## King Fahd University of Petroleum & Minerals Department of Mathematical Sciences

MATH - 201

Semester 061

Major Examination - I

## **Solution**

Venue: Bldg. 54-Hall		
Time: 10:15 p.m. to 11:45 p.m.		
Date: October 10, 2006		
Max. Marks: 40		
Name:		
ID#:		
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Section#:		

## Instructions:

- 1. Clearly indicate the theorem/result while applying it to solve a problem.
- 2. Indicate all calculations in the answer sheet.

Instructor: Dr. Qamrul Hasan Ansari

- 1. (a) Show that if a varies, then the polar equation  $r = a \sec \theta \ (\pi/2 < \theta < \pi/2)$  describes a family of lines perpendicular to the polar axis. (3 marks)
  - (b) Identify the polar curve  $r = \sec \theta \tan \theta$  by transforming it to rectangular coordinates.

(2 marks)

Solution: (a)

$$r = a \sec \theta \iff r = a \frac{1}{\cos \theta} \iff a = r \cos \theta \iff x = a$$

which represents the family of lines perpendicular to the polar axis (x-axis).

Solution: (b)

$$r = \sec \theta \tan \theta = \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} \Leftrightarrow r \cos^2 \theta = \sin \theta$$

or  $r^2 \cos^2 \theta = r \sin \theta \iff x^2 = y$  since  $x = r \cos \theta$  and  $y = r \sin \theta$ .  $y = x^2$  represents the parabola.

2. Sketch the polar curve  $r = -5 + 5\sin\theta$  and find its arc length. (5 marks)

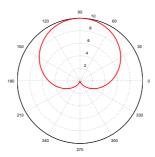
OR

Sketch the polar curve  $r = \sin 2\theta$  and find the polar equations of the tangent lines to this polar curve at the pole. (5 marks)

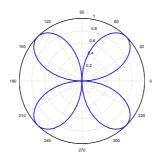
**Solution:** Since  $\frac{dr}{d\theta} = 5\cos\theta$ ,

Arc length 
$$= \int_0^{2\pi} \sqrt{(-5+5\sin\theta)^2 + (5\cos\theta)^2} d\theta$$
$$= \int_0^{2\pi} \sqrt{50(1-\sin\theta)} d\theta$$
$$= \sqrt{50} \int_0^{2\pi} \sqrt{(1-\sin\theta)} \frac{\sqrt{(1+\sin\theta)}}{\sqrt{(1+\sin\theta)}} d\theta$$
$$= \sqrt{50} \int_0^{2\pi} \frac{\cos\theta}{\sqrt{(1+\sin\theta)}} d\theta = 2\sqrt{50}$$

by using the substitution method.



<u>Solution of OR PART</u>:  $r = \sin 2\theta = 0$  when  $\theta = 0, \pi/2$ . This implies that the curve passes through the origin at  $\theta = 0, \pi/2$ .  $\frac{dr}{d\theta} = 2\cos 2\theta$  and  $\frac{dr}{d\theta}|_{\theta=0,\pi/2} \neq 0$ , we have that  $\theta = 0$  and  $\theta = \pi/2$  are the tangent lines to the curve at pole.

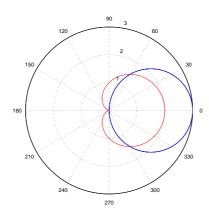


3. Sketch the polar curves  $r = 1 + \cos \theta$  and  $r = 3 \cos \theta$  and find the area of the region inside both the curves. (5 marks)

**Solution:** To find the points of intersection, we take  $1 + \cos \theta = 3 \cos \theta$  which implies that  $\cos \theta = 1/2$  and so  $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$ .

Area = 
$$2 \times \frac{1}{2} \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta + 2 \times \frac{1}{2} \int_{\pi/3}^{\pi/2} (3 \cos \theta)^2 d\theta$$
  
=  $2 \left( \frac{\pi}{4} + \frac{9\sqrt{3}}{16} + \frac{3\pi}{8} - \frac{9\sqrt{3}}{16} \right)$   
=  $\frac{5\pi}{4}$ 

(Use  $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$ ).



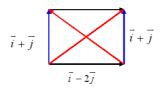
4. A sphere S has center in the first octant and is tangent to each of the three coordinate planes. The distance from the origin to the sphere is  $3 - \sqrt{3}$  units. What is the equation of the sphere? (5 marks)

Solution: Let  $P(x_0, y_0, z_0)$  and r denote the center and radius of S, respectively. In order for S to be tangent to the xy-plane, the distance  $|z_0|$  from  $P(x_0, y_0, z_0)$  to the xy-plane must equal r. Since  $P(x_0, y_0, z_0)$  is in the first octant, we conclude that  $z_0 = |z_0| = r$ . Similarly,  $x_0 = y_0 = r$  and the center of S is P(r, r, r). The distance from the origin to the center of S is then  $\sqrt{r^2 + r^2 + r^2} = \sqrt{3r^2} = \sqrt{3}r$ , from which it follows that the distance  $3 - \sqrt{3} = \sqrt{3}(\sqrt{3} - 1)$  from the origin to S is given by  $\sqrt{3}r - r = (\sqrt{3} - 1)r$ . Solving the equation  $(\sqrt{3} - 1)r = \sqrt{3}(\sqrt{3} - 1)$  yields the solution  $r = \sqrt{3}$ . Therefore, the equation of the sphere is

