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General iterative algorithms for solving mixed quasi-variational-like inclusions[☆]

Lu-Chuan Zeng [a], Qamrul Hasan Ansari [b,c,*], Jen-Chih Yao [d]

[a] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*

[b] *Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, P.O. Box 1169, Dhahran 31261, Saudi Arabia*

[c] *Department of Mathematics, Aligarh Muslim University, Aligarh, India*

[d] *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 804, Taiwan*

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ABSTRACT

In this paper, we extend the auxiliary variational inequality technique due to Ding and Yao [X.P. Ding, J.C. Yao, Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces, Comput. Math. Appl. 49 (2005) 857–869] to develop iterative algorithms for finding the approximate solutions of a mixed quasi-variational-like inclusion problem (in short, MQVLIP) in the setting of Banach spaces. We first establish a result on the existence of a solution of the equilibrium problem by virtue of the Fan–KKM lemma. Then by using this result and a result by Ding and Tan [X.P. Ding, K.K. Tan, A minimax inequality with applications to existence of equilibrium point and fixed point theorems, Colloq. Math. 63 (2) (1992) 233–247], we derive the existence of a unique solution of MQVLIP and the existence of approximate solutions generated by the proposed algorithms. Moreover, we also provide the new criteria for convergence of approximate solutions to the exact solution of MQVLIP.

1. Introduction

It is well known that variational inequalities were initially studied by Stampacchia (see [3]) and ever since have been widely generalized and applied in diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance. Up until now variational inequalities have been very effective and powerful tools of the current mathematical technology; see for example [3–6] and references therein. In variational inequality theory, one of the most important and interesting problems is the development of an efficient and implementable algorithm for solving variational inequality and its generalizations. The method based on the auxiliary principle technique was suggested by Glowinski et al. [4] (see also [7]) for solving variational inequalities. Subsequently, it has been used to solve a number of generalizations of classical variational inequalities; see for example [1,8–19] and references therein.

Let B be a real Banach space with its topological dual B^*, D be a nonempty convex subset of B, and $\langle u, v\rangle$ be the duality pairing between $u \in B^*$ and $v \in B$. Let $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, $\eta : D \times D \to B$ be mappings and $w^* \in B^*$. Let

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* Corresponding author at: Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, P.O. Box 1169, Dhahran 31261, Saudi Arabia.

E-mail addresses: zenglc@hotmail.com (L.-C. Zeng), qhansari@kfupm.edu.sa (Q.H. Ansari), yaojc@math.nsysu.edu.tw (J.-C. Yao).

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$\varphi : B \times B \to (-\infty, +\infty]$ be a real bifunction. Recently, Ding and Yao [1] considered and studied the *mixed quasi-variational-like inclusion problem* (in short, MQVLIP) which is to find $u \in D$ such that

$$\langle N(Tu, Au) - w^*, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in D. \tag{1.1}$$

We remark that if $\varphi(v, u) = f(v)$ for all $u, v \in B$ where $f : B \to (-\infty, +\infty]$ is a given function, then MQVLIP reduces to the following *mixed variational-like inequality problem* (in short, MVLIP): find $u \in D$ such that

$$\langle N(Tu, Au) - w^*, \eta(v, u)\rangle + f(v) - f(u) \geq 0, \quad \forall v \in D. \tag{1.2}$$

MVLIP and its special cases have been introduced and studied by Ding [9–12] and Fang and Huang [13] in Banach spaces, and by Lee et al. [15], Ansari and Yao [8], Zeng [17] and Schaible et al. [16] in Hilbert spaces. Moreover, if $w^* = 0$, $N(Tu, Av) = Tu - Av$, $\varphi(v, u) = f(v)$, and $\eta(v, u) = g(v) - g(u)$ for all $v, u \in D$, where $g : D \to B$ is a given mapping, then MQVLIP is equivalent to finding $u \in D$ such that

$$\langle Tu - Au, g(v) - g(u)\rangle + f(v) - f(u) \geq 0, \quad \forall v \in D.$$

This problem was introduced and studied by Yao [20] in Hilbert spaces where some existence theorems and iterative algorithms of solutions were given under suitable conditions. In addition, if $N(Tu, Au) = Tu - Au$ for all $u, v \in D$, then MQVLIP becomes the problem of finding $u \in D$ such that

$$\langle Tu - Au - w^*, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in D. \tag{1.3}$$

Problem (1.3) is called the *strongly nonlinear mixed variational-like inequality problem* considered and studied by Ding and Yao [1]. Ding [9–12] further studied the special cases of MQVLIP in reflexive Banach spaces.

In [1], Ding and Yao suggested an iterative algorithm for computing the approximate solutions of MQVLIP by using the auxiliary variational inequality technique. They also proved the existence of a unique solution to MQVLIP under weak assumptions in reflexive Banach spaces and also provided the convergence criteria of approximate solutions to the exact solution of MQVLIP.

In this paper, the Ding and Yao [1] auxiliary variational inequality technique is extended to suggest the following new iterative algorithm for computing the approximate solutions of MQVLIP.

Algorithm 1.1. Let $\eta : D \times D \to B$ be a mapping, $K : D \to (-\infty, +\infty]$ be a given Fréchet differentiable η-convex functional and $\rho > 0$ be a given positive number.

(i) At $n = 0$, start with some initial $u_0 \in D$;

(ii) At step $n + 1$, for a given iterate u_n, solve the auxiliary variational inequality problem that consists of finding $u_{n+1} \in D$ such that

$$\begin{aligned}\langle K'(u_{n+1}) - K'(u_n), \eta(v, u_{n+1})\rangle \geq{} & -\rho\langle N(Tu_n, Au_n) - w^*, \eta(v, u_{n+1})\rangle \\ & + \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in D.\end{aligned} \tag{1.4}$$

(iii) If, for given $\epsilon > 0$, $\|u_{n+1} - u_n\| \leq \epsilon$, stop. Otherwise, repeat (ii).

First, we establish a result on the existence of a solution of equilibrium problem by virtue of the Fan–KKM Lemma. By using this result and a result due to Ding and Tan [2], we establish the existence of a unique solution of MQVLIP and the existence of approximate solutions generated by Algorithm 1.1 for MQVLIP. Moreover, we also prove the convergence of approximate solutions to the exact solution of MQVLIP under a new range in which ρ takes values. Moreover, we also provide the new criteria for convergence of approximate solutions to the exact solution of MQVLIP. Compared with Theorem 4.1 of Ding and Yao [1], our results improve this result in the following aspects:

(1) Condition (i) in [[1], Theorem 4.1] is replaced by the weaker condition that for each fixed $v \in D$, the functional $u \mapsto \langle N(Tu, Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous. Indeed, conditions (i) and (iii)(b) can imply the weak continuity of $u \mapsto \langle N(Tu, Au) - w^*, \eta(u, v)\rangle$.

(2) The assumption "K' is continuous from the weak topology to the strong topology" in Theorem 4.1 in [1] is replaced by the one that the functional $w \mapsto \langle K'(w), \eta(v, w)\rangle$ is weakly upper semicontinuous on D for each fixed $v \in D$.

(3) Condition (ii) in [[1], Theorem 4.1] is replaced by the condition that T is η-cocoercive with respect to the first argument of $N(\cdot, \cdot)$ with constant $\lambda > 0$.

(4) Our convergence criteria for the approximate solutions are very different from theirs because of the appearance of a new range in which ρ takes values.

(5) The assumption "$N(\cdot, \cdot)$ is η-strongly monotone in the second argument with respect to A" in Theorem 4.1 in [1] is replaced by the one that $N(\cdot, \cdot)$ is η-relaxed monotone in the second argument with respect to A.

