



# Existence of solutions of systems of generalized implicit vector quasi-equilibrium problems

Qamrul Hasan Ansari<sup>a,b,\*,1</sup>

<sup>a</sup> *Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, PO Box 1169, Dhahran, Saudi Arabia*

<sup>b</sup> *Department of Mathematics, Aligarh Muslim University, Aligarh, India*

Received 30 May 2007

Available online 22 November 2007

Submitted by A. Dontchev

---

## Abstract

We consider five different types of systems of generalized vector quasi-equilibrium problems and establish relationships among them by using different kinds of generalized pseudomonotonicities. We prove the existence of their solutions under lower semicontinuity for a family of multivalued maps involved in the formulation of these problems. The existence of solutions of these problems is also investigated without any coercivity condition but for  $\Phi$ -condensing maps. We also establish some existence results for solutions of these problems under pseudomonotonicities in the setting of Hausdorff topological vector spaces as well as real Banach spaces.

© 2007 Elsevier Inc. All rights reserved.

**Keywords:** Systems of generalized implicit vector quasi-equilibrium problems; Generalized pseudomonotonicity; Existence results;  $u$ -Hemicontinuity;  $\mathcal{H}$ -hemicontinuity; Lower semicontinuity;  $\Phi$ -condensing maps

---

## 1. Introduction and formulations

In the last decade, systems of (vector) quasi-equilibrium problems are used as tools to establish the existence of a solution of constrained (vector) Nash equilibrium problem, also known as Debreu type (vector) equilibrium problem [15], both for nondifferentiable and (non)convex (vector valued) functions. These are also used to solve mathematical programs with equilibrium constraint [28], fixed point problem for a family of nonexpansive multivalued maps [26] and several related topics. By using different types of maximal element theorems for a family of multivalued maps and different types of fixed point theorems for a multivalued map, several authors studied the existence of solutions of different kinds of systems of (vector) quasi-equilibrium problems. See, for example, [3–7,16,18–20,26,28–30,36,37] and references therein.

---

\* Address for correspondence: Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, PO Box 1169, Dhahran, Saudi Arabia.

*E-mail address:* [qhansari@kfupm.edu.sa](mailto:qhansari@kfupm.edu.sa).

<sup>1</sup> The author expresses his thanks to the Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia for providing excellent research facilities.

Let  $I$  be any (countable or uncountable) index set. For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a Hausdorff topological vector space  $X_i$ . Throughout this paper, unless otherwise specified,  $K = \prod_{i \in I} K_i$  and  $X = \prod_{i \in I} X_i$ . For each  $i \in I$ , let  $Y_i$  be a topological vector space,  $L(X_i, Y_i)$  the space of all continuous linear operators from  $X_i$  to  $Y_i$ ,  $D_i$  a nonempty subset of  $L(X_i, Y_i)$ ,  $C_i : K \rightarrow 2^{Y_i}$  a multivalued map such that for all  $x \in K$ ,  $C_i(x)$  is a proper, closed and convex cone with apex at the origin and  $\text{int } C_i(x) \neq \emptyset$ , and  $W_i : K \rightarrow 2^{Y_i}$  a multivalued map defined as  $W_i(x) = Y_i \setminus (-\text{int } C_i(x))$  for all  $x \in K$  such that its graph is closed, where  $\text{int } C_i$  and  $2^{Y_i}$  denote the interior of  $C_i$  and the family of all subsets of  $Y_i$ , respectively. For each  $i \in I$ , let  $F_i : K_i \rightarrow 2^{Y_i}$  be a multivalued map with nonempty values,  $A_i : K \rightarrow 2^{K_i}$  a multivalued map with nonempty convex values such that  $A(x) = \prod_{i \in I} A_i(x)$ , and  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  a function. We consider the following *Systems of Generalized Implicit Vector Quasi-Equilibrium Problems* (in short, SGIVQEP):

**Problem 1.** Find  $\bar{x} \in K$  such that  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\forall \bar{u}_i \in F_i(\bar{x}): \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}), \quad \forall y_i \in A_i(\bar{x}).$$

**Problem 2.** Find  $\bar{x} \in K$  such that  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\exists \bar{u}_i \in F_i(\bar{x}): \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}), \quad \forall y_i \in A_i(\bar{x}).$$

**Problem 3.** Find  $\bar{x} \in K$  such that  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\forall y_i \in A_i(\bar{x}), \exists \bar{u}_i \in F_i(\bar{x}) \quad (\bar{u}_i \text{ depends on } y_i): \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}).$$

**Problem 4.** Find  $\bar{x} \in K$  such that  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\forall y \in A(\bar{x}) \text{ and } \forall v_i \in F_i(y): \quad \psi_i(v_i, y_i, \bar{x}_i) \notin \text{int } C_i(\bar{x}),$$

where  $y_i$  is the  $i$ th component of  $y$ .

**Problem 5.** Find  $\bar{x} \in K$  such that  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\forall y \in A(\bar{x}), \exists v_i \in F_i(y) \quad (v_i \text{ depends on } y): \quad \psi_i(v_i, y_i, \bar{x}_i) \notin \text{int } C_i(\bar{x}),$$

where  $y_i$  is the  $i$ th component of  $y$ .

**Remark 1.1.** Problem 1  $\Rightarrow$  Problem 2  $\Rightarrow$  Problem 3 and Problem 4  $\Rightarrow$  Problem 5.

The solutions of Problems 1, 2 and 3 are called general solution, strong solution and weak solution, respectively. In view of Remark 1.1, every general solution is a strong solution and every strong solution is a weak solution. But the converse assertions may not be true.

When  $A_i(x) = K_i$  for all  $x \in K$  and for each  $i \in I$ , Problems 1–5 are called *systems of generalized implicit vector equilibrium problems* (in short, SGIVEP) considered and studied by Al-Homidan et al. [1]. In this case, the existence results for solutions of these problems are investigated by introducing different kinds of generalized pseudomonotonicities. In this case, Nash equilibrium problem for vector valued functions can be solved by using Problems 1–5 but not constrained Nash equilibrium problem for vector valued functions.

Problem 3 was first considered and studied in [6]. We established the existence of a solution of Problem 3 without assuming any monotonicity condition. We showed that if for each  $i \in I$ ,  $\psi_i(u_i, x_i, y_i) = \langle u_i, \eta_i(y_i, x_i) \rangle$ , where  $\eta_i : K_i \times K_i \rightarrow X_i$  and  $\langle s_i, x_i \rangle$  denotes the evaluation of  $s_i \in L(X_i, Y_i)$  at  $x_i \in X_i$ , Problem 3 provides a sufficient condition (which is in general not necessary) for a solution of system of vector quasi-optimization problems which includes constrained Nash equilibrium problem for nondifferentiable and nonconvex functions. But, in this case, Problem 2 provides necessary and sufficient conditions for a solution of system of vector quasi-optimization problems.

If for each  $i \in I$ ,  $A_i(x) = K_i$  for all  $x \in K$ , Problem 3 is called *system of generalized implicit vector equilibrium problems* and it is introduced and studied in [8]. It is also used to give the existence of a solution of Nash equilibrium problem for nondifferentiable and nonconvex functions. Further, if  $Y_i = \mathbb{R}$  and  $C_i(x) = \mathbb{R}_-$  and  $A_i(x) = K_i$  for all  $x \in K$ , Problem 3 was studied in [10]. As an application of their results, we established some existence results for solutions of systems of optimization problems and Nash equilibrium problem [33].

When  $I$  is a singleton set,  $Y_i = \mathbb{R}$  and  $C_i(x) = \mathbb{R}_+$  for all  $x \in K$ , the existence of a solution of Problem 2 is studied in [21].

