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ON GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES

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RÉSUMÉ. Dans le présent article nous introduisons le problème vectoriel généralisé d'inéquation variationnelle et établissons l'existence de sa solution dans le contexte des espaces de Banach réflexifs. Plusieurs cas spéciaux sont aussi considérés.

ABSTRACT. In this paper, we introduce the generalized vector variational-like inequality and prove the existence of its solution in the setting of reflexive Banach spaces. Several special cases were also discussed.

1. Introduction. Because of the applications in different areas of optimization, optimal control, operations Research and economics, the vector variational inequality, which was introduced by Giannessi [7] in the finite dimensional Euclidean space in 1980, has been generalized in various directions. Since then Chen *et al.* [1–5], Siddiqi *et al.* [10, 11], Lee *et al.* [8] and Yang [13] have studied vector variational inequalities in abstract spaces. Siddiqi, Ansari and Ahmad [10] introduced and studied vector variational-like inequalities in reflexive Banach spaces and topological vector spaces. In 1990, Chen and Craven [4] introduced and studied vector variational inequalities for multivalued mappings which are called generalized vector variational inequalities. Recently, Lee *et al.* [8] studied generalized vector variational inequalities in reflexive Banach spaces.

In this paper, we introduce a more general form of vector variational inequality which includes vector variational-like inequalities and generalized vector variational inequalities as special cases. We prove the existence of its solution in the setting of reflexive Banach spaces. Several special cases were also discussed.

Let X and Y be two normed spaces and K be a nonempty closed and convex subset of X. Let $T : X \to 2^{L(X,Y)}$ be a multivalued mapping, where $L(X,Y)$ is the space of all linear continuous mappings from X into Y and $\eta : K \times K \to K$ be a continuous mapping. Let $\{C(x) : x \in K\}$ be a family of closed pointed convex cone in Y with $int\, C(x) \neq \emptyset$ for every $x \in K$, where $int\, C(x)$ is the interior of the set $C(x)$.

We consider the problem of finding $x_0 \in K$ such that for each $x \in K$, there exists $s_0 \in T(x_0)$ such that

$$\langle s_0, \eta(x, x_0)\rangle \notin -int\, C(x_0).$$

We shall call it *generalized vector variational-like inequality problem* (GVVLIP), where $\langle s_0, y\rangle$ denotes the evaluation of the linear mapping s_0 at y.

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Special Cases.

(i) If T is a single-valued mapping from X into $L(X,Y)$ then (GVVLIP) reduces to the problem of finding $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -int\, C(x_0), \quad \text{for all } x \in K$$

which is called vector variational-like inequality problem, introduced and studied by Siddiqi, Ansari and Ahmad [10].

(ii) If $\eta(x, x_0) = x - x_0$ then (GVVLIP) is equivalent to the following generalized vector variational inequality problem considered by Lee *et al.* [8]:

Find $x_0 \in K$ such that for each $x \in K$, there exists $s_0 \in T(x_0)$ such that

$$\langle s_0, x - x_0\rangle \notin -int\, C(x_0).$$

(iii) If T is a single-valued mapping from X into $L(X,Y)$ and $\eta(x, x_0) = x - x_0$ then (GVVLIP) becomes to the problem of finding $x_0 \in K$ such that

$$\langle T(x_0), x - x_0\rangle \notin -int\, C(x_0), \quad \text{for all } x \in K$$

which is known as vector variational inequality problem, studied by Chen [1].

(iv) If $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$ and T is a single-valued mapping then (GVVLIP) reduces to the problem of finding $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \geq 0, \quad \text{for all } x \in K,$$

is called variational-like inequality problem [9, 12, 14].

The special cases (i) – (iv) show that (GVVLIP) is more general and unifying setting, whose analysis is one of the main motivations of this paper.

2. Existence theory. Let X be a normed space with its dual X^*. We denote by 2^X, the set of all nonempty subsets of X and $conv(A)$, for all $A \subset X$, the convex hull of A.

We need the following concepts and results to prove our main result of this paper.

Definition 2.1. Let $T : X \to 2^{L(X,Y)}$, $\eta : K \times K \to K$ and let $C_- = \bigcap_{x \in X} C(x)$ be nonempty.

(1) T is said to be *C_--η-monotone* if $s \in T(x)$ and $t \in T(y)$, $\langle s - t, \eta(x, y)\rangle \in C_-$, for all $x, y \in X$.

