BULL. AUSTRAL. MATH. SOC. 47H10, 49J40, 90A14, 47H19, 47H04
VOL. 59 (1999) [433–442]

A FIXED POINT THEOREM AND ITS APPLICATIONS TO A SYSTEM OF VARIATIONAL INEQUALITIES

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In this paper, we first prove a fixed point theorem for a family of multivalued maps defined on product spaces. We then apply our result to prove an equilibrium existence theorem for an abstract economy. We also consider a system of variational inequalities and prove the existence of its solutions by using our fixed point theorem.

1. INTRODUCTION

By using a partition of unity, very recently Lan and Webb [8] have obtained some fixed point theorems for a family of multivalued maps defined on product spaces without compactness assumptions on the domain and range sets. Ding, Kim and Tan [6], Wu [13], Wu and Shen [14] and Yannelis and Prabhakar [15] have also studied such types of fixed point theorems with compactness assumption on the range set, which are well suited to prove equilibrium existence theorems for an abstract economy.

Pang [9] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformally modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve. He also studied the convergence of such methods. The method of decomposition was also used by Zhu and Marcotte [16] to solve a variational inequality problem defined on a set of inequality constraints. By generalising the concept of pseudomonotonicity to the product sets, Bianchi [1] proved the existence of solutions of the system of variational inequalities, that is, a family of variational inequalities defined on a product set.

Inspired by the system of variational inequalities, in the next section we establish a fixed point theorem for a family of multivalued maps defined on product spaces, which generalises the results in [6, 8, 14] and [15]. In Section 3, we apply our fixed point theorem to prove an equilibrium existence theorem for an abstract economy. In Section 4, we consider the system of variational inequalities and prove the existence of its solutions by using the result of Section 2.

Received 28th October, 1998
This research was supported by the National Science Council of the Repubic of China.

We shall use the following notation and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. If A is a non-empty subset of a topological vector space X, we shall denote by int_X the interior of A in X. If A is a subset of a vector space, $\text{co}(A)$ denotes the convex hull of A.

Let X and Y be two topological vector spaces and $T : X \longrightarrow 2^Y$ be a multivalued map. Then T is said to have the *local intersection property* [14] if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighbourhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$.

The multivalued map T is said to be *transfer open-valued* [4] if for any $x \in X$, $y \in T(x)$ there exists an $z \in X$ such that $y \in \text{int}_Y T(z)$.

The *inverse* of T, denoted by T^{-1}, is the multivalued map from $\mathcal{R}(T)$, the range of T, to X defined by

$$x \in T^{-1}(y) \quad \text{if and only if} \quad y \in T(x).$$

2. Fixed point theorem

Let I be an index set and for each $i \in I$, let E_i be a Hausdorff topological vector space. Let $\{K_i\}_{i \in I}$ be a family of non-empty convex subsets with each K_i in E_i. Let $K = \prod_{i \in I} K_i$ and $E = \prod_{i \in I} E_i$.

Theorem 1. *For each $i \in I$, let $S_i, T_i : K \longrightarrow 2^{K_i}$ be two multivalued maps. Assume that the following conditions hold.*

(i) *For each $i \in I$ and each $x \in K$, $\text{co}\big(S_i(x)\big) \subset T_i(x)$ and $S_i(x)$ is non-empty.*

(ii) *For each $i \in I$, $K = \bigcup\big\{\text{int}_K S_i^{-1}(x_i) : x_i \in K_i\big\}$.*

(iii) *If K is not compact, assume that there exist a non-empty compact convex subset C_i of K_i and a non-empty compact subset D of K such that for each $x \in K \setminus D$ there exists $\widetilde{y}_i \in C_i$ such that $x \in \text{int}_K S_i^{-1}(\widetilde{y}_i)$.*

Then there exists $\overline{x} \in K$ such that $\overline{x} \in T(\overline{x}) = \prod_{i \in I} T_i(\overline{x})$, that is, $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$, where $\overline{x}_i$ is the projection of $\overline{x}$ onto K_i.