When  $I$  is a singleton set,  $A_i(x) = K_i$  for all  $x \in K$  and  $\psi_i(u_i, x_i, y_i) = \langle u_i, \eta_i(y_i, x_i) \rangle$  (respectively  $\psi_i(u_i, x_i, y_i) = \langle u_i, y_i - x_i \rangle$ ), then Problem 2 provides necessary and sufficient conditions for solutions of vector optimization problems for nondifferentiable and nonconvex functions (respectively for nondifferentiable, but convex functions). See, for example, [2,9] and references therein. In this case, Problem 1 is considered and studied in [2,14, 24].

When  $I$  is a singleton set, Problems 2 and 3 are studied by Kum and Lee [25,31]. They proved the existence of solutions of these problems under some kind of pseudomonotonicity assumptions.

In the next section, we recall some known definitions and results which will be used in the sequel. In Section 3, we establish some relationships among Problems 1–5 by using different kinds of generalized pseudomonotonicities. Section 4 is devoted to the existence results for a solution of Problem 1 under lower semicontinuity of the family of multivalued maps involved in the formulation of the problem. The existence of a solution of Problem 1 and so Problems 2 and 3 without any coercivity condition but for  $\Phi$ -condensing maps is also established. In Section 5, we establish the existence of a strong solution of our SGVQEP by using  $\mathcal{H}$ -hemicontinuity assumption in the setting of real Banach spaces. We also prove the existence of a weak solution under generalized pseudomonotonicity and  $u$ -hemicontinuity assumptions. Basically, besides establishing existence results for solutions of Problems 1–3 without any coercivity condition but for  $\Phi$ -condensing maps, we extend the results of [1] for SGIVEP to SGIVQEP. Our results provide the existence of solutions of Problems 1–5 under some kind of pseudomonotonicity assumption and under lower semicontinuity assumption which is one of main motivations of this paper.

## 2. Preliminaries

**Definition 2.1.** (See [12].) Let  $\mathcal{X}$  and  $\mathcal{Y}$  be topological spaces. A multivalued map  $T : \mathcal{X} \rightarrow 2^{\mathcal{Y}}$  is called *upper semicontinuous* at  $x_0 \in \mathcal{X}$  if for any open set  $V$  in  $\mathcal{Y}$  containing  $T(x_0)$ , there exists an open neighborhood  $U$  of  $x_0$  in  $\mathcal{X}$  such that  $T(x) \subseteq V$  for all  $x \in U$ .

$T$  is called *lower semicontinuous* at  $x \in \mathcal{X}$  if for any  $y \in T(x)$  and for any  $x_n \in \mathcal{X}$  such that  $x_n \rightarrow x$ , there exists  $y_n \in T(x_n)$  such that  $y_n \rightarrow y$ .

It is said to be *upper (lower) semicontinuous on  $\mathcal{X}$*  if it is upper (lower) semicontinuous at every point  $x \in \mathcal{X}$ .

Further,  $T$  is said to be *continuous on  $X$*  if it is upper semicontinuous as well as lower semicontinuous on  $X$ .

**Lemma 2.1.** (See [12].)  $T$  is lower semicontinuous at  $x \in \mathcal{X}$  if and only if for any  $y \in T(x)$  and for any  $x_n \in \mathcal{X}$  such that  $x_n \rightarrow x$ , there exists  $y_n \in T(x_n)$  such that  $y_n \rightarrow y$ .

**Lemma 2.2.** (See [32].) Let  $(E, \|\cdot\|)$  be a normed vector space and  $\mathcal{H}$  be a Hausdorff metric on the collection  $\mathcal{CB}(E)$  of all nonempty, closed and bounded subsets of  $E$ ,

$$\mathcal{H}(U, V) = \max \left\{ \sup_{x \in U} \inf_{y \in V} \|x - y\|, \sup_{y \in V} \inf_{x \in U} \|x - y\| \right\},$$

for all  $U, V \in \mathcal{CB}(E)$ . If  $U$  and  $V$  are compact sets in  $E$ , then for all  $x \in U$ , there exists  $y \in V$  such that

$$\|x - y\| \leq \mathcal{H}(U, V).$$

**Definition 2.2.** (See [32].) Let  $(E, d)$  be a metric space and  $\mathcal{H}$  be a Hausdorff metric on  $\mathcal{CB}(E)$ . A multivalued map  $T : E \rightarrow \mathcal{CB}(E)$  is said to be *continuous* (in the sense of Nadler) on  $E$  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in E$

$$\mathcal{H}(T(x), T(y)) < \varepsilon \quad \text{whenever } d(x, y) < \delta.$$

**Remark 2.1.** The notion of continuity in the sense of Definitions 2.1 and 2.2 are equivalent if  $T$  is compact valued.

**Definition 2.3.** (See [38].) Let  $\Omega$  be a nonempty convex subset of a normed space  $(E, \|\cdot\|)$  and  $\mathcal{Y}$  be a normed linear space. A nonempty compact-valued multifunction  $T : \Omega \rightarrow 2^{L(E, \mathcal{Y})}$  is said to be  *$\mathcal{H}$ -hemicontinuous* if for any

$x, y \in \Omega$ , the mapping  $\alpha \mapsto \mathcal{H}(T(x + \alpha(y - x)), T(x))$  is continuous at  $0^+$ , where  $\mathcal{H}$  is the Hausdorff metric defined on  $\mathcal{CB}(E)$ .

**Definition 2.4.** (See [34,35].) Let  $E$  be a Hausdorff topological vector space and  $L$  a lattice with least element, denoted by  $\mathbf{0}$ . A mapping  $\Phi : 2^E \rightarrow L$  is called a *measure of noncompactness* provided that the following conditions hold for any  $M, N \in 2^E$ :

- (i)  $\Phi(M) = \mathbf{0}$  if and only if  $M$  is precompact (i.e., it is relatively compact).
- (ii)  $\Phi(\overline{\text{co}}\overline{\text{nv}} M) = \Phi(M)$ , where  $\overline{\text{co}}\overline{\text{nv}} M$  denotes the closed convex hull of  $M$ .
- (iii)  $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$ .

It follows from (iii) that if  $M \subseteq N$ , then  $\Phi(M) \leq \Phi(N)$ .

**Definition 2.5.** (See [34,35].) Let  $\Phi : 2^E \rightarrow L$  be a measure of noncompactness on  $E$  and  $D \subseteq E$ . A multivalued map  $T : D \rightarrow 2^E$  is called  $\Phi$ -*condensing* provided that if  $M \subseteq D$  with  $\Phi(T(M)) \geq \Phi(M)$ , then  $M$  is relatively compact.

**Remark 2.2.** Note that every multivalued map defined on a compact set is necessarily  $\Phi$ -condensing. If  $E$  is locally convex, then a compact multivalued map (i.e.,  $T(D)$  is precompact) is  $\Phi$ -condensing for any measure of noncompactness  $\Phi$ . Obviously, if  $T : D \rightarrow 2^E$  is  $\Phi$ -condensing and if  $T' : D \rightarrow 2^E$  satisfies  $T'(x) \subseteq T(x)$  for all  $x \in D$ , then  $T'$  is also  $\Phi$ -condensing.

The following particular form of a maximal element theorem for a family of multivalued maps due to Lin and Ansari (Corollary 4.4 in [27]) is the main tool to establish the existence of solutions of Problems 1–5.

**Theorem 2.1.** (See [27].) For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a Hausdorff topological vector space  $X_i$ . For each  $i \in I$ , let  $S_i, T_i : K \rightarrow 2^{K_i}$  be multivalued maps satisfying the following conditions:

- (i) For each  $i \in I$  and for all  $x \in K$ ,  $\text{co } S_i(x) \subseteq T_i(x)$ , where  $\text{co } S_i(x)$  denotes the convex hull of  $S_i(x)$ ;
- (ii) For each  $i \in I$  and for all  $x = (x_i)_{i \in I} \in K$ ,  $x_i \notin T_i(x)$ , where  $x_i$  is the  $i$ th component of  $x$ ;
- (iii) For each  $i \in I$  and for all  $y_i \in K_i$ ,  $S_i^{-1}(y_i) = \{x \in K : y_i \in S_i(x)\}$  is open in  $K$ ;
- (iv) There exist a nonempty compact subset  $M$  of  $K$  and a nonempty compact convex subset  $N_i$  of  $K_i$  for each  $i \in I$  such that for all  $x \in K \setminus M$ , there exists  $i \in I$  such that  $S_i(x) \cap N_i \neq \emptyset$ .

