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An Iterative Algorithm for Generalized Nonlinear Variational Inclusions

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Abstract—In this paper, we consider the generalized nonlinear variational inclusions for nonclosed and nonbounded valued operators and define an iterative algorithm for finding the approximate solutions of this class of variational inclusions. We also establish that the approximate solutions obtained by our algorithm converge to the exact solution of the generalized nonlinear variational inclusion.

Keywords—Variational inclusions, Generalized variational inequalities, Algorithm, Relaxed Lipschitz operators.

1. INTRODUCTION

In 1994, Hassouni and Moudafi [1] introduced a perturbed method for solving a new class of variational inequalities, known as variational inclusions. Recently, this class of variational inclusions has been extended and generalized for multivalued maps by Huang [2]. By using Hausdorff metric, he constructed an algorithm for finding the approximate solutions of his variational inclusion and proved the convergence of iterative sequences generated by this algorithm. Yao [3] solved a variational inequality involving the single-valued relaxed Lipschitz operators by using an iterative algorithm. Verma [4] generalized single-valued relaxed Lipschitz operators for multivalued maps and studied the solvability of a generalized variational inequality involving single-valued strongly monotone and multivalued relaxed Lipschitz operators. In this paper, we consider the generalized nonlinear variational inclusions with nonclosed and nonbounded valued operators and define an iterative algorithm, without using Hausdorff metric, for finding the approximate solutions of this class of variational inclusions. By the definition of multivalued relaxed Lipschitz operator, we prove that the approximate solutions obtained by this iterative algorithm converge to the exact solution of our variational inclusion.

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2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $\partial\varphi$ denote the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : \mathrm{H} \to \mathbb{R}\cup\{+\infty\}$. Given a multivalued map $T : \mathrm{H} \to 2^{\mathrm{H}}$, where 2^{H} denotes the family of nonempty subsets of H, and $f, g : \mathrm{H} \to \mathrm{H}$ be single-valued maps with $\operatorname{Im} g \cap \operatorname{dom}(\partial\varphi) \neq \emptyset$, we consider the following generalized nonlinear variational inclusion problem (GNVIP).

Find $x \in \mathrm{H}$ and $w \in T(x)$ such that $g(x) \cap \operatorname{dom}(\partial\varphi) \neq \emptyset$ and

$$\langle g(x) - f(w), y - g(x)\rangle \geq \varphi(g(x)) - \varphi(y), \qquad \forall\, y \in \mathrm{H}. \tag{2.1}$$

Inequality (2.1) is called the generalized nonlinear variational inclusion.

If $\varphi \equiv I_K$, the indicator function of a closed convex set K in H defined by

$$I_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

then GNVIP reduces to the following generalized variational inequality problem (GVIP) considered by Verma [4].

Find $x \in \mathrm{H}$ and $w \in T(x)$ such that $g(x) \in K$ and

$$\langle g(x) - f(w), y - g(x)\rangle \geq 0, \qquad \forall\, y \in K. \tag{2.2}$$

Definition 2.1. *(See [5,6].) If $G : \mathrm{H} \to 2^{\mathrm{H}}$ is a maximal monotone multivalued map, then for any fixed $\alpha > 0$, the mapping $J_\alpha^G : \mathrm{H} \to \mathrm{H}$ defined by*

$$J_\alpha^G(x) = (I + \alpha G)^{-1}(x), \qquad \forall\, x \in \mathrm{H}$$

is said to be the resolvent operator of index α of G, where I is the identity mapping on H. Furthermore, the resolvent operator J_α^G is single-valued and nonexpansive, that is,

$$\left\|J_\alpha^G(x) - J_\alpha^G(y)\right\| \leq \|x - y\|, \qquad \forall\, x, y \in \mathrm{H}.$$

Since the subdifferential $\partial\varphi$ of a proper, convex, and lower semicontinuous function $\varphi : \mathrm{H} \to \mathbb{R}\cup\{+\infty\}$ is a maximal monotone multivalued map, it follows that the resolvent operator $J_\alpha^{\partial\varphi}$ of index α of $\partial\varphi$ is given by

$$J_\alpha^{\partial\varphi}(x) = (I + \alpha\partial\varphi)^{-1}(x), \qquad \forall\, x \in \mathrm{H}.$$

3. ITERATIVE ALGORITHM

In this section, we first establish the equivalence of the generalized nonlinear variational inclusion (2.1) to a nonlinear equation. Then we define an iterative algorithm for finding the approximate solutions of GNVIP.