Proof: For each $i \in I$, we define a multivalued map $G_i : K_i \longrightarrow 2^K$ by

$$G_i(x_i) = \{x \in K : x \notin \text{int}_K S_i^{-1}(x_i)\} = K \setminus \text{int}_K S_i^{-1}(x_i).$$

Then G_i satisfies the following conditions:

(a) For each $x_i \in K_i$, $G_i(x_i)$ is closed in K.

(b) For each $i \in I$, $\bigcap_{x_i \in C_i} G_i(x_i)$ is compact in K.

Indeed, if K is compact, $\bigcap_{x_i \in C_i} G_i(x_i)$ is compact since $\bigcap_{x_i \in C_i} G_i(x_i)$ is closed in K by (a). If K is not compact,

$$\bigcap_{x_i \in C_i} G_i(x_i) = \bigcap_{x_i \in C_i} \big\{x \in K : x \notin \text{int}_K S_i^{-1}(x_i)\big\} \subset D \quad \text{by (iii)}$$

and thus is compact.

(c) Since for each $i \in I$, $K = \bigcup\{\operatorname{int}_K S_i^{-1}(x_i) : x_i \in K_i\}$, we have $\bigcap_{x_i \in K_i} G_i(x_i) = \bigcap_{x_i \in K_i} \{K \setminus \operatorname{int}_K S_i^{-1}(x_i)\} = \emptyset$, for each $i \in I$.

Now, we shall show that there exist $a_{i1}, \cdots, a_{il_i} \in K_i$ such that

$$\left(\bigcap_{x_i \in C_i} G_i(x_i)\right) \cap \left(\bigcap_{k=1}^{l_i} G_i(a_{ik})\right) = \emptyset. \tag{1}$$

Suppose that (1) is not true, then for every finite set $\{y_1, \cdots, y_n\} \subset K_i$, we have

$$\left(\bigcap_{x_i \in C_i} G_i(x_i)\right) \cap \left(\bigcap_{j=1}^{n} G_i(y_j)\right) \neq \emptyset.$$

Let $P(y) = \left(\bigcap_{x_i \in C_i} G_i(x_i)\right) \cap \left(G_i(y)\right)$ for $y \in K_i$. Then the family $\{P(y) : y \in K_i\}$ has the finite intersection property. Note that $P(y)$ is compact in K for each $y \in K_i$ because $\bigcap_{x_i \in C_i} G_i(x_i)$ is compact and $G_i(y)$ is closed in K. It follows that $\bigcap_{y \in K_i} P(y) \neq \emptyset$ and thus $\bigcap_{y \in K_i} G_i(y) \neq \emptyset$ which is a contradiction to (c).

By (1), we have

$$\left(\bigcup_{x_i \in C_i} \operatorname{int}_K S_i^{-1}(x_i)\right) \cup \left(\bigcup_{k=1}^{l_i} \operatorname{int}_K S_i^{-1}(a_{ik})\right) = K. \tag{2}$$

Let $F_i = \operatorname{co}(C_i \cup \{a_{i1}, \cdots, a_{il_i}\})$. Then F_i is compact in K_i. Let $F = \prod_{i \in I} F_i$, then F is a compact subset of K. By (2), we have

$$F \subset \left(\bigcup_{x_i \in C_i} \operatorname{int}_K S_i^{-1}(x_i)\right) \cup \left(\bigcup_{k=1}^{l_i} \operatorname{int}_K S_i^{-1}(a_{ik})\right). \tag{3}$$

Since F is compact, there exist $b_{i1}, \cdots, b_{it_i} \in C_i$ such that

$$F \subset \left(\bigcup_{j=1}^{t_i} \operatorname{int}_K S_i^{-1}(b_{ij})\right) \cup \left(\bigcup_{k=1}^{l_i} \operatorname{int}_K S_i^{-1}(a_{ik})\right). \tag{4}$$