Then there exists  $\bar{x} \in K$  such that  $S_i(\bar{x}) = \emptyset$  for all  $i \in I$ .

**Remark 2.3.** If for each  $i \in I$ ,  $K_i$  is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space  $X_i$ , then condition (iv) of Theorem 2.1 can be replaced by the following condition:

(iv)' The multivalued map  $S : K \rightarrow 2^K$  defined as  $S(x) := \prod_{i \in I} S_i(x)$  for all  $x \in K$ , is  $\Phi$ -condensing.

(See Corollary 4 in [13].)

### 3. Relationships among Problems 1–5

We recall different kinds of generalized pseudomonotonicities introduced in [1].

**Definition 3.1.** (See [1].) Let  $\{\psi_i\}_{i \in I}$  be a family of mappings  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$ . A family  $\{F_i\}_{i \in I}$  of multivalued maps  $F_i : K \rightarrow 2^{K_i}$  with nonempty values is called:

- (i) *generalized strongly pseudomonotone w.r.t.  $\{\psi_i\}_{i \in I}$*  if for all  $x, y \in K$  and for each  $i \in I$ ,

$$\forall u_i \in F_i(x) : \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x) \quad \Rightarrow \quad \forall v_i \in F_i(y) : \psi_i(v_i, y_i, x_i) \notin \text{int } C_i(x);$$

(ii) *generalized pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$  if for all  $x, y \in K$  and for each  $i \in I$ ,

$$\exists u_i \in F_i(x): \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x) \Rightarrow \forall v_i \in F_i(y): \psi_i(v_i, y_i, x_i) \notin \text{int } C_i(x);$$

(iii) *generalized weakly pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$  if for all  $x, y \in K$  and for each  $i \in I$ ,

$$\exists u_i \in F_i(x): \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x) \Rightarrow \exists v_i \in F_i(y): \psi_i(v_i, y_i, x_i) \notin \text{int } C_i(x);$$

(iv) *generalized pseudomonotone<sup>+</sup>* w.r.t.  $\{\psi_i\}_{i \in I}$  if for all  $x, y \in K$  and for each  $i \in I$ ,

$$\forall u_i \in F_i(x): \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x) \Rightarrow \exists v_i \in F_i(y): \psi_i(v_i, y_i, x_i) \notin \text{int } C_i(x);$$

(v) *u-hemicontinuous* w.r.t.  $\{\psi_i\}_{i \in I}$  if for all  $x, y \in K$  and  $\alpha \in [0, 1]$  and for each  $i \in I$ , the multivalued map

$$\alpha \mapsto \psi_i(F_i(x + \alpha(y - x)), x_i, y_i)$$

is upper semicontinuous at  $0^+$ , where

$$\psi_i(F_i(x + \alpha(y - x)), x_i, y_i) = \{\psi_i(w_i, x_i, y_i): w_i \in F_i(x + \alpha(y - x))\}.$$

**Remark 3.1.** Definition (i)  $\Rightarrow$  Definition (ii)  $\Rightarrow$  Definition (iii); Definition (iv)  $\Rightarrow$  Definition (iii); Definition (i)  $\Rightarrow$  Definition (iv); that is, Definition (i)  $\Rightarrow$  Definition (iv)  $\Rightarrow$  Definition (iii).

In the next three lemmas, we discuss the relationships among Problems 1–5.

**Lemma 3.1.**

- (a) Problem 3  $\Rightarrow$  Problem 4 if  $\{F_i\}_{i \in I}$  is *generalized pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$ .
- (b) Problem 3  $\Rightarrow$  Problem 5 if  $\{F_i\}_{i \in I}$  is *generalized weakly pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$ .
- (c) Problem 1  $\Rightarrow$  Problem 5 if  $\{F_i\}_{i \in I}$  is *generalized pseudomonotone<sup>+</sup>* w.r.t.  $\{\psi_i\}_{i \in I}$ .
- (d) Problem 1  $\Rightarrow$  Problem 4 if  $\{F_i\}_{i \in I}$  is *generalized strongly pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$ .
- (e) Problem 2  $\Rightarrow$  Problem 4 if  $\{F_i\}_{i \in I}$  is *generalized pseudomonotone* w.r.t.  $\{\psi_i\}_{i \in I}$ .

**Lemma 3.2.** For each  $i \in I$ , assume that the following conditions hold:

- (i) For all  $x \in K$  and all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \in C_i = \bigcap_{x \in K} C_i(x)$ ;
- (ii) For all  $x \in K$  and all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, \cdot)$  is  $C_i$ -convex, that is, for all  $s_i \in L(X_i, Y_i)$ ,  $x, y \in K$  and  $\alpha \in [0, 1]$ ,
 
$$\psi_i(s_i, x_i, \alpha x_i + (1 - \alpha)y_i) \in \alpha \psi_i(s_i, x_i, x_i) + (1 - \alpha)\psi_i(s_i, x_i, y_i) - C_i;$$
- (iii) For all  $s_i \in L(X_i, Y_i)$ ,  $x, y, z \in K$  and  $\alpha \in [0, 1]$ ,
 
$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1 - \alpha)\psi_i(s_i, x_i, z_i);$$
- (iv)  $\{F_i\}_{i \in I}$  is *u-hemicontinuous* w.r.t.  $\{\psi_i\}_{i \in I}$ .

Then Problem 5  $\Rightarrow$  Problem 3 as well as Problem 4  $\Rightarrow$  Problem 3.

**Proof.** We first prove that Problem 5  $\Rightarrow$  Problem 3.

Let  $\bar{x} \in K$  be a solution of Problem 5. Then  $\bar{x} \in A(\bar{x})$ . Suppose to the contrary that  $\bar{x} \in A(\bar{x})$  is not a solution of Problem 3. Then there exist an  $i \in I$  and  $\hat{y}_i \in A_i(\bar{x})$  such that for all  $\bar{u}_i \in F_i(\bar{x})$ , we have

$$\psi_i(\bar{u}_i, \bar{x}_i, \hat{y}_i) \in -\text{int } C_i(\bar{x}). \tag{3.1}$$

Let  $x_i^\alpha = \bar{x}_i + \alpha(\hat{y}_i - \bar{x}_i)$  for  $\alpha \in [0, 1]$ . Since each  $A_i(\bar{x})$  is convex, we have  $x_i^\alpha \in A_i(\bar{x})$  and so we can let  $x^\alpha = (\bar{x}_1, \dots, x_i^\alpha, \dots) \in A(\bar{x})$  such that its  $i$ th component is  $x_i^\alpha$  and the rest of the components are  $\bar{x}_j$  for all  $j \in I, j \neq i$ .

