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Systems of quasi-equilibrium problems with lower and upper bounds[☆]

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Abstract

In this work, we consider systems of quasi-equilibrium problems with lower and upper bounds and establish the existence of their solutions by using some known maximal element theorems for a family of multivalued maps. Our problems are more general than the one posed in [G. Isac, V.M. Sehgal, S.P. Singh, An alternative version of a variational inequality, Indian J. Math. 41 (1999) 25–31]. As a particular case, we also get the answer to the problem raised in [G. Isac, V.M. Sehgal, S.P. Singh, An alternative version of a variational inequality, Indian J. Math. 41 (1999) 25–31].
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1. Introduction and formulations

Given a nonempty closed subset K in a locally convex semi-reflexive topological vector space X, a mapping $F : K \times K \to \mathbb{R}$, and two real numbers α and β such that $\alpha < \beta$. In 1999, Isac et al. [9] raised the problem of finding $\bar{x} \in K$ such that $\alpha \leq F(\bar{x}, y) \leq \beta$, for all $y \in K$. Soon after, Li [11] gave the answer of this open problem by introducing and using the concept of extremal subsets. Subsequently, Chadli et al. [5] also derived some results in answering this open problem by using a fixed point theorem and the Fan–KKM lemma.

We notice that if $\alpha = 0$ and $\beta = e^{-c}$ and $F(x, y) = e^{-f(x,y)}$, where c is a constant and $f : K \times K \to \mathbb{R}$, then the above problem reduces to finding $\bar{x} \in K$ such that $f(\bar{x}, y) \geq c$, for all $y \in K$. For $c = 0$, this is known as the *equilibrium problem* which is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, etc. In the last decade, it has been extensively studied in the literature; see, for example, [2,4,6–8,19] and references therein.

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Throughout the work, unless otherwise specified, we shall use the following notation and assumptions. Let I be any index set. For each $i \in I$, let K_i be a nonempty subset of a topological vector space X_i, $F_i : K \times K_i \to \mathbb{R}$ a real valued bifunction, $A_i, B_i : K \to 2^{K_i}$ multivalued maps with nonempty values and $\alpha_i, \beta_i \in \mathbb{R}$ two real numbers such that $\alpha_i \leq \beta_i$. Let $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. We consider the following problem, more general than the one posed by Isac, Sehgal and Singh [9]:

$$(\text{SQEP})_{\text{LUB}} \begin{cases} \text{Find } \bar{x} \in K \quad \text{such that for each } i \in I, \\ \bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad \alpha_i \leq F_i(\bar{x}, y_i) \leq \beta_i, \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

We shall call this the *system of quasi-equilibrium problems with lower and upper bounds*. For I a singleton set, this problem will be called the *quasi-equilibrium problem with lower and upper bounds* (for short, $(\text{QEP})_{\text{LUB}}$).

For each $i \in I$, let X_i^* be the topological dual of X_i, K_i^* a nonempty subset of X_i^*, $\Psi_i : K_i^* \times K_i \times K_i \to \mathbb{R}$ a function and $U_i : K \to 2^{K_i^*}$ a multivalued map with nonempty values. We also consider the following problem:

$$(\text{SGIQEP})_{\text{LUB}} \begin{cases} \text{Find } \bar{x} \in K \quad \text{and} \quad \bar{u} \in U(\bar{x}) = \prod_{i\in I} U_i(\bar{x}) \quad \text{such that for each } i \in I, \\ \bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad \alpha_i \leq \Psi_i(\bar{u}_i, \bar{x}_i, y_i) \leq \beta_i, \quad \text{for all } y_i \in A_i(\bar{x}), \end{cases}$$

where u_i is the projection of u onto K_i^*. We shall call this the *system of generalized implicit quasi-equilibrium problems with lower and upper bounds*. For I a singleton set, this problem will be called the *generalized implicit quasi-equilibrium problem with lower and upper bounds* (for short, $(\text{GIQEP})_{\text{LUB}}$).

If I is a singleton set and $A_i(x) = B_i(x) = K$ for all $x \in K$, then $(\text{SQEP})_{\text{LUB}}$ reduces to the problem posed in [9].

