dots, z_{j_m}\}$ and therefore $f_j(x,y,z_{j_k}) - f_j(x,y,x_j) > 0$ for each $k = 1,2,\dots,m$. For each $k = 1,2,\dots,m$, take any $z_k \in X$ such that $\pi_j(z_k) = z_{j_k}$ and $x = (x_i)_{i\in I} \in \mathrm{co}\{z_1, z_2, \dots, z_m\}$, we get $\min_{1\leq k\leq m}\{f_j(x,y,z_{j_k}) - f_j(x,y,x_j)\} \leq 0$ by (iii), a contradiction. Therefore, for each $i \in I$ and for all $(x,y) \in X \times X$, $x_i \notin P_i'(x,y)$.

By (iv), for all $(x,y) \in X \times X \setminus K \times M$, there exists $j \in I$ such that $A_j(x) \cap P_j'(x,y) \cap C_j \neq \emptyset$ and $F_j(x) \cap D_j \neq \emptyset$. Then by Theorem 3.2, there exists $(\hat{x},\hat{y}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i'(\hat{x},\hat{y}) = \emptyset$, that is, $f_i(\hat{x},\hat{y},z_i) \leq f_i(\hat{x},\hat{y},\hat{x}_i)$ for all $z_i \in A_i(\hat{x})$. □

The following result can be easily derived from Theorem 4.1.

Corollary 4.1. *For each* $i \in I$, *let* X_i *be a nonempty convex subset of a locally convex Hausdorff topological vector space* E_i. *Let* $X = \prod_{i\in I} X_i$ *be paracompact and* $X \times X$ *be perfectly normal and paracompact. For each* $i \in I$, *let* $f_i : X \times X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ *be a function such that for each fixed* $z_i \in X_i$, $(x,y) \mapsto f_i(x,y,z_i)$ *is lower semicontinuous on* $X \times X$. *For each* $i \in I$, *assume that the conditions (i)-(ii) of Theorem 4.1 and the following conditions hold.*

(iii)′ *For each $N \in \mathrm{F}(X)$ and for all $y \in X$, $x = (x_i)_{i\in I} \in \mathrm{co}N$, $\min\{f_i(x, y, z_i)-a_i) : z_i \in N_i\} \leq 0$, where N_i is the projection of N on X_i and each a_i is any real number.*

(iv)′ *There exist nonempty compact subsets K and M of X and nonempty compact convex subsets C_i and D_i of X_i for each $i \in I$ with the property that for each $(x, y) \in X \times X \setminus K \times M$, there exist $j \in I$, $u_j \in C_j$ and $v_j \in D_j$ such that $u_j \in A_j(x)$, $v_j \in F_j(x)$ and $f_j(x, y, u_j) > a_j$.*

Then there exists $(\hat{x}, \hat{y}) \in K \times M$ such that for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ and $f_i(\hat{x}, \hat{y}, z_i) \leq a_i$ for all $z_i \in A_i(\hat{x})$.

If for each $i \in I$, $F_i(x) = X_i$ and f_i is independent of the second variable, that is, f_i is a function of x and z_i, then from Corollary 4.1, we get the following existence result for a solution of (SQEP).

Corollary 4.2. *For each $i \in I$, let X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E_i and $X = \prod_{i\in I} X_i$ be paracompact and perfectly normal. For each $i \in I$, let f_i, $g_i : X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ be functions such that for all $(x, z_i) \in X \times X_i$, $f_i(x, z_i) \leq g_i(x, z_i)$ and let A_i, $B_i : X \to 2^{X_i}$ be multivalued maps. For each $i \in I$, assume that*

(i) *for all $x \in X$, $A_i(x) \neq \emptyset$ and $\mathrm{co}A_i(x) \subseteq B_i(x)$ and A_i has open lower sections,*

(ii) *for all $z_i \in X_i$, $x \mapsto f_i(x, z_i)$ is lower semicontinuous on X,*

(iii) *for all $N \in \mathrm{F}(X)$ and for all $x \in \mathrm{co}N$, $\min\{g_i(x, z_i) : z_i \in N_i\} \leq 0$, where N_i is the projection of N on X_i, and*

(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ with the property that for each $x \in X \setminus K$, there exist $j \in I$ and $u_j \in C_j$ such that $u_j \in A_j(x)$ and $f_j(x, u_j) > 0$.*

Then there exists a solution $\hat{x} \in K$ for (SQEP).