Lemma 3.1. *Elements $x \in \mathrm{H}$ and $w \in T(x)$ are solutions of GNVIP if and only if x and w satisfy the following relation:*

$$g(x) = J_\alpha^{\partial\varphi}(g(x) - \alpha(g(x) - f(w))), \tag{3.1}$$

where $\alpha > 0$ is a constant, $J_\alpha^{\partial\varphi} = (I + \alpha\partial\varphi)^{-1}$ is the resolvent operator of index α of $\partial\varphi$ and I is the identity operator on H.

Proof. From the definition of the resolvent operator $J_\alpha^{\partial\varphi}$ of index α of $\partial\varphi$ and relation (3.1), we have

$$\begin{aligned} g(x) &= J_\alpha^{\partial\varphi}(g(x) - \alpha(g(x) - f(w))) \\ &= (I + \alpha\partial\varphi)^{-1}(g(x) - \alpha(g(x) - f(w))) \end{aligned}$$

and

$$g(x) - \alpha(g(x) - f(w)) \in g(x) + \alpha\partial\varphi(g(x)),$$

which gives $f(w) - g(x) \in \partial\varphi(g(x))$. From the definition of $\partial\varphi$, we have

$$\varphi(y) \geq \varphi(g(x)) + \langle f(w) - g(x), y - g(x)\rangle, \qquad \forall\, y \in \mathrm{H}.$$

Thus, x and w are solutions of GNVIP.

Remark 3.1. From Lemma 3.1, we see that GNVIP is equivalent to the fixed-point problem of type

$$x \in F(x), \tag{3.2}$$

where $F(x) = x - g(x) + J_\alpha^{\partial\varphi}(g(x) - \alpha(g(x) - f(w)))$.

Based on (3.1) and (3.2), we have the following iterative algorithm.

Algorithm 3.1. Given $x_0 \in \mathrm{H}$, compute x_{n+1} by the rule

$$x_{n+1} = x_n - g(x_n) + J_\alpha^{\partial\varphi}(g(x_n) - \alpha(g(x_n) - f(w_n))), \tag{3.3}$$

where $\alpha > 0$ is a constant.

4. CONVERGENCE THEOREM

We apply the Algorithm 3.1 to prove the following convergence theorem.

Theorem 4.1. *Let $g : \mathrm{H} \to \mathrm{H}$ be strongly monotone and Lipschitz continuous with corresponding constants $r \geq 0$ and $s > 0$, respectively, and $f : \mathrm{H} \to \mathrm{H}$ be Lipschitz continuous with constant $t > 0$. Let $T : \mathrm{H} \to 2^{\mathrm{H}}$ be relaxed Lipschitz with respect to f and Lipschitz continuous with corresponding constants $k \leq 0$ and $m \geq 1$, respectively. Then the sequences $\{x_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 with $x_0 \in \mathrm{H}$, $w_0 \in T(x_0)$, and*

$$\left\|\alpha - \frac{1-k+p(1-2p)}{1-2k+m^2t^2-p^2}\right\| < \frac{\sqrt{(1-k+p(1-2p))^2 - 4p(1-p)(1-2k-p^2+t^2m^2)}}{1-2k+m^2t^2-p^2}, \tag{4.1}$$

where $1-k > p(2p-1) + \sqrt{4p(1-p)(1-2k-p^2+t^2m^2)}$ and $1-2k+m^2t^2 > p^2$ for $p = \sqrt{(1-2r+s^2)} < 1/2$, converge to x and w, respectively, the solution of GNVIP.

We require the following definitions to achieve the main result.

Definition 4.1. *A mapping $g : \mathrm{H} \to \mathrm{H}$ is said to be*

(i) *strongly monotone, if there exists a constant $r \geq 0$ such that*

$$\langle g(x_1) - g(x_2), x_1 - x_2\rangle \geq r\|x_1 - x_2\|^2, \qquad \forall\, x_1, x_2 \in \mathrm{H};$$

(ii) *Lipschitz continuous, if there exists a constant $s > 0$ such that*

$$\|g(x_1) - g(x_2)\| \leq s\|x_1 - x_2\|, \qquad \forall\, x_1, x_2 \in \mathrm{H}.$$