As above, if we take for each $i \in I$, $\alpha_i = 0$ and $\beta_i = \mathrm{e}^{-c_i}$ and $F_i(x, y_i) = \mathrm{e}^{-f_i(x,y_i)}$, where c_i is a constant and $f_i : K \times K_i \to \mathbb{R}$, then $(\text{SQEP})_{\text{LUB}}$ becomes the problem of finding $\bar{x} \in K$ such that for each $i \in I$,

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}, y_i) \geq c_i, \quad \text{for all } y_i \in A_i(\bar{x}).$$

Such a problem is considered and studied in [15]. In addition, if $A_i(x) = B_i(x) = K_i$ for all $x \in K$ and for each $i \in I$, then this problem is studied by Lan and Webb [10].

If for each $i \in I$, $c_i = 0$, then the above problem is known as the *system of quasi-equilibrium problems* (for short, SQEP) which contains the system of quasi-variational inequalities, system of quasi-optimization problems, quasi-saddle point problem, constrained Nash equilibrium problem (also called the Debreu type equilibrium problem), etc., as special cases. In the recent past, SQEP has been used as a tool for solving the constrained Nash equilibrium problem for an infinite number of players; see, for example, [1,13,19,20] and references therein.

If for each $i \in I$, $\alpha_i = 0$, $\beta_i = \mathrm{e}^{-c_i}$ and $\Psi_i(u_i, x_i, y_i) = \mathrm{e}^{-\psi_i(u_i,x_i,y_i)}$, where $\psi_i : K_i^* \times K_i \times K_i \to \mathbb{R}$ is a function, then $(\text{SGIQEP})_{\text{LUB}}$ becomes the problem of finding $\bar{x} \in K$ and $\bar{u} \in U(\bar{x}) = \prod_{i\in I} U_i(\bar{x})$ such that for each $i \in I$,

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \geq c_i, \quad \text{for all } y_i \in A_i(\bar{x}).$$

When $A_i(x) = B_i(x) = K_i$ and $c_i = 0$ for each $i \in I$ and for all $x \in K$, this problem was studied by Ansari and Yao [3] and such a problem is used to solve Nash equilibrium problem for nondifferentiable and nonconvex functions.

One of the main motivations of this work is to establish existence results for a solution of $(\text{SQEP})_{\text{LUB}}$ by using some known maximal element theorems for a family of multivalued maps. By using these existence results and the selection method, we also derive the existence of a solution of $(\text{SGIQEP})_{\text{LUB}}$. The results of this work are new in the literature. As a particular case, for I a singleton set and $A_i(x) = B_i(x) = K$ for all $x \in K$, the results of this work are more general than those given by Li [11] and Chadli et al. [5].

2. Preliminaries

For a nonempty set D, we denote by 2^D the family of all subsets of D. If D is a nonempty subset of a vector space, then co$D$ denotes the convex hull of D. If D is a nonempty subset of a topological space, then the closure of D is denoted by $\overline{D}$.

A nonempty subset D of a topological space $\mathcal{X}$ is said to be *compactly open* (respectively, compactly closed) if for every nonempty compact subset C of $\mathcal{X}$, $D \cap C$ is open (respectively, closed) in C.

Throughout this work, all topological spaces are assumed to be Hausdorff.

Definition 2.1 (*[16]*). Let $\mathcal{X}$ be a topological vector space and let C be a lattice with a minimal element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^{\mathcal{X}} \to C$ is called a *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^{\mathcal{X}}$.

(a) $\Phi(\overline{\text{co}}M) = \Phi(M)$, where $\overline{\text{co}}M$ denotes the closed convex hull of M.
(b) $\Phi(M) = \mathbf{0}$ if and only if M is precompact.
(c) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 2.2 (*[16]*). Let $\mathcal{X}$ be a topological vector space, $D \subseteq \mathcal{X}$, and let Φ be a measure of noncompactness on $\mathcal{X}$. A multivalued map $T : D \to 2^{\mathcal{X}}$ is called *Φ-condensing* provided that if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$ then M is relative compact, that is, $\overline{M}$ is compact.

Remark 2.1. Note that every multivalued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If $\mathcal{X}$ is locally convex, then a compact multivalued map (i.e., $T(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : D \to 2^{\mathcal{X}}$ is Φ-condensing and $T' : D \to 2^{\mathcal{X}}$ satisfies $T'(x) \subseteq T(x)$ for all $x \in \mathcal{X}$, then T' is also Φ-condensing.

The following maximal element theorems for a family of multivalued maps will be used to establish the existence of a solution of $(\text{SQEP})_{\text{LUB}}$.