(2) T is said to *η-pseudomonotone* if there exists $s \in T(x)$ such that $\langle s, \eta(y, x)\rangle \notin -int\, C(x)$ implies that there exists $t \in T(y)$ such that

$$\langle t, \eta(y, x)\rangle \notin -int\, C(x), \quad \text{for all } x, y \in X.$$

(3) T is said to be *V-hemicontinuous* if for any $x, y \in X$, $\alpha > 0$ and $t_\alpha \in T(x + \alpha y)$, there exists $t_0 \in T(x)$ such that for any $z \in X$, $\langle t_\alpha, z\rangle \to \langle t_0, z\rangle$ as $\alpha \to 0^+$.

Definition 2.2. A mapping $T : X \to 2^X$ is called *KKM-map*, if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $conv(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} T(x_i)$.

Definition 2.3. A multivalued mapping $T : K \to 2^X$ is called *concave* if for any $x, y \in K, 0 < \alpha < 1$,

$$\alpha T(x) + (1-\alpha)T(y) \subseteq T(\alpha x + (1-\alpha)y).$$

Lemma 2.1. (KKM – Fan [6]) *Let A be an arbitrary nonempty set in a topological vector space E and $T : A \to 2^E$ be a KKK-map. If $T(x)$ is closed for all $x \in A$ and is compact for at least one $x \in A$ then*

$$\bigcap_{x \in A} T(x) \neq \emptyset.$$

Now we prove the existence theorem for (GVVLIP).

Theorem 2.1. *Let X be a reflexive Banach space and Y be a Banach space. Let K be a nonempty closed bounded convex subset of X.*

Assume that

1° *$C : K \to 2^Y$ be a multivalued mapping such that for every $x \in K$, $C(x)$ is closed pointed convex cone with $int\, C(x) \neq \emptyset$ and let $C_- = \bigcap_{x \in K} C(x)$ with a nonempty interior $int\, C_- \neq \emptyset$.*

2° *The multivalued mapping $W(x) = Y \setminus \{-int\, C(x)\}$ is upper semicontinuous and concave.*

3° *$T : X \to 2^{L(X,Y)}$ is compact valued, η-pseudomonotone and V-hemicontinuous.*

4° *$\eta : K \times K \to K$ is continuous and affine, and $\eta(x, x) = 0$, for all $x \in K$.*

Then (GVVLIP) is solvable.

Proof. Since $C(x)$ is closed pointed convex cone for every $x \in K$ and $int\, C(x) \neq \emptyset$, $\forall x \in K$ then by condition 4°,

5° $$\langle s, \eta(x, x)\rangle \notin -int\, C(x), \text{ for all } x \in K \text{ and } s \in T(x).$$

Let $F_1(y) = \{x \in K : \exists s \in T(x) \text{ such that } \langle s, \eta(y, x)\rangle \notin -int\, C(x)\}$, for any $y \in K$.

We first prove that F_1 is a KKM mapping on K. Suppose that

$$\{x_1, x_2, \ldots, x_n\} \subset K, \quad \sum_{i=1}^{n} \alpha_i = 1, \ \alpha_i \geq 0, \ i = 1, 2, \ldots, n$$

and

$$x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_1(x_i).$$

Then for any $s \in T(x)$,

$$\langle s, \eta(x_i, \sum_{i=1}^{n} \alpha_i x_i)\rangle \in -int\, C(\sum_{i=1}^{n} \alpha_i x_i), \quad i = 1, 2, \ldots, n.$$

Since η is affine, therefore we have,

$$\langle s, \eta(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i x_i)\rangle = \sum_{i=1}^{n} \alpha_i \langle s, \eta(x_i, \sum_{i=1}^{n} \alpha_i x_i)\rangle \in -int\, C(\sum_{i=1}^{n} \alpha_i x_i),$$

which is a contradiction of 5°, so we derived $Conv(\{x_1, x_2, \dots, x_n\}) \subset \bigcup_{i=1}^{n} F_1(x_i)$, so that F_1 is the KKM-mapping on K.