Definition 4.2. *Let $f : \mathrm{H} \to \mathrm{H}$ be a map. A multivalued map $T : \mathrm{H} \to 2^{\mathrm{H}}$ is said to be relaxed Lipschitz with respect to f, if for given $k \leq 0$,*

$$\langle f(w_1) - f(w_2), x_1 - x_2\rangle \leq k\|x_1 - x_2\|^2, \qquad \forall\, w_1 \in T(x_1), \quad w_2 \in T(x_2), \quad \text{and} \quad \forall\, x_1, x_2 \in \mathrm{H}.$$

The multivalued map T is called *Lipschitz continuous* [4], if for $m \geq 1$,

$$\|w_1 - w_2\| \leq m\|x_1 - x_2\|, \qquad \forall\, w_1 \in T(x_1), \quad w_2 \in T(x_2), \quad \text{and} \quad \forall\, x_1, x_2 \in \mathrm{H}.$$

Proof of Theorem 4.1. From (3.3), we have

$$\|x_{n+1}-x_n\| = \left\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) + J_\alpha^{\partial\varphi}(h(x_n)) - J_\alpha^{\partial\varphi}(h(x_{n-1}))\right\|, \tag{4.2}$$

where $h(x_n) = g(x_n) - \alpha(g(x_n) - f(w_n))$. Since the resolvent operator $J_\alpha^{\partial\varphi}$ is nonexpansive, we have

$$\begin{aligned}\left\|J_\alpha^{\partial\varphi}(h(x_n)) - J_\alpha^{\partial\varphi}(h(x_{n-1}))\right\| &\le \|h(x_n) - h(x_{n-1})\| \\ &= \|(1-\alpha)(g(x_n) - g(x_{n-1})) + \alpha(f(w_n) - f(w_{n-1}))\| \\ &\le (1-\alpha)\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ &\quad + \|(1-\alpha)(x_n - x_{n-1}) + \alpha(f(w_n) - f(w_{n-1}))\|.\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we get

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \left\|J_\alpha^{\partial\varphi}(h(x_n)) - J_\alpha^{\partial\varphi}(h(x_{n-1}))\right\| \\ &\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ &\quad + (1-\alpha)\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ &\quad + \|(1-\alpha)(x_n - x_{n-1}) + \alpha(f(w_n) - f(w_{n-1}))\| \\ &= (2-\alpha)\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ &\quad + \|(1-\alpha)(x_n - x_{n-1}) + \alpha(f(w_n) - f(w_{n-1}))\|.\end{aligned} \tag{4.4}$$

By Lipschitz continuity and strong monotonicity of g, we obtain

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \ \le\ \left(1 - 2r + s^2\right)\|x_n - x_{n-1}\|^2. \tag{4.5}$$

Since T is Lipschitz continuous and relaxed Lipschitz with respect to f and f is Lipschitz continuous, we have

$$\begin{aligned}&\|(1-\alpha)(x_n - x_{n-1}) + \alpha(f(w_n) - f(w_{n-1}))\|^2 \\ &\quad = (1-\alpha)^2\|x_n - x_{n-1}\|^2 + 2\alpha(1-\alpha)\langle f(w_n) - f(w_{n-1}), x_n - x_{n-1}\rangle \\ &\qquad + \alpha^2\|f(w_n) - f(w_{n-1})\|^2 \\ &\quad \le (1-\alpha)^2\|x_n - x_{n-1}\|^2 + 2\alpha(1-\alpha)k\ \|x_n - x_{n-1}\|^2 + \alpha^2t^2m^2\|x_n - x_{n-1}\|^2 \\ &\quad = \left((1-\alpha)^2 + 2\alpha(1-\alpha)k + \alpha^2t^2m^2\right)\|x_n - x_{n-1}\|^2.\end{aligned}$$

By combining (4.4)–(4.6), we obtain

$$\|x_{n+1}-x_n\| \ \le \theta\|x_n - x_{n-1}\|,$$

where $\theta = [(2-\alpha)p + \{(1-\alpha)^2 + 2\alpha(1-\alpha)k + \alpha^2t^2m^2\}^{1/2}]$ and $p = (1-2r+s^2)^{1/2}$. It follows from (4.1) that $\theta < 1$, and consequently, for all $q \in \mathbb{N}$,

$$\|x_{n+q} - x_n\| \ \le \left(\frac{\theta^n}{1-\theta}\right)\|x_1 - x_0\|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since H is complete, there exists an $x \in$ H such that $x_n \to x$. Now the Lipschitz continuity of T implies that $w_n \to w$. This completes the proof.

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