Theorem 2.1 (*[12]*). *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i and $S_i, T_i : K \to 2^{K_i}$ be multivalued maps. For each $i \in I$, assume that the following conditions hold:*

(a) *for all $x \in K$, $\text{co}S_i(x) \subseteq T_i(x)$;*
(b) *for all $x \in K$, $x_i \notin T_i(x)$;*
(c) *for all $y_i \in K_i$, $S_i^{-1}(y_i)$ is compactly open in K;*
(d) *there exist a nonempty compact subset D of K and a nonempty compact convex subset $C_i \subseteq K_i$ for each $i \in I$ such that for all $x \in K \setminus D$, there exists $i \in I$ such that $S_i(x) \cap C_i \neq \emptyset$.*

Then there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Theorem 2.2 (*[12]*). *For each $i \in I$, let K_i be a nonempty closed convex subset of a topological vector space X_i. Let Φ be a measure of noncompactness on $X = \prod_{i \in I} X_i$. For each $i \in I$, let $S_i, T_i : K \to 2^{K_i}$ be multivalued maps satisfying conditions* (a)–(c) *of Theorem 2.1. Further, assume that the multivalued map $T := \prod_{i \in I} T_i : K \to 2^K$ defined as $T(x) = \prod_{i \in I} T_i(x)$, is Φ-condensing. Then there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.*

3. Existence results

In this section, we first establish existence results for solutions of $(\text{SQEP})_{\text{LUB}}$ and then by using the selection method, we also derive existence results for a solution of $(\text{SGIQEP})_{\text{LUB}}$.

Theorem 3.1. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i, $F_i, P_i, Q_i : K \times K_i \to \mathbb{R}$ real valued functions, $B_i : K \to 2^{K_i}$ a multivalued map such that the set $\mathscr{F}_i = \{x \in K : x_i \in B_i(x)\}$ is compactly closed, $A_i : K \to 2^{K_i}$ a multivalued map with nonempty values such that for each $y_i \in K_i$, $A_i^{-1}(y_i)$ is compactly open in K and $\alpha_i, \beta_i \in \mathbb{R}$ two real numbers such that $\alpha_i \leq \beta_i$. For each $i \in I$, assume that the following conditions hold:*

(i) *for all $x \in K$, $\text{co}A_i(x) \subseteq B_i(x)$;*
(ii) *for all $x \in K$, $P_i(x, x_i) < \alpha_i$ or $Q_i(x, x_i) > \beta_i$;*
(iii) *for all $x \in K$ and for every nonempty finite subset $N_i \subseteq \{y_i \in K_i : F_i(x, y_i) < \alpha_i \text{ or } F_i(x, y_i) > \beta_i\}$, we have*

$$\text{co}N_i \subseteq \{y_i \in K_i : P_i(x, y_i) \geq \alpha_i \text{ and } Q_i(x, y_i) \leq \beta_i\};$$

(iv) *for all $y_i \in K_i$, the set $\{x \in K : \alpha_i \leq F_i(x, y_i) \leq \beta_i\}$ is compactly closed in K;*
(v) *there exist a nonempty compact subset D of K and a nonempty compact convex subset $C_i \subseteq K_i$ for each $i \in I$ such that for all $x \in K \setminus D$, there exist $i \in I$ and $\tilde{y}_i \in C_i$ satisfying $\tilde{y}_i \in A_i(x)$ and either $F_i(x, \tilde{y}_i) < \alpha_i$ or $F_i(x, \tilde{y}_i) > \beta_i$.*

Then there exists a solution $\bar{x} \in K$ *of* $(\text{SQEP})_{\text{LUB}}$.

Proof. For each $i \in I$ and for all $x \in K$, define two multivalued maps $G_i, H_i : K \to 2^{K_i}$ by

$$G_i(x) = \{y_i \in K_i : F_i(x, y_i) < \alpha_i \text{ or } F_i(x, y_i) > \beta_i\}$$

and

$$H_i(x) = \{y_i \in K_i : P_i(x, y_i) \geq \alpha_i \text{ and } Q_i(x, y_i) \leq \beta_i\}.$$

Condition (iii) implies that for each $i \in I$ and for all $x \in K$, $\text{co}G_i(x) \subseteq H_i(x)$. From condition (ii), we have that $x_i \notin H_i(x)$ for all $x \in K$ and for each $i \in I$.