Define a set-valued map $F_2 : K \to 2^K$ by, for any $y \in K$,

$$F_2(y) = \{x \in K : \exists t \in T(y) \text{ such that } \langle t, \eta(y, x)\rangle \notin -int\, C(x)\}.$$

Let $x \in F_1(y)$. Then there exists $s \in T(x)$ such that $\langle s, \eta(y, x)\rangle \notin -int\, C(x)$. By the η-pseudomonotonocity of T, we have there exists $t \in T(y)$ such that

$$\langle t, \eta(y, x)\rangle \notin -int\, C(x) \text{ for all } y \in K.$$

That is, $x \in F_2(y)$. Hence $F_1(y) \subseteq F_2(y)$, for any $y \in K$. Therefore F_2 is a KKM map on K.

On the other hand, for any $y \in K$, $F_2(y)$ is closed. Indeed, let $\{x_n\}$ be a sequence in $F_2(y)$ such that $x_n \to x_0 \in K$. Since $x_n \in F_2(y)$ for all n, there exists $t_n \in T(y)$ such that

$$\langle t_n, \eta(y, x_n)\rangle \notin -int\, C(x_n),$$

or

$$\langle t_n, \eta(y, x_n)\rangle \in W(x_n).$$

Since $T(y)$ is compact, without lost of generality, we can assume that there exists $t_0 \in T(y)$ such that $t_n \to t_0$. Now since $\eta(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ are continuous, $W(x)$ is upper semicontinuous and $t_n \to t_0$, $x_n \to x_0$, we have

$$\langle t_n, \eta(y, x_n)\rangle \to \langle t_o, \eta(y, x_0)\rangle \in W(x_0).$$

Hence $\langle t_0, \eta(y, x_0)\rangle \notin -int\, C(x_0)$. Therefore $x_0 \in F_2(y)$ and so $F_2(y)$ is closed.

Now we will prove that $F_2(y)$ is convex. Let $x_1, x_2 \in F_2(y)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then

$$\exists t \in T(y) \text{ such that } \langle t, \eta(y, x_1)\rangle \notin -int\, C(x_1) \qquad (*)$$

and

$$\langle t, \eta(y, x_2)\rangle \notin -int\, C(x_2). \qquad (**)$$

Multiplying $(*)$ by α and $(**)$ by β and add, we have

$$\alpha\langle t, \eta(y, x_1)\rangle + \beta\langle t, \eta(y, x_2)\rangle \in \alpha W(x_1) + \beta W(x_2).$$

Since $\eta(\cdot,\cdot)$ is affine and W is concave, we have

$$\langle t, \eta(y, \alpha x_1 + \beta x_2)\rangle \in W(\alpha x_1 + \beta x_2)$$

$$\Longrightarrow \langle t, \eta(y, \alpha x_1 + \beta x_2)\rangle \notin -int\, C(\alpha x_1 + \beta x_2).$$

Hence $\alpha x_1 + \beta x_2 \in F_2(y)$ and so $F_2(y)$ is convex.

Now we equip X with the weak topology. Then K, as a closed bounded convex subset in the reflexive Banach space X, is weakly compact. Since $F_2(y)$ is closed convex subset of a reflexive Banach space then $F_2(y)$ is weakly closed. $F_2(y) \subset K$ and weak closedness of $F_2(y)$, we have $F_2(y)$ is weakly compact. Then by KKM-Fan Lemma 2.1, we have

$$\bigcap_{y \in K} F_2(y) \neq \emptyset.$$

Let $x \in \bigcap_{y \in K} F_2(y)$. Then for any $y \in K$, there exists $t_y \in T(y)$ such that $\langle t_y, \eta(y,x)\rangle \notin -int\, C(x)$. By convexity of K, for any $\alpha \in (0,1)$ there exists $t_\alpha \in T(\alpha y + (1-\alpha)x)$ such that

$$\langle t_\alpha, \eta(\alpha y + (1-\alpha)x, x)\rangle \notin -int\, C(x).$$

Since $\eta(\cdot,\cdot)$ is affine and $\eta(x,x) = 0$, we have

$$\begin{aligned}\langle t_\alpha, \eta(\alpha y + (1-\alpha)x, x)\rangle &= \alpha\langle t_\alpha, \eta(y,x)\rangle + (1-\alpha)\langle t_\alpha, \eta(x,x)\rangle \\ &= \alpha\langle t_\alpha, \eta(y,x)\rangle.\end{aligned}$$