(6) Condition (ii) in [[1], Theorem 4.1] is replaced by the condition that $N(\cdot, \cdot)$ is Lipschitz continuous and η-strongly monotone in the first argument with respect to T, respectively.

2. Preliminaries

In this section, we first recall the following definitions and some known results. Then we establish a result on the existence of a solution of equilibrium problem by virtue of the Fan–KKM lemma.

Definition 2.1 (*[1,8]*). Let D be a nonempty subset of a Banach space B with its topological dual B^*. Let $T : D \to B^*$ and $\eta : D \times D \to B$ be two mappings. Then T is called:

(i) *η-cocoercive*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \eta(u, v)\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in D;$$

(ii) *η-monotone*, if

$$\langle Tu - Tv, \eta(u, v)\rangle \geq 0, \quad \forall u, v \in D;$$

(iii) *strictly η-monotone*, if

$$\langle Tu - Tv, \eta(u, v)\rangle > 0, \quad \forall u, v \in D \text{ with } u \neq v;$$

(iv) *η-strongly monotone*, if there exists a constant $\beta > 0$ such that

$$\langle Tu - Tv, \eta(u, v)\rangle \geq \beta\|u - v\|^2, \quad \forall u, v \in D;$$

(v) *η-relaxed monotone*, if there exists a constant $\xi \geq 0$ such that

$$\langle Tu - Tv, \eta(u, v)\rangle \leq -\xi\|u - v\|^2, \quad \forall u, v \in D;$$

(vi) *Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|Tu - Tv\| \leq L\|u - v\|, \quad \forall u, v \in D.$$

If $\eta(u, v) = u - v$ for all $u, v \in D$, then Definitions (i), (ii), (iv) and (v) reduce to the definitions of cocoercivity [21], monotonicity and strong monotonicity [22], and relaxed Lipschitz continuity [23]. It is clear that if T is η-strongly monotone with constant α, then $-T$ is η-relaxed monotone with the same constant.

Definition 2.2. Let $\eta : D \times D \to B$ be a mapping. Then η is called *Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in D.$$

Remark 2.1. The following relationships among η-monotonicity, η-strong monotonicity, η-cocoercivity and Lipschitz continuity hold:

(i) η-strong monotonicity $\Rightarrow$ η-monotonicity $\Leftarrow$ η-cocoercivity;

(ii) $\left.\begin{array}{l} T \text{ is } \eta\text{-strong monotone} \\ T \text{ is Lipschitz continuous} \end{array}\right\} \Rightarrow T$ is η-cocoercive;

(iii) $\left.\begin{array}{l} T \text{ is } \eta\text{-cocoercive} \\ \eta \text{ is Lipschitz continuous} \end{array}\right\} \Rightarrow T$ is Lipschitz continuous.

Definition 2.3. The bifunction $\varphi : B \times B \to (-\infty, +\infty]$ is said to be *skew-symmetric*, if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in B.$$

Remark 2.2. The skew-symmetric bifunctions have properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for a convex function. For the properties and applications of the skew-symmetric bifunction, we refer the reader to [24].

Definition 2.4 (*[8,25]*). Let D be a nonempty convex subset of a Banach space B and $K : D \to (-\infty, +\infty)$ be a Fréchet differentiable function. K is said to be:

(i) *η-convex*, if

$$K(v) - K(u) \geq \langle K'(u), \eta(v, u)\rangle, \quad \forall u, v \in D;$$

(ii) *η-strongly convex*, if there exists a constant $\mu > 0$ such that

$$K(v) - K(u) - \langle K'(u), \eta(v, u)\rangle \geq \frac{\mu}{2}\|u - v\|^2, \quad \forall u, v \in D.$$

In particular, if $\eta(u, v) = u - v$ for all $u, v \in D$, then K is said to be *strongly convex*.

Definition 2.5. Let D be a nonempty subset of a Banach space B with it topological dual B^*, $T, A : D \to B^*$ and $\eta : D \times D \to B$ mappings. The mapping $N : B^* \times B^* \to B^*$ is said to be:

(i) *Lipschitz continuous in the first argument*, if there exists a constant $\sigma_1 > 0$ such that

$$\|N(u, \cdot) - N(v, \cdot)\| \leq \sigma_1 \|u - v\|, \quad \forall u, v \in B^*;$$

Similarly, Lipschitz continuity of $N(\cdot, \cdot)$ in the second argument can be defined;

(ii) *Lipschitz continuous in the first argument with respect to T*, if there exists a constant $\tau > 0$ such that

$$\|N(Tu, \cdot) - N(Tv, \cdot)\| \leq \tau \|u - v\|^2, \quad \forall u, v \in D;$$

Similarly, Lipschitz continuity of $N(\cdot, \cdot)$ in the second argument with respect to A can be defined;

(iii) *η-cocoercive in the first argument with respect to T*, if there exists a constant $\alpha > 0$ such that

$$\langle N(Tu, \cdot) - N(Tv, \cdot), \eta(u, v)\rangle \geq \alpha \|Tu - Tv\|^2, \quad \forall u, v \in D;$$

(iv) *η-monotone in the first argument with respect to T*, if

$$\langle N(Tu, \cdot) - N(Tv, \cdot), \eta(u, v)\rangle \geq 0, \quad \forall u, v \in D;$$

(v) *η-strongly monotone in the first argument with respect to T*, if there exists a constant $\upsilon > 0$ such that

$$\langle N(Tu, \cdot) - N(Tv, \cdot), \eta(u, v)\rangle \geq \upsilon \|u - v\|^2, \quad \forall u, v \in D;$$

(vi) *η-relaxed monotone in the second argument with respect to A*, if there exists a constant $\xi \geq 0$ such that

$$\langle N(\cdot, Au) - N(\cdot, Av), \eta(u, v)\rangle \leq -\xi \|u - v\|^2, \quad \forall u, v \in D.$$

(vii) T is said to be *η-cocoercive with respect to the first argument of* $N(\cdot, \cdot)$, if there exists a constant $\lambda > 0$, such that

$$\langle N(Tu, \cdot) - N(Tv, \cdot), \eta(u, v)\rangle \geq \lambda \|N(Tu, \cdot) - N(Tv, \cdot)\|^2, \quad \forall u, v \in D.$$

Proposition 2.1. *Let K be a Fréchet differentiable η-strongly convex functional with constant $\mu > 0$ on a convex subset D of B, and let $\eta : D \times D \to B$ be a mapping such that $\eta(u, v) + \eta(v, u) = 0$, $\forall u, v \in D$. Then K' is η-strongly monotone with constant $\mu > 0$.*

Proof. Since K is η-strongly convex, we deduce that for each $u, v \in D$

$$K(v) - K(u) - \langle K'(u), \eta(v, u)\rangle \geq \frac{\mu}{2} \|u - v\|^2,$$
$$K(u) - K(v) - \langle K'(v), \eta(u, v)\rangle \geq \frac{\mu}{2} \|v - u\|^2.$$

Adding these two inequalities and using the condition that $\eta(u, v) + \eta(v, u) = 0$, we obtain

$$\langle K'(v) - K'(u), \eta(v, u)\rangle \geq \mu \|v - u\|^2. \quad \square$$

Proposition 2.2. *Let $\eta(v, \cdot) : D \to B$ and K' be continuous from the weak topology to the weak topology and from the weak topology to the strong topology, respectively, where v is any fixed point in D. Then the functional $g : D \to (-\infty, +\infty)$, defined as $g(u) = \langle K'(u), \eta(v, u)\rangle$ for each fixed $v \in D$, is weakly continuous on D.*

Proof. Obvious.