Since for each $i \in I$ and for all $y_i \in K_i$,

$$G_i^{-1}(y_i) = \{x \in K : F_i(x, y_i) < \alpha_i \text{ or } F_i(x, y_i) > \beta_i\}$$

we have the complement of $G_i^{-1}(y_i)$ in K,

$$[G_i^{-1}(y_i)]^c = \{x \in K : \alpha_i \leq F_i(x, y_i) \leq \beta_i\}$$

which is compactly closed in K by virtue of condition (iv). Therefore, for each $i \in I$ and for all $y_i \in K_i$, $G_i^{-1}(y_i)$ is compactly open in K.

For each $i \in I$, define other multivalued maps $S_i, T_i : K \to 2^{K_i}$ by

$$S_i(x) = \begin{cases} G_i(x) \cap A_i(x), & \text{if } x \in \mathscr{F}_i \\ A_i(x), & \text{if } x \in K \backslash \mathscr{F}_i \end{cases}$$

and

$$T_i(x) = \begin{cases} H_i(x) \cap B_i(x), & \text{if } x \in \mathscr{F}_i \\ B_i(x), & \text{if } x \in K \backslash \mathscr{F}_i. \end{cases}$$

Since for each $i \in I$ and for all $x \in K$, $\text{co}G_i(x) \subseteq H_i(x)$ and in view of condition (i), we obtain $\text{co}S_i(x) \subseteq T_i(x)$.

It is easy to see that

$$S_i^{-1}(y_i) = \left(A_i^{-1}(y_i) \cap G_i^{-1}(y_i)\right) \cup ((K \backslash \mathscr{F}_i) \cap A_i^{-1}(y_i))$$

for each $i \in I$ and for all $x \in K$. Since for each $i \in I$ and for all $y_i \in K_i$ $G_i^{-1}(y_i)$, $A_i^{-1}(y_i)$ and $K \backslash \mathscr{F}_i$ are compactly open in K, we have $S_i^{-1}(y_i)$ is compactly open in K. Also, $x_i \notin T_i(x)$ for all $x \in K$ and for each $i \in I$.

Then by Theorem 2.1, there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$. If $\bar{x} \in K \backslash \mathscr{F}_i$, then $A_i(\bar{x}) = S_i(\bar{x}) = \emptyset$ which contradicts with $A_i(x)$ is nonempty for each $i \in I$ and for all $x \in X$. Hence $\bar{x} \in \mathscr{F}_i$ for each $i \in I$. Therefore, $\bar{x}_i \in B_i(\bar{x})$ and $G_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset$ for all $i \in I$. Thus, for each $i \in I$, $\bar{x}_i \in B_i(\bar{x})$ and $\alpha_i \leq F_i(\bar{x}, y_i) \leq \beta_i$ for all $y_i \in A_i(\bar{x})$. □

Theorem 3.2. *For each* $i \in I$*, let* K_i *be a nonempty closed convex subset of a topological vector space* X_i*,* $F_i, P_i, Q_i : K \times K_i \to \mathbb{R}$ *real valued functions,* $B_i : K \to 2^{K_i}$ *a multivalued map such that the set* $\mathscr{F}_i = \{x \in K : x_i \in B_i(x)\}$ *is compactly closed,* $A_i : K \to 2^{K_i}$ *a multivalued map with nonempty values such that for each* $y_i \in K_i$*,* $A_i^{-1}(y_i)$ *is compactly open in* K *and* $\alpha_i, \beta_i \in \mathbb{R}$ *two real numbers such that* $\alpha_i \leq \beta_i$*. Let* Φ *be a measure of noncompactness on* $X = \prod_{i \in I} X_i$*. For each* $i \in I$*, assume that the conditions* (i)–(iv) *of* Theorem 3.1 *hold. Further assume that the multivalued map* $B : K \to 2^K$ *defined as* $B(x) = \prod_{i \in I} B_i(x)$ *for all* $x \in K$*, is* Φ*-condensing. Then there exists a solution* $\bar{x} \in K$ *of* $(\text{SQEP})_{\text{LUB}}$.