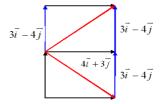
$$(x - \sqrt{3})^2 + (y - \sqrt{3})^2 + (z - \sqrt{3})^2 = 3.$$

5. Use vectors to find the lengths of the diagonals of parallelogram that has  $\overrightarrow{i} + \overrightarrow{j}$  and  $\overrightarrow{i} - 2\overrightarrow{j}$  as adjacent sides. (5 marks)

**Solution:** The diagonals are  $\overrightarrow{u_1} = \langle 1, 1 \rangle + \langle 1, -2 \rangle = \langle 2, -1 \rangle$  and  $\overrightarrow{u_2} = \langle 1, 1 \rangle - \langle 1, -2 \rangle = \langle 0, 3 \rangle$ . Therefore, the magnitude of these diagonals are  $\|\overrightarrow{u_1}\| = \sqrt{5}$  and  $\|\overrightarrow{u_2}\| = 3$ .



6. Find two unit vectors in 2-space that make an angle of 45° with  $4\overrightarrow{i} + 3\overrightarrow{j}$ . (5 marks) Solution: By inspection,  $3\overrightarrow{i} - 4\overrightarrow{j}$  is orthogonal to and has the same length as  $4\overrightarrow{i} + 3\overrightarrow{j}$  so  $\overrightarrow{u_1} = \langle 3, -4 \rangle + \langle 4, 3 \rangle = \langle 7, -1 \rangle$  and  $\overrightarrow{u_2} = -\langle 3, -4 \rangle + \langle 4, 3 \rangle = \langle 1, 7 \rangle$  make an angle of 45° with  $\langle 4, 3 \rangle$ . The unit vectors in the direction of  $\overrightarrow{u_1}$  and  $\overrightarrow{u_2}$  are  $\pm \frac{\langle 7, -1 \rangle}{\sqrt{50}}$ .



7. Find two unit vectors that are parallel to the yz-plane and orthogonal to the vector  $\overrightarrow{j} + 2\overrightarrow{k}$ . (5 marks)

OR

Using a scalar triple product to determine whether the vectors  $\overrightarrow{u} = \langle 4, -8, 1 \rangle$ ,  $\overrightarrow{v} = \langle 2, 1, -2 \rangle$  and  $\overrightarrow{w} = \langle 3, -4, 12 \rangle$  lie in the same plane. (5 marks)

Solution: A vector parallel to the yz-plane must be perpendicular to  $\overrightarrow{i}$ . Therefore  $\overrightarrow{i} \times (3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k})$  is perpendicular to both  $\overrightarrow{i}$  and  $3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}$ .

$$\overrightarrow{i} \times (3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & 0 \\ 3 & -1 & 2 \end{vmatrix} = -2\overrightarrow{j} - \overrightarrow{k}.$$

$$||\overrightarrow{i} \times (3\overrightarrow{i} - \overrightarrow{j} + 2\overrightarrow{k})|| = ||-2\overrightarrow{j} - \overrightarrow{k}|| = \sqrt{5}.$$

The unit vectors are  $\pm \frac{2\overrightarrow{j} + \overrightarrow{k}}{\sqrt{5}}$ .

## **Solution of OR PART:**

$$\begin{vmatrix} 4 & -8 & 1 \\ 2 & 1 & -2 \\ 3 & -4 & 12 \end{vmatrix} = 4(12 - 8) + 8(24 + 6) - 1(-8 - 3) = 16 + 240 - 11 \neq 0.$$

So the vectors  $\overrightarrow{u} = \langle 4, -8, 1 \rangle$ ,  $\overrightarrow{v} = \langle 2, 1, -2 \rangle$  and  $\overrightarrow{w} = \langle 3, -4, 12 \rangle$  do not lie in the same plane.

8. Show that the following lines are skew.

$$L_1$$
:  $x = 2 + 8t$ ,  $y = 6 - 8t$ ,  $z = 10t$ 

$$L_2$$
:  $x = 3 + 8t$ ,  $y = 5 - 3t$ ,  $z = 6 + t$ .

**Solution:** The vectors  $\overrightarrow{v_1} = \langle 8, -8, 10 \rangle$  and  $\overrightarrow{v_2} = \langle 8, -3, 1 \rangle$  are parallel to the lines  $L_1$  and  $L_2$ , respectively. Since  $\overrightarrow{v_1}$  and  $\overrightarrow{v_2}$  are not scalar multiple of each other,  $\overrightarrow{v_1}$  and  $\overrightarrow{v_2}$  are not parallel an so lines  $L_1$  and  $L_2$  are not parallel.

Let us re-write  $L_1$  and  $L_2$  as follows:

$$L_1$$
:  $x = 2 + 8t$ ,  $y = 6 - 8t$ ,  $z = 10t$ 

$$L_2$$
:  $x = 3 + 8s$ ,  $y = 5 - 3s$ ,  $z = 6 + s$ .

They are intersecting if there exist values of s and t such that

$$2 + 8t = 3 + 8s \tag{1}$$

$$6 - 8t = 5 - 3s \tag{2}$$

$$10t = 6 + s \tag{3}$$

From (1) and (2), we obtain s = 0 and t = 1/8 but these values of s and t does not satisfy (3). Hence  $L_1$  and  $L_2$  are not intersecting. Thus  $L_1$  and  $L_2$  are skew.