Let $\{c_{i1}, \cdots, c_{in_i}\} = \{a_{i1}, \cdots, a_{il_i}, b_{i1}, \cdots, b_{it_i}\}$. We rewrite (4) as follows

$$F \subset \bigcup_{k=1}^{n_i} \operatorname{int}_K S_i^{-1}(c_{ik}). \tag{5}$$

Let $X_i = \operatorname{co}\left(\{c_{i1}, \cdots, c_{in_i}\}\right)$ and $X = \prod_{i \in I} X_i$. We denote by Δ_i the vector subspace of E_i generated by X_i. Then Δ_i is locally convex since it is a finite dimensional subspace. We

note that X is a compact set in $\prod_{i\in I}\Delta_i$, and $X \subset F \subset \bigcup_{k=1}^{n_i} \operatorname{int}_K S_i^{-1}(c_{ik})$. Therefore

$$X \subset \left(\bigcup_{k=1}^{n_i} \operatorname{int}_K S_i^{-1}(c_{ik})\right) \cap X \subseteq \bigcup_{k=1}^{n_i} \operatorname{int}_X S_i^{-1}(c_{ik}) \subset X$$

and hence $X = \bigcup_{k=1}^{n_i} \operatorname{int}_X S_i^{-1}(c_{ik})$.

Since X is compact, there exists a partition of unity $\{g_{i1}, \cdots, g_{in_i}\}$ subordinated to this finite subcovering such that

(I) for each $k = 1, \cdots, n_i$, $g_{ik} : X \longrightarrow [0,1]$ is continuous,

(II) for each $k = 1, \cdots, n_i$, $g_{ik}(x) = 0$, for $x \notin \operatorname{int}_X S_i^{-1}(c_{ik})$,

(III) for each $x \in X$, $\sum_{k=1}^{n_i} g_{ik}(x) = 1$.

For each $i \in I$, we define a map $f_i : X \longrightarrow X_i$ by $f_i(x) = \sum_{k=1}^{n_i} g_{ik}(x)c_{ik}$, for all $x \in X$. Obviously, for each $i \in I$, f_i is continuous function. For each $x \in X$ and each k with $g_{ik}(x) \neq 0$, we have $x \in \operatorname{int}_X S_i^{-1}(c_{ik}) \subset S_i^{-1}(c_{ik})$ and so that $c_{ik} \in S_i(x)$ for each $i \in I$. By (i) and the definition of f_i, we have for each $i \in I$, $f_i(x) \in \operatorname{co}\big(S_i(x)\big) \subset T_i(x)$, for all $x \in X$.

Define a map $h : X \longrightarrow X$ by $h(x) = \big(f_i(x)\big)_{i\in I}$. Since for each $x \in X$, $f_i(x) \in X_i$, it follows that h is well-defined and continuous. By Tychonoff's fixed point theorem [12], h has a fixed point $\overline{x} = (\overline{x}_i)_{i\in I} \in X$. This implies that $\overline{x}_i = f_i(\overline{x})$ for all $i \in I$. Since for each $i \in I$, $f_i(x) \in T_i(x)$ for all $x \in X$, we have $\overline{x}_i \in T_i(\overline{x})$. □

REMARK. (a) Theorem 1 generalises in [6, Theorem 2] and [14], respectively.

(b) The assumtion (ii) in Theorem 1 can be replaced by any one of the following conditions:

(ii)′ For each $i \in I$, S_i^{-1} is transfer open-valued.

(ii)″ For each $i \in I$, S_i has the local intersection property.

(ii)‴ For each $i \in I$ and $x_i \in K_i$, $S_i^{-1}(x_i)$ is open in K.

PROOF: For the proof of (ii)′, we refer to [4, Lemma 2.1].

(ii)″ Since for each $i \in I$, $S_i(x) \neq \emptyset$, for all $x \in K$, there exists a neighbourhood $N_i(x)$ of x such that $\bigcap_{z\in N_i(x)} S_i(z) \neq \emptyset$. Let $y_i \in \bigcap_{z\in N_i(x)} S_i(z)$, then $y_i \in S_i(z)$, for all $z \in N_i(x)$ and hence $z \in S_i^{-1}(y_i)$, for all $z \in N_i(x)$. Therefore, $x \in N_i(x) \subset S_i^{-1}(y_i)$, so that $x \in \operatorname{int}_K S_i^{-1}(y_i)$. This implies that $K \subset \bigcup\{\operatorname{int}_K S_i^{-1}(y_i) : y_i \in K_i\} \subset K$, and hence $K = \bigcup\{\operatorname{int}_K S_i^{-1}(y_i) : y_i \in K_i\}$.