For all $D \subset B$, we denote by co(D) the convex hull of D. A point-to-set mapping $F : D \to 2^B$ is said to be a *KKM mapping* if, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subset D$,

$$\text{co}(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i),$$

where 2^B denotes the family of all nonempty subsets of B. $\square$

Fan–KKM Lemma 2.1 ([26]). *Let D be a nonempty subset of a topological vector space E and let $F : D \to 2^E$ be a KKM mapping. If $F(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$, then $\bigcap_{x \in D} F(x) \neq \emptyset$.*

In order to obtain one of our main results, we establish the following result on the existence of a solution to the *equilibrium problem* which consists of finding $\hat{u} \in D$ such that:

$$\psi(v, \hat{u}) \leq 0, \quad \text{for all } v \in D, \tag{2.1}$$

where ψ is a real bifunction defined on $D \times D$.

Compared with Ding and Yao [[1], Lemma 2.1], the following result is very convenient to use.

Lemma 2.2. *Let D be a nonempty convex subset of a topological vector space E and let $\psi : D \times D \to [-\infty, +\infty]$ be such that*

(i) *for each $v \in D$, $u \to \psi(v, u)$ is lower semicontinuous on each nonempty compact subset of D;*
(ii) *for each finite set $\{v_1, \ldots, v_n\} \subset D$ and for each $u = \sum_{i=1}^n \lambda_i v_i$ $(\lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1)$, $\min_{1\leq i\leq n} \psi(v_i, u) \leq 0$;*
(iii) *there exists a nonempty compact convex subset D_0 of D such that for some $v_0 \in D_0$, there holds:*

$$\psi(v_0, u) > 0, \quad \text{for all } u \in D \setminus D_0.$$

Then equilibrium problem (2.1) has a solution $\hat{u} \in D_0$, that is, $\psi(v, \hat{u}) \leq 0$, for all $v \in D$.

Proof. For any fixed finite subset $A \subset D$, we define the subset $X = \text{co}(A \cup D_0)$ of D. Note that D_0 is a nonempty compact convex subset of D. Hence X is a nonempty compact convex subset of D. Now, we define a point-to-set mapping $G : X \to 2^X$ as follows:

$$G(v) = \{u \in X : \psi(v, u) \leq 0\}, \quad \text{for all } v \in X.$$

From condition (ii) it follows that $v \in G(v)$ for all $v \in X$. We assert that G is a KKM mapping. Indeed, suppose to the contrary, that there exist a finite subset $\{v_1, v_2, \ldots, v_n\}$ of X and a pool $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of nonnegative numbers with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\bar{u} = \sum_{i=1}^{n} \lambda_i v_i \notin G(v_i), \quad \text{for all } i.$$

Then, we have

$$\psi(v_i, \bar{u}) > 0, \quad \text{for all } i,$$

which together with condition (ii), implies that

$$0 \geq \min_{1\leq i\leq n} \psi(v_i, \bar{u}) > 0.$$

This leads to a contradiction. Thus, G is a KKM mapping.

Since condition (i) implies that G is lower semicontinuous on a compact subset $X \subset D$, $G(v)$ is a closed subset of X for each $v \in X$. So $G(v)$ is compact for each $v \in X$. Thus, by Fan–KKM Lemma 2.1, we derive

$$\bigcap_{v\in X} G(v) \neq \emptyset.$$

Let $\bar{u} \in \bigcap_{v\in X} G(v)$. Then, by using condition (iii) we can see that $\bar{u} \in G(v_0) \subset D_0$. Meanwhile, it is also easy to see that $\bar{u} \in \bigcap_{v\in A}\{u \in D_0 : \psi(v, u) \leq 0\}$, and hence,

$$\bigcap_{v\in A}\{u \in D_0 : \psi(v, u) \leq 0\} \neq \emptyset.$$

Now, we also define a point-to-set mapping $F : D \to 2^{D_0}$ as follows:

$$F(v) = \{u \in D_0 : \psi(v, u) \leq 0\}, \quad \text{for all } v \in D.$$

Since D_0 is compact, it follows from condition (i) that $F(v)$ is compact for all $v \in D$. Consequently, we conclude that

$$\bigcap_{v\in D} F(v) \neq \emptyset.$$

This shows that there exists $\hat{u} \in D_0$ such that

$$\psi(v, \hat{u}) \leq 0, \quad \text{for all } v \in D,$$

that is, equilibrium problem has a solution $\hat{u} \in D_0$. □

Remark 2.3. Lemma 2.2 is proved by Ding and Tan [2] with the following condition in place of condition (iii).

(iii)′ there exist a nonempty compact convex subset D_0 of D and a nonempty compact subset $\mathcal{K}$ of D such that for each $u \in D \setminus \mathcal{K}$, there is a $v \in \text{co}(D_0 \cup \{u\})$ with $\psi(v, u) > 0$.

3. Main results

In this section, by using Lemma 2.2 we derive the existence and uniqueness of a solution to MQVLIP and the existence of the approximate solutions generated by Algorithm 1.1 for MQVLIP under weaker assumptions in reflexive Banach spaces. Moreover, we also prove the convergence of approximate solutions to the exact solution of MQVLIP under mild conditions.

Theorem 3.1. *Let D be a nonempty closed convex subset of a reflexive Banach space B with the dual space B^*. Let $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$ be mappings. Let $w^* \in B^*$ and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that for each $v \in B$, $\mathrm{int}\{v \in D : \varphi(v, v) < \infty\} \neq \emptyset$ and $\varphi(\cdot, v)$ is proper convex. Suppose that K is η-strongly convex with constant μ and the functional $w \mapsto \langle K'(w), \eta(v, w)\rangle$ is weakly upper semicontinuous on D for each fixed $v \in D$. Suppose also that:*

(i) *for each fixed $v \in D$, the functional $u \mapsto \langle N(Tu, Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous;*
(ii) *T is η-cocoercive with respect to the first argument of $N(\cdot, \cdot)$ with constant $\lambda > 0$;*
(iii) *η is Lipschitz continuous with constant $\delta > 0$ such that;*
  (a) *$\eta(u, v) = \eta(u, z) + \eta(z, v)$ for each $u, v, z \in D$,*
  (b) *for each fixed $v \in D$, $u \to \eta(u, v)$ is continuous from the weak topology to the weak topology,*
  (c) *for each fixed $u, w \in D$, $v \mapsto \langle N(Tu, Au), \eta(w, v)\rangle$ is concave,*
(iv) *$N(\cdot, \cdot)$ is Lipschitz continuous and η-strongly monotone in the second argument with respect to A with constants $\beta > 0$ and $\xi > 0$, respectively.*

Then,

(I) *there exists a unique solution $\hat{u} \in D$ of MQVLIP,*
(II) *for each $\rho > 0$, there exists a unique solution $u_{n+1} \in D$ to problem (1.4),*
(III) *if*

$$0 < \rho < \frac{2\lambda\mu\xi}{\delta^2(\lambda\beta^2 + \xi)}, \tag{3.1}$$

then the sequence $\{u_n\}$ defined by Algorithm 1.1 converges strongly to a unique solution $\hat{u}$ of MQVLIP.

Proof. We divide the proof into three steps.

Step 1. We claim the existence of a unique solution of MQVLIP.