Proof. In view of Theorem 2.2, it is sufficient to show that the multivalued map $T : K \to 2^K$ defined as $T(x) = \prod_{i \in I} T_i(x)$ for all $x \in K$, is Φ-condensing, where the T_i's are the same as defined in the proof of Theorem 3.1. By the definition of T_i, $T_i(x) \subseteq B_i(x)$ for each $i \in I$ and for all $x \in K$ and therefore $T(x) \subseteq B(x)$ for all $x \in K$. Since B is Φ-condensing, by Remark 2.1, we have that T is also Φ-condensing. □

Remark 3.1. (a) Condition (iv) of Theorems 3.1 and 3.2 can be replaced by the following stronger condition:

(iv)′ for each $i \in I$ and for all $y_i \in K_i$, the function $F_i(\cdot, y_i) : K \to \mathbb{R}$ is continuous on each compact subset of K.

(b) Conditions (ii) and (iii) of Theorems 3.1 and 3.2 can be replaced simultaneously by the following conditions:

(ii)$'$ for each $i \in I$ and for all $x \in K$, $P_i(x, x_i) \geq \alpha_i$ and $Q_i(x, x_i) \leq \beta_i$; and
(iii)$'$ for all $x \in K$ and for every nonempty finite subset $N_i \subseteq \{y_i \in K_i : F_i(x, y_i) < \alpha_i$ or $F_i(x, y_i) > \beta_i\}$, we have

$$\text{co}N_i \subseteq \{y_i \in K_i : P_i(x, y_i) < \alpha_i \text{ or } Q_i(x, y_i) > \beta_i\}.$$

In this case, the multivalued map H_i in the proof of Theorem 3.1 will be defined as

$$H_i(x) = \{y_i \in K_i : P_i(x, y_i) < \alpha_i \text{ or } Q_i(x, y_i) > \beta_i\}.$$

(c) In Theorems 3.1 and 3.2, if for each $i \in I$, P_i and Q_i are identically equal to F_i, then condition (ii)$'$ becomes $\alpha_i \leq F_i(x, x_i) \leq \beta_i$ for all $x \in K$, and condition (iii)$'$ reduces to the condition that "for all $x \in K$, the set $\{y_i \in K_i : F_i(x, y_i) < \alpha_i \quad \text{or} \quad F_i(x, y_i) < \beta_i\}$ is empty or convex". In view of Corollary 2.4 in [11], it is equivalent to say that the set $\{y_i \in K_i : \alpha_i \leq F_i(x, y_i) \leq \beta_i\}$ is an extremal subset of K.
(d) When I is a singleton set, Theorem 3.1 gives the existence of a solution of $(\text{QEP})_{\text{LUB}}$. In this case, Theorem 3.1 extends and generalizes Theorem 2.2 in [5] and Theorem 3.1 in [11].

Now, we derive the existence results for a solution of $(\text{SGIQEP})_{\text{LUB}}$ by using Theorems 3.1 and 3.2.

Let $\mathcal{Z}$ be topological vector spaces and W a nonempty subset of $\mathcal{Z}$. Let $U : W \to 2^{\mathcal{Z}}$ and $t : W \to \mathcal{Z}$. Recall that t is a *selection* of U on W if $t(x) \in U(x)$ for all $x \in W$. Furthermore the function t is called a *continuous selection* of U on W if it is continuous on W and a selection of U on W.

For results on the existence of a continuous selection, we refer the reader to [14,17,18] and references therein.

Theorem 3.3. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space X_i, X_i^* the topological dual of X_i, K_i^* a nonempty subset of X_i^* $\Psi_i : K_i^* \times K_i \times K_i \to \mathbb{R}$ a real valued function, $B_i : K \to 2^{K_i}$ a multivalued map such that the set $\mathscr{F}_i = \{x \in K : x_i \in B_i(x)\}$ is compactly closed, $A_i : K \to 2^{K_i}$ a multivalued map with nonempty values such that for each $y_i \in K_i$, $A_i^{-1}(y_i)$ is compactly open in K, $U_i : K \to 2^{K_i^*}$ a multivalued map having a selection $t_i : K \to K_i^*$ on K and $\alpha_i, \beta_i \in \mathbb{R}$ two real numbers such that $\alpha_i \leq \beta_i$. For each $i \in I$, assume that the following conditions hold:*