(ii)‴ From assumption (i) in Theorem 1, we have $K = \bigcup\{S_i^{-1}(y_i) : y_i \in K_i\}$ and since for each $y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K, we get $K = \bigcup\{\operatorname{int}_K S_i^{-1}(y_i) : y_i \in K_i\}$. □

When the index set I is a singleton, we have the following result which generalises the well-known Browder fixed point theorem [3], [10, Theorem 1], [14, Corollary 3] and [15, Theorem 3.3].

COROLLARY 1. *Let K be a non-empty convex subset of a Hausdorff topological vector space E, and let $S, T : K \longrightarrow 2^K$ be two multivalued maps. Assume that the following conditions hold.*

(i) *For each $x \in K$, $\operatorname{co}\big(S(x)\big) \subset T(x)$ and $S(x)$ is non-empty.*

(ii) $K = \bigcup\big\{\operatorname{int}_K S^{-1}(y) : y \in K\big\}$.

(iii) *If K is not compact, assume that there exist a non-empty compact convex subset C of K and a non-empty compact subset D of K such that for each $x \in K \setminus D$ there exists $\widetilde{y} \in C$ such that $x \in \operatorname{int}_K S^{-1}(\widetilde{y})$.*

Then there exists $\overline{x} \in K$ such that $\overline{x} \in T(\overline{x})$.

REMARK. The assumption (ii) in Corollary 1 is equivalent to the following condition of Tarafdar [10]:

(ii)′ For each $y \in K$, $S^{-1}(y) = \big\{x \in K : y \in S(x)\big\}$ contains a relatively open subset O_y of K (O_y could be empty for some y) such that $\bigcup\limits_{y \in K} O_y = K$.

3. EQUILIBRIUM EXISTENCE THEOREM

In this section, we prove an equilibrium existence theorem for a non-compact abstract economy with an infinite number of commodities and an infinite number of agents.

Let I be a (possibly uncountable) set of agents. An *abstract economy* $\Gamma = (K_i, A_i, B_i, P_i)$ [6], where $A_i, B_i : K = \prod\limits_{j \in I} K_j \longrightarrow 2^{K_i}$ are constraint correspondences and $P_i : K \longrightarrow 2^{K_i}$ is a preference correspondence. An equilibrium [6] for Γ is a point $\overline{x} \in K$ such that for each $i \in I$,

$$\overline{x}_i \in B_i(\overline{x}) \quad \text{and} \quad A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset.$$

When $A_i = B_i$ for each $i \in I$, these definitions of an abstract economy and an equilibrium coincide with the standard definitions, see for example [2] and [15].

THEOREM 2. *Let $\{K_i\}_{i \in I}$ be a family of non-empty convex subsets with each K_i in a Hausdorff topological vector space E_i and let $\Gamma = (K_i, A_i, B_i, P_i)$ be an abstract economy. Assume that the following conditions hold.*

(i) *For each $i \in I$ and each $x \in K$, $\operatorname{co}\big(A_i(x)\big) \subset B_i(x)$ and $A_i(x)$ is non-empty.*

(ii) *For each $i \in I$, $K = \bigcup\limits_{y_i \in K_i} \operatorname{int}_K\big[\big\{(\operatorname{co} P_i)^{-1}(y_i) \cup W_i\big\} \cap A_i^{-1}(y_i)\big]$, where $W_i = \big\{x \in K : A_i(x) \cap P_i(x) = \emptyset\big\}$.*

(iii) *For each $i \in I$ and each $x \in K$, $x_i \notin \operatorname{co}\big(P_i(x)\big)$.*

(iv) *If K is not compact, assume that there exist a non-empty compact convex subset C_i of K_i and a non-empty compact subset D of K such that for each $x \in K \setminus D$ there exists $\widetilde{y}_i \in C_i$ such that $x \in \operatorname{int}_K\big[\{(\operatorname{co}P_i)^{-1}(\widetilde{y}_i) \cup W_i\} \cap A_i^{-1}(\widetilde{y}_i)\big]$.*

Then there exists a point $\overline{x} \in K$ such that $\overline{x}_i \in B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$, for all $i \in I$.