Indeed, we define a function $\varphi : D \times D \to [-\infty, +\infty]$ by

$$\psi(v, u) = \langle N(Tu, Au) - w^*, \eta(u, v)\rangle + \varphi(u, u) - \varphi(v, u).$$

Since $\varphi(\cdot, \cdot)$ is a weakly continuous functional, and since for each fixed $v \in D$ the functional $u \mapsto \langle N(Tu, Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous, for each $v \in D$ the functional $u \mapsto \psi(v, u)$ is weakly lower semicontinuous. This shows that condition (i) in Lemma 2.2 holds. Now we claim that $\psi(v, u)$ satisfies condition (ii) in Lemma 2.2. If this is false, then there exist a finite subset $\{v_1, \ldots, v_m\} \subset D$ and $u = \sum_{i=1}^m \lambda_i v_i$ with $\sum_{i=1}^m \lambda_i = 1$ for some $\lambda_1, \ldots, \lambda_m \geq 0$ such that $\psi(v_i, u) > 0$ for each $i = 1, \ldots, m$, that is, for each $i = 1, \ldots, m$,

$$\langle N(Tu, Au) - w^*, \eta(u, v_i)\rangle + \varphi(u, u) - \varphi(v_i, u) > 0, \quad \text{for all } i = 1, \ldots, m.$$

It follows that

$$\sum_{i=1}^m \lambda_i \langle N(Tu, Au) - w^*, \eta(u, v_i)\rangle + \varphi(u, u) - \sum_{i=1}^m \lambda_i \varphi(v_i, u) > 0.$$

Note that for each $v \in B$, $\varphi(\cdot, v)$ is a convex functional. Hence it follows that

$$\sum_{i=1}^m \lambda_i \langle N(Tu, Au) - w^*, \eta(u, v_i)\rangle > 0.$$

From condition (iii)(a), (c), we have $0 = \langle N(Tu, Au) - w^*, \eta(u, u)\rangle > 0$, a contradiction. Therefore, condition (ii) in Lemma 2.2 holds. Since for each $v \in B$, $u \mapsto \varphi(u, v)$ is a proper convex weakly lower semicontinuous functional and $\mathrm{int}\{v \in D : \varphi(v, v) < \infty\} \neq \emptyset$, we take $v^* \in \mathrm{int}\{v \in D : \varphi(v, v) < \infty\}$. By Proposition I.2.6 of Pascali and Sburlan [[27], p. 27], $\varphi(\cdot, v^*)$ is subdifferentiable at v^*. Hence we have

$$\varphi(u, v^*) - \varphi(v^*, v^*) \geq \langle r, u - v^*\rangle, \quad \forall r \in \partial\varphi(v, v^*)|_{v=v^*}, u \in B. \tag{3.2}$$

Noting that $\varphi(\cdot, \cdot)$ is skew-symmetric, we have

$$\varphi(u, u) - \varphi(v^*, u) \geq \varphi(u, v^*) - \varphi(v^*, v^*) \geq \langle r, u - v^*\rangle, \quad \forall r \in \partial\varphi(v, v^*)|_{v=v^*}, u \in B. \tag{3.3}$$

By using conditions (ii) and (iv), and equality $\eta(u, v) = -\eta(v, u)$, we have

$$\begin{aligned}\psi(v^*, u) &= \langle N(Tu, Au) - w^*, \eta(u, v^*)\rangle + \varphi(u, u) - \varphi(v^*, u)\\ &\geq \langle N(Tv^*, Av^*) - N(Tu, Au), \eta(v^*, u)\rangle - \langle N(Tv^*, Av^*) - w^*, \eta(v^*, u)\rangle + \langle r, u - v^*\rangle\\ &= \langle N(Tv^*, Av^*) - N(Tu, Av^*), \eta(v^*, u)\rangle + \langle N(Tu, Av^*) - N(Tu, Au), \eta(v^*, u)\rangle\\ &\quad - \langle N(Tv^*, Av^*) - w^*, \eta(v^*, u)\rangle + \langle r, u - v^*\rangle\\ &\geq \lambda\|N(Tv^*, Av^*) - N(Tu, Av^*)\|^2 + \xi\|v^* - u\|^2 - \delta\|N(Tv^*, Av^*) - w^*\|\|v^* - u\| - \|r\|\|u - v^*\|\\ &\geq \xi\|v^* - u\|^2 - \delta\|N(Tv^*, Av^*) - w^*\|\|v^* - u\| - \|r\|\|u - v^*\|\\ &= \|u - v^*\|[\xi\|u - v^*\| - \delta\|N(Tv^*, Av^*) - w^*\| - \|r\|].\end{aligned}$$

Let $R = (1/\xi)[\delta\|N(Tv^*, Av^*) - w^*\| + \|r\|]$ and $D_0 = \{u \in D : \|u - v^*\| \leq R\}$. Then D_0 is a weakly compact convex subset of D. Putting $v_0 = v^*$, we have that $\varphi(v_0, u) > 0$ for all $u \in D \setminus D_0$ and so condition (iii) in Lemma 2.2 is satisfied. By Lemma 2.2, there exists $\hat{u} \in D$ such that $\psi(v, \hat{u}) \leq 0$ for all $v \in D$, that is,

$$\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(v, \hat{u})\rangle + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0, \quad \forall v \in D.$$

So $\hat{u}$ is a solution of MQVLIP. Utilizing the similar argument as in the proof of Ding and Yao [[1], Theorem 3.2], we can readily prove that $\hat{u}$ is the unique solution of MQVLIP.

Step 2. We claim the existence and uniqueness of a solution to problem (1.4) for each given iterate $u_n \in D$.

Indeed, for the sake of simplicity, we rewrite (1.4) as follows: find $w \in D$ such that

$$\langle \rho N(Tu_n, Au_n) - \rho w^* + K'(w) - K'(u_n), \eta(v, w)\rangle + \rho\varphi(v, w) - \rho\varphi(w, w) \geq 0, \quad \forall v \in D. \tag{3.4}$$

Let $\hat{u} \in D$ be a unique solution of MQVLIP. For each fixed $\rho > 0$ and $u_n \in D$, define a functional $\psi : D \times D \to [-\infty, +\infty]$ by

$$\psi(v, w) = \langle K'(u_n) - K'(w) - \rho(N(Tu_n, Au_n) - w^*), \eta(v, w)\rangle + \rho\varphi(w, w) - \rho\varphi(v, w).$$

Note that for each fixed $v \in D$, the functional $w \mapsto \langle K'(w), \eta(v, w)\rangle$ is weakly upper semicontinuous on D. Also, note that for each fixed $v \in D$, $u \to \eta(u, v)$ is continuous from the weak topology to the weak topology and $\varphi(\cdot, \cdot)$ is weakly continuous. Thus, it is easy to see that for each fixed $v \in D$, the function $w \mapsto \psi(v, w)$ is weakly lower semicontinuous continuous on each weakly compact subset of D and so condition (i) in Lemma 2.2 is satisfied. We claim that condition (ii) in Lemma 2.2 holds. If this is false, then there exist a finite set $\{v_1, \ldots, v_n\} \subset D$ and a $w = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, such that

$$\psi(v_i, w) = \langle K'(u_n) - K'(w) - \rho(N(Tu_n, Au_n) - w^*), \eta(v_i, w)\rangle + \rho\varphi(w, w) - \rho\varphi(v_i, w) > 0, \quad \forall i = 1, \ldots, n.$$