Define a multivalued map  $H_i : [0, 1] \rightarrow 2^{Y_i}$  by

$$H_i(\alpha) = \{\psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i): u_i^\alpha \in F_i(x^\alpha)\}.$$

Then from (3.1)

$$H_i(0) = \psi_i(F_i(\bar{x}), \bar{x}_i, \hat{y}_i) \subseteq -\text{int } C_i(\bar{x}).$$

Since  $\{F_i\}_{i \in I}$  is  $u$ -hemicontinuous w.r.t.  $\{\psi_i\}_{i \in I}$ , there exists  $\delta \in (0, 1]$  such that for all  $\alpha \in (0, \delta)$ ,

$$H_i(\alpha) \subseteq -\text{int } C_i(\bar{x}).$$

Therefore, for all  $\alpha \in (0, \delta)$  and all  $u_i^\alpha \in F_i(x^\alpha)$ , we have

$$\psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i) \in -\text{int } C_i(\bar{x}). \quad (3.2)$$

Fix  $\alpha \in (0, \delta)$ . Then by conditions (i)–(iii), we have for all  $u_i^\alpha \in F_i(x^\alpha)$

$$\psi_i(u_i^\alpha, x_i^\alpha, x_i^\alpha) \in \alpha \psi_i(u_i^\alpha, x_i^\alpha, \hat{y}_i) + (1 - \alpha) \psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) - C_i,$$

or

$$\begin{aligned} -(1 - \alpha) \psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) &\in \alpha \psi_i(u_i^\alpha, x_i^\alpha, \hat{y}_i) - \psi_i(u_i^\alpha, x_i^\alpha, x_i^\alpha) - C_i \\ &\subseteq \alpha(1 - \alpha) \psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i) - C_i - C_i \\ &\subseteq -\text{int } C_i(\bar{x}) - C_i(\bar{x}) - C_i(\bar{x}) \\ &\subseteq -\text{int } C_i(\bar{x}). \end{aligned}$$

Thus for all  $x^\alpha \in A(\bar{x})$ , we have  $\psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) \in \text{int } C_i(\bar{x})$  for all  $u_i^\alpha \in F_i(x^\alpha)$  which contradicts our supposition that  $\bar{x}$  is a solution of Problem 5. This completes the proof.

The proof of the second part lies on the lines of the proof of the first part. Therefore we omit it.  $\square$

**Proposition 3.1.** *Under the conditions of Lemma 3.1(a) and Lemma 3.2, Problems 3–5 are equivalent.*

**Lemma 3.3.** *For each  $i \in I$ , let  $(X_i, \|\cdot\|)$  and  $Y_i$  be real Banach spaces and  $K_i$  be a nonempty convex subset of  $X_i$ . For each  $i \in I$ , assume that the following conditions hold:*

- (i) For all  $x \in K$  and all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \in C_i = \bigcap_{x \in K} C_i(x)$ ;
- (ii) For all  $x \in K$  and all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, \cdot)$  is  $C_i$ -convex, that is, for all  $s_i \in L(X_i, Y_i)$ ,  $x, y \in K$  and  $\alpha \in [0, 1]$ ,

$$\psi_i(s_i, x_i, \alpha x_i + (1 - \alpha)y_i) \in \alpha \psi_i(s_i, x_i, x_i) + (1 - \alpha) \psi_i(s_i, x_i, y_i) - C_i;$$

- (iii) For all  $s_i \in L(X_i, Y_i)$ ,  $x, y, z \in K$  and  $\alpha \in [0, 1]$ ,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1 - \alpha) \psi_i(s_i, x_i, z_i);$$

- (iv)  $\psi_i$  is continuous in the first argument;

- (v)  $F_i$  is  $\mathcal{H}$ -hemicontinuous and for all  $x \in K$ ,  $F_i(x)$  is a nonempty compact set in  $Y_i$ ;

- (vi) The family  $\{F_i\}_{i \in I}$  is generalized pseudomonotone w.r.t.  $\{\psi_i\}_{i \in I}$ .

Then Problems 2 and 4 are equivalent.

**Proof.** Problem 2  $\Rightarrow$  Problem 4 follows from condition (vi).

Problem 4  $\Rightarrow$  Problem 2: Let  $\bar{x} \in K$  be a solution of Problem 4. Then  $\bar{x} \in A(\bar{x})$  and for each  $i \in I$ ,

$$\forall y \in A(\bar{x}) \text{ and } \forall v_i \in F_i(y): \quad \psi_i(v_i, y_i, \bar{x}_i) \notin \text{int } C_i(\bar{x}), \quad (3.3)$$

where  $y_i$  is the  $i$ th component of  $y$ . For any given  $y \in A(\bar{x})$ , we know that  $y^\alpha = \alpha y + (1 - \alpha)\bar{x} \in A(\bar{x})$  for all  $\alpha \in (0, 1)$  since each  $A_i(\bar{x})$  is convex and so  $A(\bar{x})$ . Then from (3.3), we have

$$\forall i \in I, \forall v_i^\alpha \in F_i(y^\alpha): \quad \psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) \notin \text{int } C_i(\bar{x}),$$

where  $y_i^\alpha$  is the  $i$ th component of  $y^\alpha$ . Then we have

$$\forall i \in I, \forall v_i^\alpha \in F_i(y^\alpha): \quad \psi_i(v_i^\alpha, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}). \quad (3.4)$$

Suppose that (3.4) does not hold. Then there exist an  $i \in I$  and  $v_i^\alpha \in F_i(y^\alpha)$  such that

$$\psi_i(v_i^\alpha, \bar{x}_i, y_i) \in -\text{int } C_i(\bar{x}). \tag{3.5}$$

Fix this  $i$  and  $v_i^\alpha \in F_i(y^\alpha)$ . Then by conditions (i)–(iii), we have

$$\psi_i(v_i^\alpha, y_i^\alpha, y_i) \in \alpha \psi_i(v_i^\alpha, y_i^\alpha, y_i) + (1 - \alpha) \psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) - C_i,$$

or

$$\begin{aligned} -(1 - \alpha) \psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) &\in \alpha \psi_i(v_i^\alpha, y_i^\alpha, y_i) - \psi_i(v_i^\alpha, y_i^\alpha, y_i^\alpha) - C_i \\ &\subseteq \alpha(1 - \alpha) \psi_i(v_i^\alpha, \bar{x}_i, y_i) - C_i - C_i \\ &\subseteq -\text{int } C_i(\bar{x}) - C_i(\bar{x}) - C_i(\bar{x}) \\ &\subseteq -\text{int } C_i(\bar{x}). \end{aligned}$$

Therefore,  $\psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) \in \text{int } C_i(\bar{x})$  which contradicts to (3.3), and hence (3.4) holds.

Since  $F_i(y^\alpha)$  and  $F_i(\bar{x})$  are compact. From Lemma 2.1 we have that for each fixed  $v_i^\alpha \in F_i(y^\alpha)$ , there exists  $u_i^\alpha \in F_i(\bar{x})$  such that

$$\|v_i^\alpha - u_i^\alpha\| \leq \mathcal{H}(F_i(y^\alpha), F_i(\bar{x})).$$

Since each  $F_i(\bar{x})$  is compact, without loss of generality, we may assume that  $u_i^\alpha \rightarrow \bar{u}_i \in F_i(\bar{x})$  as  $\alpha \rightarrow 0^+$ . Since for each  $i \in I$ ,  $F_i$  is  $\mathcal{H}$ -hemicontinuous,

$$\mathcal{H}(F_i(y^\alpha), F_i(\bar{x})) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0^+.$$

Therefore,

$$\|v_i^\alpha - \bar{u}_i\| \leq \|v_i^\alpha - u_i^\alpha\| + \|u_i^\alpha - \bar{u}_i\| \leq \mathcal{H}(F_i(y^\alpha), F_i(\bar{x})) + \|u_i^\alpha - \bar{u}_i\| \rightarrow 0 \quad \text{as } \alpha \rightarrow 0^+.$$

Since for each  $i \in I$ ,  $\psi_i$  is continuous in the first argument and  $W_i(\bar{x})$  is closed, we have

$$\psi_i(\bar{u}_i, \bar{x}_i, y_i) \in Y_i \setminus \{-\text{int } C_i(\bar{x})\} = W_i(\bar{x}) \Leftrightarrow \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}),$$

that is,  $\bar{x}$  is a solution of Problem 1.  $\square$

**Remark 3.2.** If  $A_i(x) = K_i$  for each  $i \in I$ , then Lemmas 3.1, 3.2 and 3.3 are considered in [1].

#### 4. Existence results under lower semicontinuity

Rest of the paper, unless otherwise specified, we assume that  $I$  is any index set and for each  $i \in I$ ,  $Y_i$  is a topological vector space,  $K = \prod_{i \in I} K_i$ ,  $C_i : K \rightarrow 2^{Y_i}$  is a multivalued map such that for all  $x \in K$ ,  $C_i(x)$  is a proper, closed and convex cone with apex at the origin and  $\text{int } C_i(x) \neq \emptyset$ , and the graph of the multivalued map  $W_i : K \rightarrow 2^{Y_i}$  defined by  $W_i(x) = Y_i \setminus \{-\text{int } C_i(x)\}$  for all  $x \in K$ , is closed. For each  $i \in I$ , we also assume that  $A_i : K \rightarrow 2^{K_i}$  is a multivalued map such that for all  $x \in K$ ,  $A_i(x)$  is nonempty and convex,  $A_i^{-1}(y_i)$  is open in  $K$  for all  $y_i \in K_i$  and the set  $\mathcal{F}_i := \{x \in K : x_i \in A_i(x)\}$  is closed in  $K$ , where  $x_i$  is the  $i$ th component of  $x$ .

