King Fahd University of Petroleum & Minerals

Department of Mathematical Sciences

MATH - 202 Semester 062 Major Examination - I

Solution

Venue: Bldg. 5, Room 201
Time: 6:30 p.m. to 8:00 p.m.
Date: March 21, 2007
Max. Marks: 40

Name:____

ID#:____

S. No#:____

Section#:____

Instructions:

1. *Programmable calculators and mobile phones are* **NOT** *allowed in the examination hall.*
2. *Clearly indicate the theorem/result while applying it to solve a problem.*
3. *Write all calculations in the answer sheet.*

Instructor: Dr. Qamrul Hasan Ansari

1. (a) Let $y = c_1e^x + c_2e^{-x}$ be a two parameter family of solutions of second order ODE $y'' - y = 0$. Find a solution of the IVP consisting of this ODE and initial conditions $y(0) = 1$ and $y'(0) = 0$.

(3 marks)

(b) Use the family $y = c_1 + c_2x^2$ to find the solution of ODE $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1$ and $y'(1) = 6$. (3 marks)

Solution: (a)

- Differentiating $y = c_1e^x + c_2e^{-x}$ once, we get

$$y' = c_1e^x - c_2e^{-x}.$$

- By initial conditions, we obtain

$$y(0) = 1 \quad \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(0) = 0 \quad \Rightarrow \quad c_1 - c_2 = 0$$

Thus, $c_1 = 1/2 = c_2$.

- By substituting values of c_1 and c_2 in y, we get the solution of IVP: $y = (1/2)e^x + (1/2)e^{-x}$.

Solution: (b)

- Since $y = c_1 + c_2x^2$, we have $y' = 2c_2x$.
- By boundary conditions, we obtain

$$y(0) = 1 \quad \Rightarrow \quad c_1 = 1$$

$$y'(1) = 6 \quad \Rightarrow \quad c_2 = 3$$

Thus, $c_1 = 0 = c_2$.

- By substituting values of c_1 and c_2 in y, we get the solution of IVP: $y = 1 + 3x^2$.

2. Verify that the relation $x^2 + y^2 = 25$ is an implicit solution of the ODE $\dfrac{dy}{dx} = -\dfrac{x}{y}$ on the interval $-5 < x < 5$. (4 marks)

Solution: By implicit differentiation, we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad \text{(given ODE)}$$

Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfying the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $-5 < x < 5$.

3. Find a solution of ODE $x\dfrac{dy}{dx} = y^2 - y$ that passes through the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. (5 marks)

OR

Solve ODE $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$. (5 marks)

Solution:

- We write ODE as $\dfrac{dy}{y(y-1)} = \dfrac{dx}{x}$.
- Partial fraction LHS gives $\dfrac{1}{y(y-1)} = \dfrac{-1}{y} + \dfrac{1}{y-1}$ (See class notes).
- We re-write the ODE as $\left(\dfrac{-1}{y} + \dfrac{1}{y-1}\right) dy = \dfrac{dx}{x}$.
- Integrating above equation, we obtain

$$-\ln y + \ln(y-1) = \ln x + \ln c \quad \Rightarrow \quad y = \frac{1}{1 - cx}.$$

- Solution passing through $\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$\frac{1}{2} = \frac{1}{1 - c(1/2)} \quad \Rightarrow \quad c = -2.$$

- The solution is $y = \dfrac{1}{1 + 2x}$.

OR PART

- We write the ODE as

$$\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\left(\frac{y-2}{y+3}\right) dy = \left(\frac{x-1}{x+4}\right) dx \quad \Leftrightarrow \quad \left(\frac{(y+3)-5}{(y+3)}\right) dy = \left(\frac{(x+4)-5}{(x+4)}\right) dx$$

- Re-write as

$$\left(1 - \frac{5}{(y+3)}\right) dy = \left(1 - \frac{5}{(x+4)}\right) dx$$

- By integrating, we get the solution of the given ODE

$$y - 5\ln(y+3) = x - \ln(x+4) + c.$$

4. Find the solution of the ODE $y' + (\tan x)y = \cos^2 x$ subject to $y(0) = -1$. (5 marks)

Solution:

- Since the form of the given ODE is linear, we have $P(x) = \tan x$.
- $\int P(x)dx = \int \tan x dx = \ln(\sec x)$.
- Integrating factor, IF = $e^{\int P(x)dx} = 2^{\ln(\sec x)} = \sec x$.
- Multiplying ODE by IF, we obtain

$$\frac{d}{dx}\left((\sec x)y\right) = \cos x.$$

- Integrating both sides, we have

$$y = \sin x \cos x + c\cos x \quad \text{for } \pi/2 \leq x \leq \pi/2.$$

- By using initial conditions, we get $c = -1$.
- By putting this value of c in y, we get the solution of the IVP: $y = \sin x \cos x - \cos x$.