Note that condition (iii)(a) implies $\eta(u, u) = 0$ and $\eta(u, v) = -\eta(v, u)$ for all $u, v \in D$. Hence, from condition (iii)(c) and the convexity of $\varphi(\cdot, w)$ it follows that

$$\begin{aligned}0 &< \sum_{i=1}^{n} \lambda_i[\langle K'(u_n) - K'(w) - \rho(N(Tu_n, Au_n) - w^*), \eta(v_i, w)\rangle + \rho\varphi(w, w) - \rho\varphi(v_i, w)]\\ &\leq \langle K'(u_n) - K'(w) - \rho(N(Tu_n, Au_n) - w^*), \eta(w, w)\rangle + \rho\varphi(w, w) - \rho\sum_{i=1}^{n} \lambda_i\varphi(v_i, w)\\ &\leq 0,\end{aligned}$$

which is a contradiction. Thus, condition (ii) in Lemma 2.2 is satisfied. Note that the η-strong convexity of K implies that K' is η-strongly monotone with constant $\mu > 0$. By using the similar argument as in the proof of Step 1, we can readily prove that condition (iii) of Lemma 2.2 is also satisfied. By Lemma 2.2 there exists a point $w \in D$, such that $\psi(v, w) \leq 0$ for all $v \in D$. Hence inequality (3.4) holds. This also shows that problem (1.4) has a solution in D. Now we prove that the solution of problem (3.4) is unique. Let w_1 and w_2 be two solutions of problem (3.4). Then, for all $v \in D$,

$$\langle \rho N(Tu_n, Au_n) - \rho w^* + K'(w_1) - K'(u_n), \eta(v, w_1)\rangle + \rho\varphi(v, w_1) - \rho\varphi(w_1, w_1) \geq 0, \quad \forall v \in D \tag{3.5}$$

and

$$\langle \rho N(Tu_n, Au_n) - \rho w^* + K'(w_2) - K'(u_n), \eta(v, w_2)\rangle + \rho\varphi(v, w_2) - \rho\varphi(w_2, w_2) \geq 0, \quad \forall v \in D. \tag{3.6}$$

Taking $v = w_2$ in (3.5) and $v = w_1$ in (3.6), and adding these two inequalities, we obtain

$$\langle K'(w_2) - K'(w_1), \eta(w_1, w_2)\rangle \geq 0,$$

since $\eta(w_2, w_1) + \eta(w_1, w_2) = 0$ holds, and $\varphi(\cdot, \cdot)$ is skew-symmetric. Therefore, we have

$$\mu\|w_1 - w_2\|^2 \leq \langle K'(w_1) - K'(w_2), \eta(w_1, w_2)\rangle \leq 0,$$

because of the η-strong monotonicity of K'. This shows that $w_1 = w_2$. Letting $u_{n+1} = w$, the unique solution of problem (3.4), we finish the proof of Step 2.

Step 3. We claim the convergence of sequence $\{u_n\}$ generated by Algorithm 1.1 to a unique solution of MQVLIP.

Indeed, let $\hat{u} \in D$ be a unique solution of MQVLIP. Define a functional $\Lambda : D \to (-\infty, +\infty]$ by

$$\Lambda(u) = K(\hat{u}) - K(u) - \langle K'(u), \eta(\hat{u}, u)\rangle.$$

By the η-strong convexity of K, we have

$$\Lambda(u) = K(\hat{u}) - K(u) - \langle K'(u), \eta(\hat{u}, u)\rangle \geq \frac{\mu}{2}\|u - \hat{u}\|^2. \tag{3.7}$$

Note that $\eta(u, v) = -\eta(v, u)$ for all $u, v \in D$ and $\varphi(\cdot, \cdot)$ is skew-symmetric. Since $u_{n+1} \in D$ and $\hat{u}$ is a unique solution of MQVLIP, from the η-strong convexity of K and (3.4) with $w = u_{n+1}$ and $v = \hat{u}$ it follows that

$$\begin{aligned}
\Lambda(u_n) - \Lambda(u_{n+1}) &= k(u_{n+1}) - K(u_n) - \langle K'(u_n), \eta(u_{n+1}, u_n)\rangle + \langle K'(u_{n+1}) - K'(u_n), \eta(\hat{u}, u_{n+1})\rangle \\
&\geq \frac{\mu}{2}\|u_n - u_{n+1}\|^2 + \rho\langle N(Tu_n, Au_n) - w^*, \eta(u_{n+1}, \hat{u})\rangle + \rho[\varphi(u_{n+1}, u_{n+1}) - \varphi(\hat{u}, u_{n+1})] \\
&\geq \frac{\mu}{2}\|u_n - u_{n+1}\|^2 + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\
&\quad + \rho[\langle N(T\hat{u}, A\hat{u}) - w^*, \eta(u_{n+1}, \hat{u})\rangle + \varphi(u_{n+1}, \hat{u}) - \varphi(\hat{u}, \hat{u})] \\
&\geq \frac{\mu}{2}\|u_n - u_{n+1}\|^2 + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\
&= \frac{\mu}{2}\|u_n - u_{n+1}\|^2 + Q,
\end{aligned} \tag{3.8}$$

where $Q = \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle$.

Observe that

$$\begin{aligned}
Q &= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\
&= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\
&= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_n, \hat{u})\rangle + \rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\
&\quad + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_{n+1}, u_n)\rangle + \rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\
&\geq \rho\lambda\|N(Tu_n, Au_n) - N(T\hat{u}, Au_n)\|^2 + \rho\xi\|u_n - \hat{u}\|^2 - \rho\delta\|N(Tu_n, Au_n) - N(T\hat{u}, Au_n)\|\|u_{n+1} - u_n\| \\
&\quad - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\
&= \rho[\lambda\|N(Tu_n, Au_n) - N(T\hat{u}, Au_n)\|^2 - \delta\|N(Tu_n, Au_n) - N(T\hat{u}, Au_n)\|\|u_{n+1} - u_n\|] \\
&\quad - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| + \rho\xi\|u_n - \hat{u}\|^2 \\
&\geq \rho\left[-\frac{\delta^2}{4\lambda}\right]\|u_{n+1} - u_n\|^2 - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| + \rho\xi\|u_n - \hat{u}\|^2.
\end{aligned} \tag{3.9}$$

Therefore, we have

$$\begin{aligned}
\Lambda(u_n) - \Lambda(u_{n+1}) &\geq \frac{1}{2}\left(\mu - \frac{\rho\delta^2}{2\lambda}\right)\|u_{n+1} - u_n\|^2 - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| + \rho\xi\|u_n - \hat{u}\|^2 \\
&\geq \left[\rho\xi - \frac{\rho^2\beta^2\delta^2}{2(\mu - \rho\delta^2/2\lambda)}\right]\|u_n - \hat{u}\|^2.
\end{aligned} \tag{3.10}$$

Condition (3.1) and inequality (3.10) show that the sequence $\{\Lambda(u_n)\}$ is strictly decreasing (unless $u_n = \hat{u}$) and is nonnegative by (3.7). Hence it converges to some number. Thus, the difference of two consecutive terms of the sequence $\{\Lambda(u_n)\}$ goes to zero, and so the sequence $\{u_n\}$ converges strongly to $\hat{u}$. This completes the proof. □

Remark 3.1. If $N(\cdot, \cdot)$ is Lipschitz continuous in the first argument with constant σ_1 and η-cocoercive in the first argument with respect to T with constant α, then T is η-cocoercive with respect to the first argument of $N(\cdot, \cdot)$ with constant α/σ_1^2. Indeed, observe that for all $u, v \in D$,

$$\langle N(Tu, \cdot) - N(Tv, \cdot), \eta(u, v)\rangle \geq \alpha\|Tu - Tv\|^2 \geq \frac{\alpha}{\sigma_1^2}\|N(Tu, \cdot) - N(Tv, \cdot)\|^2.$$

This shows that condition (ii) in Theorem 3.1 is different from the condition [[1], Theorem 4.1] that $N(\cdot, \cdot)$ is η-cocoercive in the first argument with respect to T.