Let us recall the following definitions.

**Definition 4.1.** (See [1].) For each  $i \in I$ , let  $F_i : K \rightarrow 2^{D_i}$  be a multivalued map with nonempty values. A family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  is called  $C_i(x)$ -quasiconvex-like w.r.t.  $\{F_i\}_{i \in I}$  if for all  $x \in K$ ,  $y'_i, y''_i \in K_i$  and  $\alpha \in [0, 1]$ , we either have  $\forall u_i \in F_i(x)$ ,

$$\psi_i(u_i, x_i, \alpha y'_i + (1 - \alpha) y''_i) \in \psi_i(u_i, x_i, y'_i) - \text{int } C_i(x),$$

or

$$\psi_i(u_i, x_i, \alpha y'_i + (1 - \alpha) y''_i) \in \psi_i(u_i, x_i, y''_i) - \text{int } C_i(x).$$

**Definition 4.2.** (See [1].) For each  $i \in I$ , let  $F_i : K \rightarrow 2^{D_i}$  be multivalued map with nonempty values. A family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  is called *simultaneously  $C_i(x)$ -quasiconvex-like w.r.t.  $\{F_i\}_{i \in I}$*  if for all  $x \in K$ ,  $y'_i, y''_i \in K_i$  and  $\alpha \in [0, 1]$ , we either have  $\forall u'_i, u''_i \in F_i(x)$ ,

$$\psi_i(\alpha u'_i + (1 - \alpha)u''_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u'_i, x_i, y'_i) - \text{int } C_i(x),$$

or

$$\psi_i(\alpha u'_i + (1 - \alpha)u''_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u''_i, x_i, y''_i) - \text{int } C_i(x).$$

Now we establish an existence result for a solution of Problem 1 under lower semicontinuity of the family of multivalued maps involved in the formulation of the problem.

**Theorem 4.1.** For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a Hausdorff topological vector space  $X_i$ . For each  $i \in I$ , let  $F_i : K \rightarrow 2^{K_i}$  be a lower semicontinuous multivalued map with nonempty convex values and  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  be a function such that the following conditions are satisfied:

- (i) For all  $x \in K$ , the family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i$  is simultaneously  $C_i(x)$ -quasiconvex-like w.r.t.  $\{F_i\}_{i \in I}$ ;
- (ii) For all  $x \in K$  and for all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \notin -\text{int } C_i(x)$ ;
- (iii) For each fixed  $y_i$ , the map  $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$  is continuous on  $D_i \times K_i$ ;
- (iv) There exist a nonempty compact subset  $M$  of  $K$  and a nonempty compact convex subset  $N_i$  of  $K_i$  for each  $i \in I$  such that for all  $x \in K \setminus M$ , there exist  $i \in I$  and  $\tilde{y}_i \in N_i$  such that  $\tilde{y}_i \in A_i(x)$  and  $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int } C_i(x)$  for all  $u_i \in F_i(x)$ .

Then Problem 1 has a solution.

**Proof.** For all  $x \in K$  and for each  $i \in I$ , define a multivalued map  $P_i : K \rightarrow 2^{K_i}$  by

$$P_i(x) = \{y_i \in K_i : \exists u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \in -\text{int } C_i(x)\}.$$

Since for all  $x \in K$  and for all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \notin -\text{int } C_i(x)$ , we have  $x_i \notin P_i(x)$ .

For all  $x \in K$ ,  $P_i(x)$  is convex. Indeed, let  $y'_i, y''_i \in P_i(x)$ . Then

$$\exists u'_i \in F_i(x) \quad \text{such that } \psi_i(u'_i, x_i, y'_i) \in -\text{int } C_i(x) \tag{4.1}$$

and

$$\exists u''_i \in F_i(x) \quad \text{such that } \psi_i(u''_i, x_i, y''_i) \in -\text{int } C_i(x). \tag{4.2}$$

Since  $F_i(x)$  is convex, we have  $\hat{u}_i = \alpha u'_i + (1 - \alpha)u''_i \in F_i(x)$  for all  $\alpha \in [0, 1]$ . For all  $\alpha \in [0, 1]$ , from condition (i) and (4.1)–(4.2) we either have

$$\psi_i(\alpha u'_i + (1 - \alpha)u''_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u'_i, x_i, y'_i) - \text{int } C_i(x) \subseteq -\text{int } C_i(x)$$

or

$$\psi_i(\alpha u'_i + (1 - \alpha)u''_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u''_i, x_i, y''_i) - \text{int } C_i(x) \subseteq -\text{int } C_i(x).$$

In either case, we have

$$\exists \hat{u}_i = \alpha u'_i + (1 - \alpha)u''_i \in F_i(x) : \quad \psi_i(\hat{u}_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in -\text{int } C_i(x).$$

That is,  $\alpha y'_i + (1 - \alpha)y''_i \in P_i(x)$ , and so  $P_i(x)$  is convex.

The complement of  $P_i^{-1}(y_i)$  in  $K$ ,

$$[P_i^{-1}(y_i)]^c = \{x \in K : \forall u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x)\}$$

is closed in  $K$ .



Indeed, let  $\{x^n\}$  be a net in  $[P_i^{-1}(y_i)]^c$  such that  $x^n \rightarrow x^* \in K$  (componentwise). Then for each  $i \in I$ , and  $\forall u_i^n \in F_i(x^n)$  we have  $\psi_i(u_i^n, x_i^n, y_i) \notin -\text{int } C_i(x^n)$ , that is

$$\psi_i(u_i^n, x_i^n, y_i) \in W_i(x^n) = Y_i \setminus \{-\text{int } C_i(x^n)\}. \tag{4.3}$$

By lower semicontinuity of  $F_i$ , for any  $u_i^* \in F_i(x^*)$ , there exists  $\tilde{u}_i^n \in F_i(x^n)$  such that  $\{\tilde{u}_i^n\}$  converges to  $u_i^*$ . Since (4.3) is true for all  $u_i^n \in F_i(x^n)$ , therefore, it also holds for  $\tilde{u}_i^n \in F_i(x^n)$ , that is

$$\psi_i(\tilde{u}_i^n, x_i^n, y_i) \in W_i(x^n).$$

Since  $\tilde{u}_i^n \rightarrow u_i^*$ ,  $x_i^n \rightarrow x^*$  and  $\psi_i(\cdot, \cdot, y_i)$  is continuous on  $D_i \times K_i$ , we have

$$\psi_i(\tilde{u}_i^n, x_i^n, y_i) \rightarrow \psi_i(u_i^*, x_i^*, y_i).$$

Since the graph of  $W_i$  is closed, we have

$$\psi_i(u_i^*, x_i^*, y_i) \in W_i(x^*) \Rightarrow \psi_i(u_i^*, x_i^*, y_i) \notin -\text{int } C_i(x^*),$$

that is,

$$\forall u_i^* \in F_i(x^*), \quad \psi_i(u_i^*, x_i^*, y_i) \notin -\text{int } C_i(x^*).$$

Hence  $x^* \in [P_i^{-1}(y_i)]^c$  and thus  $[P_i^{-1}(y_i)]^c$  is closed in  $K$ . Therefore,  $P_i^{-1}(y_i)$  is open in  $K$ .