(i) *for all $x \in K$, $\text{co}A_i(x) \subseteq B_i(x)$;*
(ii) *for all $x \in K$ and $u_i \in U_i(x)$, $\Psi_i(u_i, x_i, x_i) < \alpha_i$ or $\Psi_i(u_i, x_i, x_i) > \beta_i$;*
(iii) *for all $x \in K$, $u_i \in U_i(x)$ and for every nonempty finite subset $N_i \subseteq \{y_i \in K_i : \Psi_i(u_i, x_i, y_i) < \alpha_i$ or $\Psi_i(u_i, x_i, y_i) > \beta_i\}$ we have*

$$\text{co}N_i \subseteq \{y_i \in K_i : \alpha_i \leq \Psi_i(u_i, x_i, y_i) \leq \beta_i\};$$

(iv) *for all $y_i \in K_i$, the set $\{x \in K : \alpha_i \leq \Psi_i(t_i(x), x_i, y_i) \leq \beta_i\}$ is compactly closed in K;*
(v) *there exist a nonempty compact subset D of K and a nonempty compact convex subset $C_i \subseteq K_i$ for each $i \in I$ such that for all $x \in K \setminus D$, there exist $i \in I$ and $\tilde{y}_i \in C_i$ satisfying $\tilde{y}_i \in A_i(x)$ and either $\Psi_i(u_i, x_i, \tilde{y}_i) < \alpha_i$ or $\Psi_i(u_i, x_i, \tilde{y}_i) > \beta_i$ for all $u_i \in U_i(x)$.*

Then $(\text{SGIQEP})_{\text{LUB}}$ has a solution.

Proof. Since for each $i \in I$, U_i has a selection t_i on K, we have for all $x \in K$, $t_i(x) \in U_i(x)$. Define a function $F_i : K \times K_i \to \mathbb{R}$ by

$$F_i(x, y_i) = \Psi_i(t_i(x), x_i, y_i), \quad \text{for all } (x, y_i) \in K \times K_i.$$

Then by Theorem 3.1, there exists $\bar{x} \in K$ such that for each $i \in I$,

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad \alpha_i \leq \Psi_i(t_i(\bar{x}), \bar{x}_i, y_i) \leq \beta_i, \quad \text{for all } y_i \in A_i(\bar{x}).$$

For each $i \in I$, set $\bar{u}_i = t_i(\bar{x}) \in U_i(\bar{x})$. Then there exists $\bar{x} \in K$ and $\bar{u} \in U(\bar{x}) = \prod_{i \in I} U_i(\bar{x})$ such that for each $i \in I$,

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad \alpha_i \leq \Psi_i(\bar{u}_i, \bar{x}_i, y_i) \leq \beta_i, \quad \text{for all } y_i \in A_i(\bar{x}).$$

This completes the proof. □

By using the same argument as in the proof of the above theorem and Theorem 3.2, we have the following existence result for a solution of $(\text{SGIQEP})_{\text{LUB}}$ with Φ-condensing multivalued maps B_i in the formulation of the problem.

Theorem 3.4. *For each $i \in I$, let K_i be a nonempty closed convex subset of a topological vector space X_i, X_i^* the topological dual of X_i, K_i^* a nonempty subset of X_i^* $\Psi_i : K_i^* \times K_i \times K_i \to \mathbb{R}$ a real valued function, $B_i : K \to 2^{K_i}$ a multivalued map such that the set $\mathscr{F}_i = \{x \in K : x_i \in B_i(x)\}$ is compactly closed, $A_i : K \to 2^{K_i}$ a multivalued map with nonempty values such that for each $y_i \in K_i$, $A_i^{-1}(y_i)$ is compactly open in K, $U_i : K \to 2^{K_i^*}$ a multivalued map having a selection $t_i : K \to K_i^*$ on K and $\alpha_i, \beta_i \in \mathbb{R}$ two real numbers such that $\alpha_i \le \beta_i$. Let Φ be a measure of noncompactness on $X = \prod_{i \in I} X_i$. For each $i \in I$, assume that the conditions* (i)–(iv) *of Theorem* 3.3 *hold. Further assume that the multivalued map $B : K \to 2^K$ defined as $B(x) = \prod_{i \in I} B_i(x)$ for all $x \in K$ is Φ-condensing. Then* $(\text{SGIQEP})_{\text{LUB}}$ *has a solution.*

Remark 3.2. (a) In Theorems 3.3 and 3.4, we only assumed that the multivalued map U_i has a selection (not necessarily continuous). But in most of the papers that have appeared in the literature, the continuity of the selection is needed; see, for example [14,18] and references therein.
(b) When I is a singleton set, Theorems 3.3 and 3.4 give the existence of a solution of $(\text{GIQEP})_{\text{LUB}}$.

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