As an application of Theorem 3.4, we have the following result.

Theorem 4.2. *For each $i \in I$, let $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ where $\{C_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space E_i. For each $i \in I$, let $f_i : X \times X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that for each fixed $z_i \in X_i$, $(x, y) \mapsto f_i(x, y, z_i)$ is continuous on $X \times X$. For each $i \in I$, assume that the conditions (i)-(iii) of Theorem 4.1 and the following condition hold.*

(iv)′ *For each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ which is escaping from $X \times X$ relative to $\{C_n \times C_n\}_{n=1}^{\infty}$ where $C_n = \prod_{i\in I} C_{i,n}$ for each $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $(u_m, v_m) \in C_m$ such that for each $i \in I$, $\pi_i(u_m) \in A_i(x_m)$, $\pi_i(v_m) \in F_i(x_m)$, and $f_i(x_m, y_m, \pi_i(u_m)) > f_i(x_m, y_m, \pi_i(x_m))$.*

Then there exists a solution $(\hat{x}, \hat{y}) \in X \times X$ *for (SQMI).*

Proof. Applying Theorem 3.4 and following the same argument as in the proof of Theorem 4.1, we obtain the conclusion. □

The following result can be easily derived from Theorem 4.2.

Corollary 4.3. *For each* $i \in I$, *let* $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ *where* $\{C_{i,j}\}_{j=1}^{\infty}$ *is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space* E_i. *For each* $i \in I$, *let* $f_i : X \times X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ *be a function such that for each fixed each* $z_i \in X_i$, $(x, y) \mapsto f_i(x, y, z_i)$ *is lower semicontinuous on* $X \times X$. *For each* $i \in I$, *assume that the conditions (i)-(iii) of Corollary 4.1 and the following conditions hold.*

- (iv)′ *For each sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *in* $X \times X$ *which is escaping from* $X \times X$ *relative to* $\{C_n \times C_n\}_{n=1}^{\infty}$ *where* $C_n = \prod_{i \in I} C_{i,n}$ *for each* $n \in \mathbb{N}$, *there exist* $m \in \mathbb{N}$ *and* $(u_m, v_m) \in C_m$ *such that for each* $i \in I$, $\pi_i(u_m) \in A_i(x_m)$, $\pi_i(v_m) \in F_i(x_m)$ *and* $f_i(x_m, y_m, \pi_i(u_m)) > a_i$.

Then there exists $(\hat{x}, \hat{y}) \in X \times X$ *such that for each* $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$, $\hat{y}_i \in \overline{F_i}(\hat{x})$ *and* $f_i(\hat{x}, \hat{y}, z_i) \leq a_i$ *for all* $z_i \in A_i(\hat{x})$.

If for each $i \in I$, $F_i(x) = X_i$ and f_i is independent of the second variable, that is, f_i is a function of x and z_i, then from Corollary 4.3, we get the following result.

Corollary 4.4. *For each* $i \in I$, *let* $X_i = \bigcup_{j=1}^{\infty} C_{i,j}$ *where* $\{C_{i,j}\}_{j=1}^{\infty}$ *is an increasing sequence of nonempty compact convex subsets of a locally convex topological vector space* E_i. *For each* $i \in I$, *let* f_i, $g_i : X \times X_i \to \mathbb{R} \cup \{\pm\infty\}$ *be functions such that for all* $(x, z_i) \in X \times X_i$, $f_i(x, z_i) \leq g_i(x, z_i)$ *and let* A_i, $B_i : X \to 2^{X_i}$ *be multivalued maps. For each* $i \in I$, *assume that the conditions (i)-(iii) of Corollary 4.2 and the following condition hold.*

- (iv)′ *For each sequence* $\{x_n\}_{n=1}^{\infty}$ *in* X *which is escaping from* $X \times X$ *relative to* $\{C_n\}_{n=1}^{\infty}$ *where* $C_n = \prod_{i \in I} C_{i,n}$ *for each* $n \in \mathbb{N}$, *there exist* $m \in \mathbb{N}$ *and* $u_m \in C_m$ *such that for each* $i \in I$, $\pi_i(u_m) \in A_i(x_m)$ *and* $f_i(x_m, \pi_i(u_m)) > a_i$.

Then there exists a solution $\hat{x} \in X$ *for (SQEP).*

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