For each  $i \in I$  and for all  $x \in K$ , define another multivalued map  $S_i : K \rightarrow 2^{K_i}$  by

$$S_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in \mathcal{F}_i, \\ A_i(x), & \text{if } x \in K \setminus \mathcal{F}_i. \end{cases}$$

Then, clearly for each  $i \in I$  and for all  $x \in K$ ,  $S_i(x)$  is convex and  $x_i \notin \text{co } S_i(x)$  since  $x_i \notin P_i(x)$ . Since for each  $i \in I$  and for all  $y_i \in K_i$ ,

$$S_i^{-1}(y_i) = (A_i^{-1}(y_i) \cap P_i^{-1}(y_i)) \cup ((K \setminus \mathcal{F}_i) \cap A_i^{-1}(y_i))$$

(see, for example, the proof of Lemma 2.3 in [17]) and  $A_i^{-1}(y_i)$ ,  $P_i^{-1}(y_i)$  and  $K \setminus \mathcal{F}_i$  are open in  $K$ , we have  $S_i^{-1}(y_i)$  is open in  $K$ .

Condition (iii) of Theorem 2.1 is followed from condition (iv). Then all the conditions of Theorem 2.1 are satisfied and hence there exists  $\bar{x} \in K$  such that  $S_i(\bar{x}) = \emptyset$  for each  $i \in I$ . Since for each  $i \in I$  and for all  $x \in K$ ,  $A_i(x)$  is nonempty, we have  $A_i(\bar{x}) \cap S_i(\bar{x}) = \emptyset$  for each  $i \in I$ . Therefore for each  $i \in I$ ,  $\bar{x}_i \in A_i(\bar{x})$  and for all  $\bar{u}_i \in F_i(\bar{x})$  satisfying

$$\psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(x), \quad \forall y_i \in A_i(\bar{x}),$$

and so  $\bar{x} \in K$  is a solution of Problem 1.  $\square$

Now we present the existence result for a solution of Problem 1 without any coercivity condition but for  $\Phi$ -condensing maps.

**Theorem 4.2.** For each  $i \in I$ , let  $K_i$  be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space  $X_i$  and let the multivalued map  $A = \prod_{i \in I} A_i : K \rightarrow 2^K$  defined as  $A(x) = \prod_{i \in I} A_i(x)$  for all  $x \in K$ , be  $\Phi$ -condensing. Assume that the conditions (i)–(iii) of Theorem 4.1 hold. Then Problem 1 has a solution.

**Proof.** In view of Remark 2.2, it is sufficient to show that the multivalued map  $S : K \rightarrow 2^K$  defined as  $S(x) = \prod_{i \in I} S_i(x)$  for all  $x \in K$ , is  $\phi$ -condensing, where  $S_i$ 's are the same as defined in the proof of Theorem 4.1. By the definition of  $S_i$ ,  $S_i(x) \subseteq A_i(x)$  for each  $i \in I$  and for all  $x \in K$  and therefore  $S(x) \subseteq A(x)$  for all  $x \in K$ . Since  $A$  is  $\Phi$ -condensing, by Remark 2.1, we have  $S$  is also  $\Phi$ -condensing.  $\square$

**Remark 4.1.** Best of our knowledge, there is no result on the existence of a solution of system of vector quasi-equilibrium problems under lower semicontinuity assumption. Therefore, Theorems 4.1 and 4.2 are new in the literature.

### 5. Existence results under pseudomonotonicity

Now we prove the existence of a solution of (SGIVQEP) under generalized pseudomonotonicity assumption.

**Theorem 5.1.** *For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a Hausdorff topological vector space  $X_i$ . For each  $i \in I$ , let  $F_i : K \rightarrow 2^{K_i}$  be a multivalued map with nonempty values and  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  be a function such that the following conditions are satisfied:*

- (i) *The family  $\{F_i\}_{i \in I}$  of multivalued maps  $F_i$  is generalized pseudomonotone w.r.t.  $\{\psi_i\}_{i \in I}$ ;*
- (ii) *For all  $x \in K$ , the family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i$  is  $C_i(x)$ -quasiconvex-like w.r.t.  $\{F_i\}_{i \in I}$ ;*
- (iii) *For all  $x \in K$  and for all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \notin -\text{int } C_i(x)$ ;*
- (iv) *For each fixed  $(v_i, y_i) \in D_i \times K_i$ , the map  $x_i \mapsto \psi_i(v_i, y_i, x_i)$  is continuous on  $K_i$ ;*
- (v) *There exist a nonempty compact subset  $M$  of  $K$  and a nonempty compact convex subset  $N_i$  of  $K_i$  for each  $i \in I$  such that for all  $x \in K \setminus M$ , there exist  $i \in I$  and  $\tilde{y}_i \in N_i$  such that  $\tilde{y}_i \in A_i(x)$  and  $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int } C_i(x)$  for all  $u_i \in F_i(x)$ .*

Then Problem 4 has a solution.

**Proof.** For all  $x \in K$  and for each  $i \in I$ , define two multivalued maps  $P_i, Q_i : K \rightarrow 2^{K_i}$  by

$$P_i(x) = \{y_i \in K_i : \exists v_i \in F_i(y) \text{ such that } \psi_i(v_i, y_i, x_i) \in \text{int } C_i(x)\}$$

and

$$Q_i(x) = \{y_i \in K_i : \forall u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \in -\text{int } C_i(x)\}.$$

From condition (iii), we have  $\psi_i(u_i, x_i, x_i) \notin -\text{int } C_i(x)$  and so  $x_i \notin Q_i(x)$ .

For all  $x \in K$  and for each  $i \in I$ ,  $P_i(x) \subseteq Q_i(x)$  by generalized pseudomonotonicity of  $\{F_i\}_{i \in I}$  w.r.t.  $\{\psi_i\}_{i \in I}$ .

If for all  $x \in K$  and for each  $i \in I$ ,  $Q_i(x)$  is convex, then  $\text{co } P_i(x) \subseteq \text{co } Q_i(x) = Q_i(x)$ . Indeed, let  $y'_i, y''_i \in Q_i(x)$ , then  $\forall u_i \in F_i(x)$ , we have

$$\psi_i(u_i, x_i, y'_i) \in -\text{int } C_i(x) \quad \text{and} \quad \psi_i(u_i, x_i, y''_i) \in -\text{int } C_i(x). \tag{5.1}$$

Since  $\{\psi_i\}_{i \in I}$  is  $C_i(x)$ -quasiconvex-like and from (5.1), for all  $\alpha \in [0, 1]$  and  $\forall u_i \in F_i(x)$ , we either have

$$\psi_i(u_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u_i, x_i, y'_i) - \text{int } C_i(x) \subseteq -\text{int } C_i(x)$$

or

$$\psi_i(u_i, x_i, \alpha y'_i + (1 - \alpha)y''_i) \in \psi_i(u_i, x_i, y''_i) - \text{int } C_i(x) \subseteq -\text{int } C_i(x).$$

In either case, we have  $\alpha y'_i + (1 - \alpha)y''_i \in Q_i(x)$  for all  $\alpha \in [0, 1]$  and so  $Q_i(x)$  is convex.

The complement of  $P_i^{-1}(y_i)$  in  $K$ ,

$$[P_i^{-1}(y_i)]^c = \{x \in K : \forall v_i \in F_i(y) \text{ such that } \psi_i(v_i, y_i, x_i) \notin \text{int } C_i(x)\}$$

is closed in  $K$ .