PROOF: For each $i \in I$, let $V_i = \{x \in K : A_i(x) \cap P_i(x) \neq \emptyset\}$ and for each $x \in K$, let $I(x) = \{i \in I : A_i(x) \cap P_i(x) \neq \emptyset\}$. For each $i \in I$, we define two multivalued maps $S_i, T_i : K \longrightarrow 2^{K_i}$ by

$$S_i(x) = \begin{cases} \operatorname{co}P_i(x) \cap A_i(x), & \text{for } i \in I(x) \\ A_i(x), & \text{for } i \notin I(x) \end{cases}$$

and

$$T_i(x) = \begin{cases} \operatorname{co}P_i(x) \cap B_i(x), & \text{for } i \in I(x) \\ B_i(x), & \text{for } i \notin I(x). \end{cases}$$

Then for each $i \in I$ and each $x \in K$, $\operatorname{co}\big(S_i(x)\big) \subset T_i(x)$ and $S_i(x)$ is non-empty. Now for each $i \in I$ and $y_i \in K_i$,

$$\begin{aligned} S_i^{-1}(y_i) &= \big[\{(\operatorname{co}P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)\} \cap V_i\big] \cup \big[A_i^{-1}(y_i) \cap W_i\big] \\ &= \big[(\operatorname{co}P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)\big] \cup \big[A_i^{-1}(y_i) \cap W_i\big] \\ &= \big[(\operatorname{co}P_i)^{-1}(y_i) \cup W_i\big] \cap A_i^{-1}(y_i). \end{aligned}$$

From (ii), we have

$$K = \bigcup_{y_i \in K_i} \operatorname{int}_K\big[\{(\operatorname{co}P_i)^{-1}(y_i) \cup W_i\} \cap A_i^{-1}(y_i)\big] = \bigcup_{y_i \in K_i} \operatorname{int}_K S_i^{-1}(y_i).$$

Hence all the conditions of Theorem 1 are satisfied, therefore there exists $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$ for all $i \in I$. By (iii) and the definition of T_i, we have $\overline{x}_i \in B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$, for all $i \in I$. □

4. System of variational inequalities

Let I be an index set and for each $i \in I$, let E_i be a Hausdorff topological vector space with its topological dual E_i^*. Let $\{K_i\}_{i \in I}$ be a family of non-empty convex subsets with each K_i in E_i. Let $K = \prod_{i \in I} K_i$, $K^i = \prod_{j \neq i,\ j \in I} K_j$ and $E = \prod_{i \in I} E_i$. For each $i \in I$, let $A_i : K \longrightarrow E_i^*$ be a given function. Then we consider the *system of variational inequalities* (in short, SVI) which is to find $\overline{x} = (\overline{x}_i, \overline{x}^i) \in K$ such that for each $i \in I$,

$$\langle A_i(\overline{x}), y_i - \overline{x}_i \rangle \geqslant 0, \quad \text{for all } y_i \in K_i, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between E_i^* and E_i.

The (SVI) was considered by Pang [9] with applications in equilibrium problems. He showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be modelled as the (SVI). He also investigated the local and global convergence of various iterative methods for solving the (SVI). But he did not discuss the existence of solutions of the (SVI). Later, Bianchi [1] proved the existence of solutions of the (SVI) by generalising the concept of pseudomonotonicity to the product sets and using the Fan-KKM Theorem [7] The (SVI) was also studied by Cohen and Chaplais [5], and Zhu and Marcotte [16].

First we shall prove an existence theorem for a more general system.