Therefore,

$$\alpha\langle t_\alpha, \eta(y,x)\rangle \notin -int\, C(x).$$

Dividing by α, we get

$$\langle t_\alpha, \eta(y,x)\rangle \notin -int\, C(x).$$

By the V-hemicontinuity of T, there exists $t_0 \in T(x)$ such that

$$\langle t_0, \eta(y,x)\rangle \notin -int\, C(x).$$

Hence $x \in \bigcap_{y \in K} F_1(y)$. Thus $\bigcap_{y \in K} F_1(y) \neq \emptyset$. Consequently, there exists $x_0 \in K$ such that for each $x \in K$, there exists $s_0 \in T(x_0)$ such that

$$\langle s_0, \eta(x, x_0)\rangle \notin -int\, C(x_0). \qquad \square$$

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Résumé substantiel en français. Soient X et Y deux espaces normés et K un sous-ensemble convexe fermé non-vide de X. Étant données une application multivoque $T : X \to 2^{L(X,Y)}$, où $L(X,Y)$ désigne l'espace des fonctions continues de X dans Y, et une application continue $\eta : K \times K \to K$, considérons une famille $\{C(x) : x \in K\}$ de cônes convexes pointés fermés dans Y pour lesquels $int\, C(x) \neq \emptyset$ où $int\, C(x)$ désigne l'intérieur de l'ensemble $C(x)$.

Nous étudions le problème de trouver $x_0 \in K$ tel que pour tout $x \in K$, il existe $s_0 \in T(x_0)$ satisfaisant

$$\langle s_0, \eta(x, x_0)\rangle \notin -int\, C(x_0)$$

que nous appellerons *problème vectoriel généralisé d'inéquations de type variationnel* (PVGITV); en anglais: *generalized vector variational-like inequalities problem* (GVVLIP), où $\langle s_0, y\rangle$ désigne l'évaluation en y de l'application linéaire s_0.

Cas spéciaux.

(i) Si T est une application de X dans $L(X,Y)$, alors le (PVGITV) se réduit au problème de trouver $x_0 \in K$ tel que

$$\langle T(x_0), \eta(x,x_0)\rangle \notin -int\, C(x_0), \quad \text{pour tout } x \in K$$

appelé *problème vectoriel d'inéquations de type variationnel*, introduit et étudié dans [10] par Siddiqi, Ansari et Ahmad.

(ii) Si $\eta(x,x_0) = x - x_0$, alors le (PVGITV) est équivalent au *problème vectoriel généralisé d'inéquations variationnelles* suivant considéré par Lee *et al.* [8]:
Trouver $x_0 \in K$ tel que pour tout $x \in K$, il existe $s_0 \in T(x_0)$ tel que

$$\langle s_0, x - x_0\rangle \notin -int\, C(x_0).$$

(iii) Si T est une application univoque de X dans $L(X,Y)$ et $\eta(x,x_0) = x - x_0$, alors le (PVGITV) se réduit au *problème vectoriel d'inéquations variationnelles*, étudié par Chen [1], qui consiste à trouver $x_0 \in K$ tel que

$$\langle T(x_0), x - x_0\rangle \notin -int\, C(x_0), \quad \text{pour tout } x \in K.$$

(iv) Si $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$ et T est une application univoque, alors le (PVGITV) se réduit au *problème d'inéquations variationnelles* [9, 12, 14] qui consiste à trouver $x_0 \in K$ tel que

$$\langle T(x_0), \eta(x,x_0)\rangle \geq 0, \quad \text{pour tout } x \in K.$$

Les cas spéciaux (i) – (iv) illustrent l'aspect général et unificateur du (PVGITV), qui est l'objet principal du présent travail. Nous démontrons ici le théorème suivant:

Théorème 2.1. *Soient X et Y deux espaces de Banach, X étant réflexif. Soit K un sous-ensemble borné fermé convexe non-vide de X.*

Supposons de plus que

1) *$C : K \to 2^Y$ est une application multivoque telle que pour tout $x \in K$, $C(x)$ est un cône convexe pointé fermé d'intérieur non-vide et que $C_- = \bigcap_{x\in K} C(x)$ possède un intérieur non-vide,*
2) *la fonction multivoque $W(x) = Y \setminus \{-int\, C(x)\}$ est concave semi-continue supérieurement,*
3) *$T : X \to 2^{L(X,Y)}$ est η-pseudomonotone et V-hémicontinue,*
4) *$\eta : K \times K \to K$ est continue affine et $\eta(x,x) = 0$ pour tout $x \in K$.*

Alors le (PVGITV) est résoluble.

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