Example 3.1. Let $B = \mathbb{R}^2$ be a two-dimensional real Euclidean space with inner product $\langle .,. \rangle$ and norm $\| \cdot \|$ defined as follows:

$$\langle x, y\rangle = ac + bd \quad \text{and} \quad \|x\| = \sqrt{a^2+b^2}, \quad \forall x = (a, b) \in \mathbb{R}^2, y = (c, d) \in \mathbb{R}^2.$$

Let $D = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$. Define the mappings $T, A : D \to \mathbb{R}^2$, $N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : D \times D \to \mathbb{R}^2$ as follows: For all $x = (a, b) \in \mathbb{R}^2$ and $y = (c, d) \in \mathbb{R}^2$

$$Tx = \left(a - \frac{1}{2}\sin a, b - \frac{1}{2}\sin b\right), \quad Ax = -\frac{1}{2}Tx, \quad N(x, y) = x + y,$$

and

$$\eta(x, y) = \left(a - c - \frac{1}{4}(b-d), b - d + \frac{1}{4}(a-c)\right).$$

Then T is η-cocoercive with respect to the first argument of $N(\cdot,\cdot)$ with constant $\lambda > 0$. Indeed, it is clear that T is Lipschitz continuous. Also, observe that

$$\begin{aligned} &\langle N(Tx,\cdot) - N(Ty,\cdot), \eta(x,y)\rangle \\ &= \left(a - c - \frac{1}{2}(\sin a - \sin c)\right)\left(a - c - \frac{1}{4}(b-d)\right) + \left(b - d - \frac{1}{2}(\sin b - \sin d)\right)\left(b - d + \frac{1}{4}(a-c)\right) \\ &\ge \|x-y\|^2 - \frac{5}{4\sqrt{2}}\|x-y\|^2 = \frac{4\sqrt{2}-5}{4\sqrt{2}}\|x-y\|^2. \end{aligned}$$

This together with the Lipschitz continuity of T implies that T is η-cocoercive with respect to the first argument of $N(\cdot,\cdot)$ with constant $\lambda > 0$. Moreover, it is easy to see that $N(\cdot,\cdot)$ is Lipschitz continuous and η-strongly monotone in the second argument with respect to A with constants $\beta > 0$ and $\xi > 0$, respectively. Further, we can see that η satisfies all the conditions of Theorem 3.1.

As a special case of Theorem 3.1, we derive the following result.

Corollary 3.1. *Let D be a nonempty closed convex subset of a reflexive Banach space B with the dual space B^*. Let $T, A : D \to B^*$ and $\eta : D \times D \to B$ be three mappings. Let $w^* \in B^*$ and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that* $\operatorname{int}\{v \in D : \varphi(v, v) < \infty\} \neq \emptyset$ *and $\varphi(\cdot, v)$ is proper convex for each $v \in D$. Suppose that $K : B \to (-\infty, +\infty)$ is a Fréchet differentiable and η-strongly convex functional with constant $\mu > 0$ such that the functional $w \mapsto \langle K'(w), \eta(v, w)\rangle$ is weakly upper semicontinuous on D for each fixed $v \in D$. Suppose also that:*

(i) *for each fixed $v \in D$, the functional $u \mapsto \langle (Tu - Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous;*
(ii) *T is η-cocoercive with constant $\lambda > 0$;*
(iii) *condition* (iii) *in Theorem* 3.1 *holds;*
(iv) *A is Lipschitz continuous and η-relaxed monotone with constants $\beta > 0$ and $\xi \ge 0$, respectively.*

Then,

(I) *mixed variational-like inequality problem (1.3) has a unique solution $\hat{u} \in D$,*
(II) *for each $\rho > 0$ and each given iterate u_n, there exists a unique solution $u_{n+1} \in D$ to problem (1.4) with $N(Tu, Av) = Tu - Av$ for all $u, v \in D$,*
(III) *if*

$$0 < \rho < \frac{2\mu\lambda\xi}{\delta^2(\lambda\beta^2 + \xi)},$$

then the sequence $\{u_n\}$ defined by Algorithm 1.1 *with $N(Tu, Av) = Tu - Av$ for all $u, v \in D$ converges strongly to a unique solution $\hat{u}$ of problem (1.3).*

Proof. Let $N(Tu, Av) = Tu - Av$ for all $u, v \in D$. It is easy to verify that $N(\cdot,\cdot)$, T, A, η, and $\varphi(\cdot,\cdot)$ satisfy all conditions of Theorem 3.1. Thus, the conclusions (I)–(III) of Corollary 3.1 follow immediately from Theorem 3.1. □

Now, by using Lemma 2.2 along with condition (iii)′ in place of condition (iii), we establish the following result.

Theorem 3.2. *Let D be a nonempty closed convex subset of a reflexive Banach space B with the dual space B^*. Let $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$ be mappings. Let $w^* \in B^*$ and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that for each $v \in B$,* $\operatorname{int}\{v \in D : \varphi(v, v) < \infty\} \neq \emptyset$ *and $\varphi(\cdot, v)$ is proper convex. Suppose that for each fixed $v \in D$, the functional $u \mapsto \langle N(Tu, Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous. Suppose also that:*

(i) *$N(\cdot,\cdot)$ is Lipschitz continuous and η-strongly monotone in the first argument with respect to T with constants $\tau > 0$ and $\upsilon > 0$, respectively;*

(ii) $N(\cdot,\cdot)$ *is Lipschitz continuous and* η*-relaxed monotone in the second argument with respect to A with constants* $\beta > 0$ *and* $\xi \geq 0$*, respectively;*
(iii) η *is Lipschitz continuous with constant* $\delta > 0$ *such that;*
  (a) $\eta(u, v) = \eta(u, z) + \eta(z, v)$ *for each* $u, v, z \in D$*,*
  (b) $\eta(\cdot,\cdot)$ *is affine in the first variable,*
  (c) *for each fixed* $v \in D$*,* $u \to \eta(v, u)$ *is continuous from the weak topology to the weak topology,*
(iv) $K : D \to R$ *is* η*-strongly convex with constant* $\mu > 0$*, and* K' *is continuous from the weak topology to the strong topology.*

Then,

(I) *there exists a unique solution* $\hat{u} \in D$ *of MQVLIP,*
(II) *for each* $\rho > 0$*, there exists a unique solution* $u_{n+1} \in D$ *to problem* (1.4)*,*
(III) *if*

$$\begin{cases} 0 < \rho < \dfrac{2\mu}{\delta^2} \cdot \max\left\{ \dfrac{\upsilon + \xi - 2\beta\delta}{(\tau+\beta)^2}, \dfrac{\xi(\upsilon - 2\beta\delta)}{\beta^2(\upsilon - 2\beta\delta) + \tau^2\xi} \right\}, \\ 2\beta\delta < \upsilon, \end{cases} \tag{3.11}$$

then the sequence $\{u_n\}$ *defined by Algorithm* 1.1 *converges strongly to a unique solution* $\hat{u}$ *of MQVLIP.*

Proof. As in Theorem 3.1, we divide the proof into three steps.

Step 1. We claim the existence of a unique solution of MQVLIP.

The function $\varphi : D \times D \to [-\infty, +\infty]$ is defined as in the proof of Theorem 3.1. Conditions (i) and (ii) of Lemma 2.2 are satisfied by using the same argument as in the proof of Theorem 3.1.