Indeed, let  $\{x^n\}$  be a net in  $[P_i^{-1}(y_i)]^c$  such that  $x^n \rightarrow x^* \in K$  (componentwise). Then for each  $i \in I$  and  $\forall v_i \in F_i(y)$ , we have  $\psi_i(v_i, y_i, x_i^n) \notin \text{int } C_i(x^n)$ , that is,

$$\psi_i(v_i, y_i, x_i^n) \in W_i(x^n) = Y_i \setminus \{\text{int } C_i(x^n)\}.$$

Since  $\psi_i(v_i, y_i, \cdot)$  is continuous on  $K_i$  and the graph of  $W_i$  is closed, we have,

$$\psi_i(v_i, y_i, x_i^n) \rightarrow \psi_i(v_i, y_i, x_i^*) \in W_i(x^*) \implies \psi_i(v_i, y_i, x_i^*) \notin \text{int } C_i(x^*).$$

That is,  $x^* \in [P_i^{-1}(y_i)]^c$  and thus  $[P_i^{-1}(y_i)]^c$  is closed in  $K$ . Therefore,  $P_i^{-1}(y_i)$  is open in  $K$ .

For each  $i \in I$  and for all  $x \in K$ , define other multivalued maps  $S_i, T_i : K \rightarrow 2^{K_i}$  as

$$S_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in \mathcal{F}_i, \\ A_i(x), & \text{if } x \in K \setminus \mathcal{F}_i, \end{cases}$$

and

$$T_i(x) = \begin{cases} A_i(x) \cap Q_i(x), & \text{if } x \in \mathcal{F}_i, \\ A_i(x), & \text{if } x \in K \setminus \mathcal{F}_i. \end{cases}$$

Then, clearly for each  $i \in I$  and for all  $x \in K$ ,  $\text{co } S_i(x) \subseteq T_i(x)$  since  $\text{co } P_i(x) \subseteq Q_i(x)$  and  $A_i(x)$  is convex. Also, for each  $i \in I$  and for all  $x \in K$ ,  $x_i \notin T_i(x)$  because  $x_i \notin Q_i(x)$ . Since for each  $i \in I$  and for all  $y_i \in K_i$ ,

$$S_i^{-1}(y_i) = (A_i^{-1}(y_i) \cap P_i^{-1}(y_i)) \cup ((K \setminus \mathcal{F}_i) \cap A_i^{-1}(y_i))$$

and  $A_i^{-1}(y_i)$ ,  $P_i^{-1}(y_i)$  and  $K \setminus \mathcal{F}_i$  are open in  $K$ , we have  $S_i^{-1}(y_i)$  is open in  $K$ .

Condition (iv) of Theorem 2.1 is followed from condition (v). Then all the conditions of Theorem 2.1 are satisfied and hence there exists  $\bar{x} \in K$  such that  $S_i(\bar{x}) = \emptyset$  for each  $i \in I$ . Since for each  $i \in I$  and for all  $x \in K$ ,  $A_i(x)$  is nonempty, we have  $A_i(\bar{x}) \cap S_i(\bar{x}) = \emptyset$  for each  $i \in I$ . Therefore for each  $i \in I$ ,  $\bar{x}_i \in A_i(\bar{x})$  and for all  $\tilde{u}_i \in F_i(\bar{x})$  satisfying

$$\psi_i(v_i, y_i, \bar{x}_i) \notin \text{int } C_i(x), \quad \forall y_i \in A_i(\bar{x}),$$

and so  $\bar{x} \in K$  is a solution of Problem 4.  $\square$

**Theorem 5.2.** For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a Hausdorff topological vector space  $X_i$ . For each  $i \in I$ , let  $F_i : K \rightarrow 2^{K_i}$  be a multivalued map with nonempty values and  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  be a function such that the following conditions are satisfied:

- (i) The family  $\{F_i\}_{i \in I}$  of multivalued maps  $F_i$  is  $u$ -hemicontinuous and generalized pseudomonotone w.r.t.  $\{\psi_i\}_{i \in I}$ ;
- (ii) The family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i$  is  $C_i$ -convex in the third argument;
- (iii) For all  $s_i \in L(X_i, Y_i)$ ,  $x, y, z \in X$  and  $\alpha \in [0, 1]$ ,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1 - \alpha)\psi_i(s_i, x_i, z_i);$$

- (iv) For all  $x \in K$  and for all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \in \tilde{C}_i$ ;
- (v) For each fixed  $(v_i, y_i) \in D_i \times K_i$ , the map  $x_i \mapsto \psi_i(v_i, y_i, x_i)$  is continuous on  $K_i$ ;
- (vi) There exist a nonempty compact subset  $M$  of  $K$  and a nonempty compact convex subset  $N_i$  of  $K_i$  for each  $i \in I$  such that for all  $x \in K \setminus M$ , there exist  $i \in I$  and  $\tilde{y}_i \in N_i$  such that  $\tilde{y}_i \in A_i(x)$  and  $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int } C_i(x)$  for all  $u_i \in F_i(x)$ .

Then Problem 3 has a solution.

**Proof.** For each  $i \in I$ , let  $P_i$ ,  $Q_i$ ,  $S_i$  and  $T_i$  be the same as defined in the proof of Theorem 3.1. Then by using conditions (ii)–(iv), it is easy to see that for all  $x \in K$ ,  $Q_i(x)$  is convex and so is  $T_i(x)$ .

From the proof of Theorem 5.1, there exists a solution  $\bar{x} \in K$  of Problem 4. In view of Lemmas 3.1 and 3.2,  $\bar{x} \in K$  is a solution of Problem 3.  $\square$

Now we prove the existence of a strong solution of Problem 2.

**Theorem 5.3.** For each  $i \in I$ , let  $K_i$  be a nonempty convex subset of a real Banach space  $X_i$  and  $Y_i$  be a real Banach space. For each  $i \in I$ , let  $F_i : K \rightarrow 2^{K_i}$  be a multivalued map with nonempty compact values and  $\psi_i : D_i \times K_i \times K_i \rightarrow Y_i$  be a function such that the following conditions are satisfied:

- (i) The family  $\{F_i\}_{i \in I}$  of multivalued maps  $F_i$  is  $\mathcal{H}$ -hemicontinuous and generalized pseudomonotone w.r.t.  $\{\psi_i\}_{i \in I}$ ;
- (ii) The family  $\{\psi_i\}_{i \in I}$  of functions  $\psi_i$  is  $C_i$ -convex in the third argument;
- (iii) For all  $s_i \in L(X_i, Y_i)$ ,  $x, y, z \in X$  and  $\alpha \in [0, 1]$ ,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1 - \alpha)\psi_i(s_i, x_i, z_i);$$

- (iv) For all  $x \in K$  and for all  $u_i \in F_i(x)$ ,  $\psi_i(u_i, x_i, x_i) \in \tilde{C}_i$ ;
- (v) For each fixed  $(v_i, y_i) \in D_i \times K_i$ , the map  $x_i \mapsto \psi_i(v_i, y_i, x_i)$  is continuous on  $K_i$ ;

- (vi) *There exist a nonempty compact subset  $M$  of  $K$  and a nonempty compact convex subset  $N_i$  of  $K_i$  for each  $i \in I$  such that for all  $x \in K \setminus M$ , there exist  $i \in I$  and  $\tilde{y}_i \in N_i$  such that  $\tilde{y}_i \in A_i(x)$  and  $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int } C_i(x)$  for all  $u_i \in F_i(x)$ .*

*Then Problem 2 has a solution.*

**Proof.** For each  $i \in I$ , let  $P_i$ ,  $Q_i$ ,  $S_i$  and  $T_i$  be the same as defined in the proof of Theorem 5.1. Then by using conditions (ii)–(iv), it is easy to see that for all  $x \in K$ ,  $Q_i(x)$  is convex and so is  $T_i(x)$ .