THEOREM 3. *For each $i \in I$, let f_i and g_i be two real-valued functions defined on $K \times K_i$. Assume that the following conditions hold.*

(i) *For each $i \in I$, $x \mapsto f_i(x, y_i)$ is upper semicontinuous on K, for all $y_i \in K_i$.*

(ii) *For each $i \in I$ and each finite subset $\{y_i^1, \cdots, y_i^n\}$ of K_i we have $f_i(x, y_i^j) < 0$, for all $j = 1, \cdots, n$ imply $g_i(x, \widehat{y}_i) < 0$, where $\widehat{y}_i = \sum_{j=1}^{n} \alpha^j y_i^j$, $\alpha^j \geqslant 0$ for all $j = 1, \cdots, n$ and $\sum_{j=1}^{n} \alpha^j = 1$.*

(iii) *For each $i \in I$ and all $x \in K$, $x_i \in K_i$, $g_i(x, x_i) \geqslant 0$.*

(iv) *If K is not compact, assume that there exist a non-empty compact convex subset C_i of K_i and a non-empty compact subset D of K such that for each $x \in K \setminus D$ there exists $\widetilde{y}_i \in C_i$ such that $f_i(x, \widetilde{y}_i) < 0$.*

Then there exists $\overline{x} \in K$ such that for each $i \in I$, $f_i(\overline{x}, y_i) \geqslant 0$, for all $y_i \in K_i$.

PROOF: For each $y_i \in K_i$, we define a multivalued map $H_i : K_i \longrightarrow K$ by

$$H_i(y_i) = \{x \in K : f_i(x, y_i) \geqslant 0\}.$$

Then from (i) we have, for each $i \in I$ and $y_i \in K_i$, $H_i(y_i)$ is closed in K.

Suppose that the conclusion of this theorem is not true. Then for each $x \in K$,

$$\{y_i \in K_i : f_i(x, y_i) < 0\} = \{y_i \in K_i : x \notin H_i(y_i)\} \neq \emptyset, \text{ for each } i \in J \subseteq I.$$

Now, we define two multivalued maps $S_i, T_i : K \longrightarrow 2^{K_i}$ by

$$S_i(x) = \{y_i \in K_i : f_i(x, y_i) < 0\} \text{ and } T_i(x) = \{y_i \in K_i : g_i(x, y_i) < 0\}, \text{ respectively.}$$

Then clearly $S_i(x)$ is non-empty, for all $x \in K$ and each $i \in J$.

Let $\{y_i^1, \cdots, y_i^n\}$ be a finite subset of K_i such that $f_i(x, y_i^j) < 0$ for all $j = 1, \cdots, n$ and $x \in K$. Then from (ii), we have $g_i(x, \widehat{y}_i) < 0$, where $\widehat{y}_i = \sum_{j=1}^{n} \alpha^j y_i^j$, $\alpha^j \geqslant 0$ for all

$j = 1, \cdots, n$ and $\sum_{j=1}^{n} \alpha^j = 1$. Hence any convex combination of points of $S_i(x)$ lies in $T_i(x)$. Thus $\operatorname{co}\big(S_i(x)\big) \subset T_i(x)$, for all $x \in K$ and each $i \in J$.

Since $H_i(y_i)$ is closed for all $y_i \in K_i$ and each $i \in J$, we have $S_i^{-1}(y_i) = \{x \in K : f_i(x, y_i) < 0\} = \big[H_i(y_i)\big]^c$ (the complement of $H_i(y_i)$ in K) is open in K and hence $\operatorname{int}_K S_i^{-1}(y_i) = S_i^{-1}(y_i)$ for all $y_i \in K_i$. Since for each $x \in K$, $S_i(x)$ is non-empty, we have $K = \bigcup_{y_i \in K_i} S_i^{-1}(y_i) = \bigcup_{y_i \in K_i} \operatorname{int}_K S_i^{-1}(y_i)$. Then S_i and T_i, for each $i \in J$, satisfy all the conditions of Theorem 1. Hence there exists $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$, that is, there exists $\overline{x} \in K$ such that $g_i(\overline{x}, \overline{x}_i) < 0$ for each $i \in J$,which is a contradiction of assumption (iii). The result is proved. □

THEOREM 4. *For each $i \in I$, let f_i and g_i be two real-valued functions defined on $K \times K_i$. Assume that the following conditions hold.*

(i) *For each $i \in I$, $x \mapsto f_i(x, y_i)$ is upper semicontinuous on K, for all $y_i \in K_i$.*

(ii) *For each $i \in I$, $f_i(x, y_i) < 0$ implies $g_i(x, y_i) < 0$, for all $(x, y_i) \in K \times K_i$.*

(iii) *For each $i \in I$, either f_i or g_i is quasiconvex in their second variables.*

(iv) *For each $i \in I$ and all $x \in K$, $x_i \in K_i$, $g_i(x, x_i) \geqslant 0$.*

(v) *If K is not compact, assume that there exist a non-empty compact convex subset C_i of K_i and a non-empty compact subset D of K such that for each $x \in K \setminus D$ there exists $\widetilde{y}_i \in C_i$ such that $f_i(x, \widetilde{y}_i) < 0$.*

Then there exists $\overline{x} \in K$ such that for each $i \in I$, $f_i(\overline{x}, y_i) \geqslant 0$, for all $y_i \in K_i$.