By using (3.2)–(3.2), conditions (i)–(ii) and $\eta(u, v) = -\eta(v, u)$, we have

$$\begin{aligned} \psi(v^*, u) &= \langle N(Tu, Au) - w^*, \eta(u, v^*)\rangle + \varphi(u, u) - \varphi(v^*, u) \\ &\geq \langle N(Tv^*, Av^*) - N(Tu, Au), \eta(v^*, u)\rangle - \langle N(Tv^*, Av^*) - w^*, \eta(v^*, u)\rangle + \langle r, u - v^*\rangle \\ &= \langle N(Tv^*, Av^*) - N(Tu, Av^*), \eta(v^*, u)\rangle + \langle N(Tu, Au) - N(Tu, Av^*), \eta(v^*, u)\rangle \\ &\quad + 2\langle N(Tu, Av^*) - N(Tu, Au), \eta(v^*, u)\rangle - \langle N(Tv^*, Av^*) - w^*, \eta(v^*, u)\rangle + \langle r, u - v^*\rangle \\ &\geq \upsilon\|v^* - u\|^2 + \xi\|v^* - u\|^2 - 2\beta\delta\|v^* - u\|^2 - \delta\|N(Tv^*, Av^*) - w^*\|\|v^* - u\| - \|r\|\|u - v^*\| \\ &= \|u - v^*\|[(\upsilon + \xi - 2\beta\delta)\|u - v^*\| - \delta\|N(Tv^*, Av^*) - w^*\| - \|r\|]. \end{aligned}$$

Let $R = (1/(\upsilon + \xi - 2\beta\delta))[\delta\|N(Tv^*, Av^*) - w^*\| + \|r\|]$ and $\mathcal{K} = \{u \in D : \|u - v^*\| \leq R\}$. Then $\mathcal{K}$ and $D_0 = \{v^*\}$ are both weakly compact convex subsets of D. For each $u \in D \setminus \mathcal{K}$, there exists $v^* \in \mathrm{co}(D_0 \cup \{u\})$ such that $\varphi(v^*, u) > 0$ and so condition (iii)′ of Lemma 2.2 is satisfied. Then, as in the proof of Theorem 3.1, there exists a unique solution $\hat{u}$ of MQVLIP.

Step 2. We claim the existence of a unique solution of problem (1.4) for each given iterate $u_n \in D$.

Let $\psi : D \times D \to [-\infty, +\infty]$ be the same as in the proof of Theorem 3.1. Since for each fixed $v \in D$, $u \mapsto \eta(v, u)$ is continuous from the weak topology to the weak topology, and since K' is continuous from the weak topology to the strong topology, from Proposition 2.2 we know that for each fixed $v \in D$ the functional $w \mapsto \langle K'(w), \eta(v, w)\rangle$ is weakly continuous on D. Note that $\varphi(\cdot,\cdot)$ is weakly continuous. Thus, it is easy to see that for each fixed $v \in D$, the function $w \mapsto \psi(v, w)$ is weakly continuous on each weakly compact subset of D and so condition (i) in Lemma 2.2 is satisfied. The rest of this step is exactly the same as in the proof of Step 2 in Theorem 3.1.

Step 3. We claim the convergence of sequence $\{u_n\}$ generated by Algorithm 1.1 to a unique solution of MQVLIP.

Let Λ, Q be the same as in the proof of Theorem 3.1. Then

$$\begin{aligned} Q &= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, \hat{u})\rangle \\ &= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle \\ &= \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_n, \hat{u})\rangle + \rho\langle N(T\hat{u}, A\hat{u}) - N(T\hat{u}, Au_n), \eta(u_n, \hat{u})\rangle \\ &\quad + 2\rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_n, \hat{u})\rangle + \rho\langle N(Tu_n, Au_n) - N(T\hat{u}, Au_n), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \rho\langle N(T\hat{u}, Au_n) - N(T\hat{u}, A\hat{u}), \eta(u_{n+1}, u_n)\rangle \\ &\geq \rho\upsilon\|u_n - \hat{u}\|^2 + \rho\xi\|u_n - \hat{u}\|^2 - 2\rho\beta\delta\|u_n - \hat{u}\|^2 \\ &\quad - \rho\tau\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\|. \end{aligned} \tag{3.12}$$

Now, we discuss two cases for the range of the ρ value.

Case 1:

$$0 < \rho < \frac{2\mu(\upsilon + \xi - 2\beta\delta)}{\delta^2(\tau+\beta)^2}. \tag{3.13}$$

By using (3.11) and (3.12), we have

$$\begin{aligned} \Lambda(u_n) - \Lambda(u_{n+1}) &\geq \frac{\mu}{2}\|u_{n+1} - u_n\|^2 + \rho[(\upsilon + \xi - 2\beta\delta)\|u_n - \hat{u}\|^2 - \delta(\tau + \beta)\|u_n - \hat{u}\|\|u_{n+1} - u_n\|] \\ &\geq \rho(\upsilon + \xi - 2\beta\delta)\|u_n - \hat{u}\|^2 - \frac{\rho^2\delta^2(\tau + \beta)^2}{4 \cdot \frac{\mu}{2}}\|u_n - \hat{u}\|^2 \\ &\geq \rho\left((\upsilon + \xi - 2\beta\delta) - \frac{\rho\delta^2(\tau + \beta)^2}{2\mu}\right)\|u_n - \hat{u}\|^2. \end{aligned} \tag{3.14}$$

Therefore, it follows from (3.13) and (3.14) that the sequence $\{\Lambda(u_n)\}$ is strictly decreasing (unless $u_n = \hat{u}$) and is nonnegative by (3.7). Hence it converges to some number. Thus, the difference of two consecutive terms of the sequence $\{\Lambda(u_n)\}$ goes to zero, and so the sequence $\{u_n\}$ converges strongly to $\hat{u}$.

Case 1:

$$0 < \rho < \frac{2\mu\xi(\upsilon - 2\beta\delta)}{\delta^2(\beta^2(\upsilon - 2\beta\delta) + \tau^2\xi)}. \tag{3.15}$$

By using (3.11) and (3.13), we also have

$$\begin{aligned} \Lambda(u_n) - \Lambda(u_{n+1}) &\geq \frac{\mu}{2}\|u_{n+1} - u_n\|^2 + \rho\upsilon\|u_n - \hat{u}\|^2 + \rho\xi\|u_n - \hat{u}\|^2 \\ &\quad - 2\rho\beta\delta\|u_n - \hat{u}\|^2 - \rho\tau\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| - \rho\beta\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\ &\geq \frac{\mu}{2}\|u_{n+1} - u_n\|^2 - \frac{\rho\beta^2\delta^2}{4\xi}\|u_{n+1} - u_n\|^2 + \rho\upsilon\|u_n - \hat{u}\|^2 \\ &\quad - 2\rho\beta\delta\|u_n - \hat{u}\|^2 - \rho\tau\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| \\ &= \frac{2\mu\xi - \rho\beta^2\delta^2}{4\xi}\|u_{n+1} - u_n\|^2 - \rho\tau\delta\|u_n - \hat{u}\|\|u_{n+1} - u_n\| + \rho(\upsilon - 2\beta\delta)\|u_n - \hat{u}\|^2 \\ &\geq -\frac{\rho^2\tau^2\delta^2}{4 \cdot \frac{2\mu\xi - \rho\beta^2\delta^2}{4\xi}}\|u_n - \hat{u}\|^2 + \rho(\upsilon - 2\beta\delta)\|u_n - \hat{u}\|^2 \\ &= \rho\left[\upsilon - 2\beta\delta - \frac{\rho\xi\tau^2\delta^2}{2\mu\xi - \rho\beta^2\delta^2}\right]\|u_n - \hat{u}\|^2. \end{aligned} \tag{3.16}$$