5. Find the value of k so that the ODE $(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$ is exact. (6 marks)

OR

Is $(\tan x - \sin x \sin y)dx + \cos x \cos y dy = 0$ exact? If so, solve it. (6 marks)

Solution:

- The given ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. From the given ODE, we have $M(x, y) = y^3 + kxy^4 - 2x$ and $N(x, y) = 3xy^2 + 20x^2y^3$.
- $\frac{\partial M}{\partial y} = 3y^2 + 4kxy^3$ and $\frac{\partial N}{\partial x} = 3y^2 + 40xy^3$.
- Since the given ODE is exact, we must have

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3$$

which gives $k = 10$.

OR PART

- From the given ODE, we set $M(x, y) = \tan x - \sin x \sin y$ and $N(x, y) = \cos x \cos y$
- $\frac{\partial M}{\partial y} = -\sin x \cos y$ and $\frac{\partial N}{\partial x} = -\sin x \cos y$
- Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given ODE is exact

- Since the given ODE is exact, there exists a function $f(x,y)$ such that $\dfrac{\partial f}{\partial x} = M(x,y) = \tan x - \sin x \sin y$ and $\dfrac{\partial f}{\partial y} = N(x,y)$
- To find $f(x,y)$, we integrate first expression w.r.t. x,
$$f(x,y) = \ln|\sec x| + \cos x \sin y + g(y)$$
where $g(y)$ is the integrating constant (a function of y).
- Taking the partial derivative of this expression w.r.t. y, we get
$$\frac{\partial f}{\partial y} = \cos x \cos y + g'(y)$$
- Setting $\dfrac{\partial f}{\partial y}$ equal to $N(x,y)$, we obtain
$$\cos x \cos y + g'(y) = \cos x \cos y \;\Rightarrow\; g'(y) \;\Rightarrow\; g(y) = 0$$
- Putting this value of $g(y)$ in $f(x,y)$, we get
$$f(x,y) = \ln|\sec x| + \cos x \sin y$$
- The solution of ODE is
$$\ln|\sec x| + \cos x \sin y = c.$$

6. Is $(y^2 + xy^3)dx + (5y^2 - xy + y^3 \sin y)dy = 0$ exact? If not, find the integrating factor and make it exact. (5 marks)

Solution:

- We set $M(x,y) = y^2 + xy^3$ and $N(x,y) = 5y^2 - xy + y^3 \sin y$, then $\dfrac{\partial M}{\partial y} = 2y + 3xy^2$ and $\dfrac{\partial N}{\partial x} = -y$
- Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the given ODE is not exact.
- We note that $\dfrac{N_x - M_y}{M} = -\dfrac{3}{y} = f(y)$, then the integrating factor is $\mu(y) = e^{-\int (3/y)dy} = 1/y^3$.
- Multiplying the given ODE by IF $\mu(y)$, we get
$$\frac{1}{y^3}(y^2 + xy^3)dx + \frac{1}{y^3}(5y^2 - xy + y^3 \sin y)dy = 0$$
or
$$\left(\frac{1}{y} + x\right)dx + \left(\frac{5}{y} - \frac{x}{y^2} + \sin y\right)dy = 0 \qquad (*)$$
- We set $\tilde{M} = \dfrac{1}{y} + x$ and $\tilde{N} = \dfrac{5}{y} - \dfrac{x}{y^2} + \sin y$, then
$$\frac{\partial \tilde{M}}{\partial y} = -\frac{1}{y^2} = \frac{\partial \tilde{N}}{\partial x}$$
and hence (*) is exact.

7. Solve the ODE $\frac{dy}{dx} = \sin(x+y)$ by an appropriate substitution. (5 marks)

Solution:

- We substitute $u = x + y$ in the given ODE so that $\frac{du}{dx} = 1 + \frac{dy}{dx}$. Then

$$\frac{du}{dx} - 1 = \sin u \quad \text{or} \quad \frac{1}{1+\sin u} du = dx.$$

- Multiplying last equation by $\frac{1+\sin u}{1-\sin u}$, we get

$$\left(\frac{1-\sin u}{\cos^2 u}\right) du = dx \quad \Leftrightarrow \quad \left(\sec^2 u - \tan u \sec u\right) du = dx.$$

- Integrating both sides, we obtain

$$\tan u - \sec u = x + c.$$

- Re-substitute the value of u, we obtain the solution of the given ODE

$$\tan(x+y) - \sec(x+y) = x + c.$$

8. Describe the population growth model. Write it in the form of an IVP. (4 marks)

OR

Describe the Newton's law of cooling / warming model. Write it in the form of an ODE. (4 marks)

Solution: See the Class Notes