From the proof of Theorem 5.1, there exists a solution  $\bar{x} \in K$  of Problem 4. From Lemma 3.3,  $\bar{x} \in K$  is a solution of Problem 2.  $\square$

**Remark 5.1.** By using the technique of [3,4,6,7,10,11,22–24,28], it is easy to derive the existence of a solution of constrained Nash equilibrium problem for nondifferentiable and nonconvex functions from Theorems 4.1, 4.2, 5.1, 5.2 and 5.3. By using the technique of [2,9,14], one can easily establish the equivalence between systems of vector quasi-optimization problems and Problem 1 or Problem 2. Since Theorems 4.1, 4.2 and 5.3 provide the existence of a solution of Problem 1 and so Problem 2, we will have necessary and sufficient conditions for a solution of system of vector quasi-optimization problems. By using the technique of [26,29], we can derive further applications of SGIVQEP to the fixed point theory of nonexpansive maps and mathematical programs with equilibrium constraint.

## References

- [1] S. Al-Homidan, Q.H. Ansari, S. Schaible, Existence of solutions of systems of generalized implicit vector variational inequalities, *J. Optim. Theory Appl.* 134 (3) (2007) 515–531.
- [2] Q.H. Ansari, A note on generalized vector variational-like inequalities, *Optimization* 41 (1997) 197–205.
- [3] Q.H. Ansari, W.K. Chan, X.Q. Yang, The system of vector quasi-equilibrium problems with applications, *J. Global Optim.* 29 (2004) 45–57.
- [4] Q.H. Ansari, W.K. Chan, X.Q. Yang, Weighted quasi-variational inequalities and constrained Nash equilibrium problems, *Taiwanese J. Math.* 10 (2006) 361–380.
- [5] Q.H. Ansari, A. Idzik, J.C. Yao, Coincidence and fixed point theorems with applications, *Topological Methods Nonlinear Anal.* 15 (2000) 191–202.
- [6] Q.H. Ansari, Z. Khan, System of generalized vector quasi-equilibrium problems with applications, in: S. Nanda, G.P. Rajasekhar (Eds.), *Math. Anal. Appl.*, Narosa Publishing House, New Delhi, India, 2004, pp. 1–13.
- [7] Q.H. Ansari, L.J. Lin, L.B. Su, Systems of simultaneous generalized vector quasiequilibrium problems and their applications, *J. Optim. Theory Appl.* 127 (2005) 27–44.
- [8] Q.H. Ansari, S. Schaible, J.C. Yao, System of generalized vector equilibrium problems with applications, *J. Global Optim.* 22 (2002) 3–16.
- [9] Q.H. Ansari, A.H. Siddiqi, A generalized vector variational-like inequality and optimization over an efficient set, in: M. Brokate, A.H. Siddiqi (Eds.), *Functional Analysis with Current Applications in Science, Technology and Industry*, in: Pitman Research Notes in Mathematics Series, vol. 377, Addison Wesley Longman Limited, Essex, 1998, pp. 177–191.
- [10] Q.H. Ansari, J.C. Yao, Systems of generalized variational inequalities and their applications, *Appl. Anal.* 76 (3–4) (2000) 203–217.
- [11] Q.H. Ansari, J.C. Yao, On nondifferentiable and nonconvex vector optimization problems, *J. Optim. Theory Appl.* 106 (3) (2000) 487–500.
- [12] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston–Basel–Berlin, 1990.
- [13] S. Chebbi, M. Florenzano, Maximal elements and equilibria for condensing correspondences, *Nonlinear Anal.* 38 (1999) 995–1102.
- [14] G.Y. Chen, B.D. Craven, A vector variational inequality and optimization over an efficient set, *Math. Methods Oper. Res.* 3 (1990) 1–12.
- [15] G. Debreu, A social equilibrium existence theorem, *Proc. Natl. Acad. Sci. USA* 38 (1952) 886–893.
- [16] P. Deguire, K.K. Tan, G.X.-Z. Yuan, The study of maximal element, fixed point for  $L_S$ -majorized mappings and their applications to minimax and variational inequalities in the product topological spaces, *Nonlinear Anal.* 37 (1999) 933–951.
- [17] X.P. Ding, Existence of solutions for quasi-equilibrium problems in noncompact topological spaces, *Comput. Math. Appl.* 39 (2000) 13–21.
- [18] X.P. Ding, System of generalized vector quasi-equilibrium problems in locally  $FC$ -spaces, *Acta Math. Sin. (Engl. Ser.)* 22 (2006) 1529–1538.
- [19] X.-P. Ding, J.-C. Yao, Maximal element theorems with applications to generalized games and system of generalized vector quasi-equilibrium problems in  $G$ -convex spaces, *J. Optim. Theory Appl.* 126 (2005) 571–588.
- [20] X.-P. Ding, J.-C. Yao, L.-J. Lin, Solutions of system of generalized vector quasi-equilibrium problems in locally  $G$ -convex uniform spaces, *J. Math. Anal. Appl.* 298 (2004) 389–410.
- [21] J.-Y. Fu, Generalized vector quasi-equilibrium problems, *Math. Methods Oper. Res.* 52 (2000) 57–64.
- [22] G.M. Lee, On relations between vector variational inequality and vector optimization problem, in: X.Q. Yang, et al. (Eds.), *Progress in Optimization*, Kluwer Academic, Dordrecht–Boston–London, 2000, pp. 167–179.
- [23] G.M. Lee, D.S. Kim, Existence of solutions for vector optimization problems, *J. Math. Anal. Appl.* 220 (1998) 90–98.
- [24] G.M. Lee, D.S. Kim, B.S. Lee, Generalized vector variational inequality, *Appl. Math. Lett.* 9 (1) (1996) 39–42.
- [25] G.M. Lee, S. Kum, On implicit vector variational inequalities, *J. Optim. Theory Appl.* 104 (2) (2000) 409–425.
- [26] L.-J. Lin, System of generalized vector quasi-equilibrium problems with applications to fixed point theorems for a family of nonexpansive multivalued maps, *J. Global Optim.* 34 (2006) 15–32.

- [27] L.-J. Lin, Q.H. Ansari, Collective fixed points and maximal elements with applications to abstract economies, *J. Math. Anal. Appl.* 296 (2004) 455–472.
- [28] L.-J. Lin, H.-W. Hsu, Existence theorems for systems of generalized vector quasiequilibrium problems and optimization problems, *J. Global Optim.* 37 (2007) 195–213.
- [29] L.-J. Lin, Y.H. Liu, Existence theorems for systems of generalized vector quasi-equilibrium problems and mathematical programs with equilibrium constraint, *J. Optim. Theory Appl.* 130 (2006) 461–475.
- [30] Z. Lin, J. Yu, The existence of solutions for the systems of generalized vector quasi-equilibrium problems, *Appl. Math. Lett.* 18 (2005) 415–422.
- [31] S. Kum, G.M. Lee, Remarks on implicit vector variational inequalities, *Taiwanese J. Math.* 6 (3) (2002) 369–382.
- [32] S.B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.* 30 (1969) 475–488.
- [33] J. Nash, Non-cooperative games, *Ann. of Math.* 54 (1951) 286–295.
- [34] W.V. Petryshyn, P.M. Fitzpatrick, Fixed point theorems of multivalued noncompact inward maps, *J. Math. Anal. Appl.* 46 (1974) 756–767.
- [35] W.V. Petryshyn, P.M. Fitzpatrick, Fixed point theorems of multivalued noncompact acyclic mappings, *Pacific J. Math.* 54 (1974) 17–23.
- [36] G.X.-Z. Yuan, G. Isac, K.-K. Tan, J. Yu, The study of minimax inequalities, abstract economies and applications to variational inequalities and Nash equilibria, *Acta Appl. Math.* 54 (1998) 135–166.
- [37] G.X.-Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, 1999.
- [38] L.-C. Zeng, J.C. Yao, Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces, *J. Global Optim.* 36 (2006) 483–497.