Therefore, it follows from (3.15) and (3.16) that the sequence $\{\Lambda(u_n)\}$ is strictly decreasing (unless $u_n = \hat{u}$) and is nonnegative by (3.7). Hence it converges to some number. Thus, the difference of two consecutive terms of the sequence $\{\Lambda(u_n)\}$ goes to zero, and so the sequence $\{u_n\}$ converges strongly to $\hat{u}$. This completes the proof. □

Remark 3.2. Condition (i) in Theorem 3.2 implies the η-cocoercivity of the mapping $x \mapsto N(Tx, Az)$ for each fixed $z \in D$. Indeed, observe that for all $x, y, z \in D$,

$$\langle N(Tx, Az) - N(Ty, Az), \eta(x, y)\rangle \geq \upsilon\|x - y\|^2 \geq \frac{\upsilon}{\tau^2}\|N(Tx, Az) - N(Ty, Az)\|^2.$$

This shows that condition (i) in Theorem 3.2 is different from the condition [[1], Theorem 4.1] that $N(\cdot, \cdot)$ is η-cocoercive in the first argument with respect to T. Moreover, the following conditions are very different: one condition in Theorem 3.2 that $N(\cdot, \cdot)$ is η-relaxed monotone in the second argument with respect to A, and the other condition in [[1], Theorem 4.1] that $N(\cdot, \cdot)$ is η-strongly monotone in the second argument with respect to A.

The following result can be easily derived from the above theorem.

Corollary 3.2. *Let D be a nonempty closed convex subset of a reflexive Banach space B with the dual space B^*. Let $T, A : D \to B^*$ and $\eta : D \times D \to B$ be three mappings. Let $w^* \in B^*$ and $\varphi : B \times B \to (-\infty, +\infty]$ be skew-symmetric and weakly continuous such that* $\operatorname{int}\{v \in D : \varphi(v, v) < \infty\} \neq \emptyset$ *and $\varphi(\cdot, v)$ is proper convex for each $v \in D$. Suppose that $K : B \to (-\infty, +\infty]$ is a Fréchet differentiable and η-strongly convex functional with constant $\mu > 0$ such that its derivative K' is continuous from the weak topology to the strong topology and that for each fixed $v \in D$, the functional $u \mapsto \langle (Tu - Au) - w^*, \eta(u, v)\rangle$ is weakly lower semicontinuous. Suppose also that:*

(i) *T is Lipschitz continuous and η-strongly monotone with constants $\tau > 0$ and $\upsilon > 0$, respectively;*
(ii) *A is Lipschitz continuous and η-strongly monotone with constants $\beta > 0$ and $\xi > 0$, respectively;*
(iii) *η is Lipschitz continuous with constant $\delta > 0$, such that conditions* (iii)(a), (b), (c) *in Theorem 3.2 hold.*

Then,

(I) *mixed variational-like inequality problem* (1.3) *has a unique solution* $\hat{u} \in D$,

(II) *for each* $\rho > 0$ *and each given iterate* u_n, *there exists a unique solution* $u_{n+1} \in D$ *to problem* (1.4) *with* $N(Tu, Av) = Tu - Av$ *for all* $u, v \in D$,

(III) *if*

$$\begin{cases} 0 < \rho < \dfrac{2\mu}{\delta^2} \cdot \max\left\{ \dfrac{\upsilon + \xi - 2\beta\delta}{(\tau+\beta)^2}, \dfrac{\xi(\upsilon - 2\beta\delta)}{\beta^2(\upsilon - 2\beta\delta) + \tau^2\xi} \right\}, \\ 2\beta\delta < \upsilon, \end{cases}$$

then the sequence $\{u_n\}$ *defined by* Algorithm 1.1 *with* $N(Tu, Av) = Tu - Av$ *for all* $u, v \in D$ *converges strongly to a unique solution* $\hat{u}$ *of problem* (1.3).

Proof. Let $N(Tu, Av) = Tu - Av$ for all $u, v \in D$. It is easy to verify that $N(\cdot,\cdot)$, T, A, η, and $\varphi(\cdot,\cdot)$ satisfy all conditions of Theorem 3.2. Thus, conclusions (I)–(III) follow immediately from Theorem 3.2. □

Remark 3.3. (1) We emphasize that our algorithm and convergence analysis are different from those of Ding and Yao [1]. Meanwhile, Corollaries 3.1 and 3.2 greatly improve Theorem 3.2 of Ansari and Yao [8] in the following aspects:

(a) D may be a unbounded subset of a reflexive Banach space;

(b) For the case of unbounded domain, the condition in Remark 3.2 of Ansari and Yao [[8], p. 537] is unnecessary;

(c) Corollaries 3.1 and 3.2 give not only the iterative schemes for solving MVLIP (1.3) but also the existence and uniqueness of solutions of MVLIP (1.3).

(2) Corollaries 3.1 and 3.2 also improve Theorem 4.1 of Ding [9–11] in several ways. Hence, Theorems 3.1 and 3.2 further generalize the results of Ansari and Yao [8] to a more general mixed quasi-variational-like inclusion problem (1.1). We can also generalize Theorem 3.3 and Corollaries 3.1 and 3.2 in [8] to reflexive Banach spaces under weaker assumptions; for example, we give the following algorithm and convergence theorem.

Modified algorithm. Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \geq 0 (\forall n \geq 0)$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\eta : D \times D \to B$ be a mapping, $K : D \to (-\infty, +\infty]$ be a given Fréchet differentiable η-convex functional, and $\rho > 0$ be a given positive number.

(i) At $n = 0$, start with some initial $u_0 \in D$;

(ii) At step $n + 1$, for a given iterate x_n, solve the auxiliary variational inequality problem that consists of finding $u_{n+1} \in D$ such that

$$\begin{aligned} &\langle K'(u_{n+1}) - K'(u_n) + \rho(N(Tu_n, Au_n) - w^*), \eta(v, u_{n+1})\rangle \\ &\quad + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq -\varepsilon_n, \quad \forall v \in D. \end{aligned} \tag{3.17}$$

(iii) If, for given $\epsilon > 0$, $\|u_{n+1} - u_n\| \leq \epsilon$, stop. Otherwise, repeat (ii).

Theorem 3.3. *Assume that all the conditions of* Theorem 3.2 *are satisfied. Let* $\{\varepsilon_n\}$ *be a sequence of nonnegative numbers such that* $\lim_{n\to\infty} \varepsilon_n = 0$. *Then, the sequence* $\{u_n\}$ *generated by the above Modified Algorithm converges strongly to a unique solution of MQVLIP.*

Proof. From the proof of Theorem 3.2, we have

$$\Lambda(u_n) - \Lambda(u_{n+1}) \geq \theta\|u_n - \hat{u}\|^2 + \varepsilon_n,$$

where $\theta = \rho\left((\upsilon + \xi - 2\beta\delta) - \frac{\rho\delta^2(\tau+\beta)^2}{2\mu}\right)$ or $\rho\left(\upsilon - 2\beta\delta - \frac{\rho\xi\tau^2\delta^2}{2\mu\xi - \rho\beta^2\delta^2}\right)$, and

$$\lim_{n\to\infty} (\theta\|u_n - \hat{u}\|^2 + \varepsilon_n) = 0.$$

Now, observe that

$$\lim_{n\to\infty} \theta\|u_n - \hat{u}\|^2 = \lim_{n\to\infty} [(\theta\|u_n - \hat{u}\|^2 + \varepsilon_n) - \varepsilon_n] = \lim_{n\to\infty} (\theta\|u_n - \hat{u}\|^2 + \varepsilon_n) - \lim_{n\to\infty} \varepsilon_n = 0,$$

and hence $\{u_n\}$ converges strongly to $\hat{u}$, a unique solution of the MQVLIP (1.1). This completes the proof. □

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