PROOF: The result will follow from Theorem 3 if we show that $\operatorname{co}\big(S_i(x)\big) \subset T_i(x)$, for all $x \in K$ and each $i \in I$, where S_i and T_i are defined as in the proof of Theorem 3.

Let $\{y_i^1, \cdots, y_i^n\}$ be a finite subset of K_i such that $f_i(x, y_i^j) < 0$, for all $j = 1, \cdots, n$. Assume that for each $i \in I$, f_i is quasiconvex in the second variable. Then we have $f_i(x, \widehat{y}_i) < 0$, where $\widehat{y}_i = \sum_{j=1}^{n} \alpha^j y_i^j$, $\alpha^j \geqslant 0$, for all $j = 1, \cdots, n$ and $\sum_{j=1}^{n} \alpha^j = 1$. From (ii), we have $g_i(x, \widehat{y}_i) < 0$ and hence $\widehat{y}_i \in T_i(x)$, that is, the convex combinations of points of $S_i(x)$ lie in $T_i(x)$. Thus $\operatorname{co}\big(S_i(x)\big) \subset T_i(x)$, for all $x \in K$ and each $i \in I$.

Now let for each $i \in I$, g_i be quasiconvex in the second variable. Since $f_i(x, y_i^j) < 0$, for all $j = 1, \cdots, n$, from (ii) we have $g_i(x, y_i^j) < 0$, for all $j = 1, \cdots, n$. Since g_i is quasiconvex in the second variable, we have $g_i(x, \widehat{y}_i) < 0$, and hence $\widehat{y}_i \in T_i(x)$. Thus $\operatorname{co}\big(S_i(x)\big) \subset T_i(x)$, for all $x \in K$ and each $i \in I$. □

REMARK. When I is a singleton, Theorem 3 generalises [**11**, Lemma 2.1] while Theorem 4 reduces to [**11**, Lemma 2.1].

COROLLARY 2. *For each $i \in I$, let A_i be a function defined on K into E_i^*. Assume that the following conditions hold.*

(i) *For each $i \in I$, A_i is upper semicontinuous on K.*

(ii) *There exists a family* $\{g_i\}_{i\in I}$ *of real-valued functions defined on* $K \times K_i$ *such that*

(a) *for each* $i \in I$, $\langle A_i(x), y_i - x_i\rangle < 0$ *imply* $g_i(x, y_i) < 0$, *for all* $x \in K$ *and* $x_i, y_i \in K_i$,

(b) *for each* $i \in I$ *and all* $x \in K$, $x_i \in K_i$, $g_i(x, x_i) \geqslant 0$.

(iii) *If* K *is not compact, assume that there exist a non-empty compact convex subset* C_i *of* K_i *and a non-empty compact subset* D *of* K *such that for each* $x \in K \setminus D$ *there exists* $\widetilde{y}_i \in C_i$ *such that* $\langle A_i(x), \widetilde{y}_i - x_i\rangle < 0$.

Then there exists a solution $\overline{x} \in K$ *of the (SVI).*

PROOF: For each $i \in I$, we define a real-valued function f_i on $K \times K_i$ by

$$f_i(x, y_i) = \langle A_i(x), y_i - x_i\rangle, \quad \text{for all } (x, y_i) = \big((x_i, x^i), y_i\big) \in K \times K_i.$$

Then by Theorem 4, there exists $\overline{x} \in K$ such that

$$\langle A_i(\overline{x}), y_i - \overline{x}_i\rangle \geqslant 0, \quad \text{for all } y_i \in K_i.$$

□

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